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A SIMPLE REDUCTION OF SINGLE LINEAR DIFFERENTIAL EQUATIONS TO BIRKHOFF AND SCHLESINGER'S CANONICAL SYSTEMS

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In the paper [4], H. L. Turrittin asked, reviewing Okubo's problem, whether a single linear differential equation with a reguler singularity and an irregular singularity of rank 1 can always be reduced to a system of linear differential equations of the Birkhoff form $tY'=(A+tB)Y$. K. Okubo answered his question for this case in a private letter. However, in a general case in which a single linear differential equation has an irregular singularity of an arbitrary rank, such a reduction problem seemed to be difficult and to be left unsolved.

Several years ago, the author tried to solve the reduction problem, together with another reduction of every single Fuchsian differential equation to the so-called hypergeometric system [3], in a lecture given at Professor R. Gérard's seminar of Strasbourg University, but the author has so far laid the result aside.

In recent years, the computer becomes very familiar even to mathematicians, who can now leave some difficult and tedious calculations to the computer through programs for algebraic manupulation by REDUCE, muMATH, MACSYMA, etc. The algorithm of our method of reduction seems to be quite suitable to algebraic computation by the computer. And this paper, which is based on the lecture note, is probably useful for establishing an effective algebraic computation system for the reduction of single differential equations to systems of the Birkhoff or Schlesinger form.

1. The purpose of this paper is to show *a simple and effective method* for the reduction of single linear differential equations with singularities to systems of linear differential equations which have the simple form as possible and have the same characteristic properties at the singularities. To illustrate the method, we treat of two reduction problems in this paper.

We first consider the reduction of the single linear differential equation

$$t^n \frac{d^n x}{dt^n} = \sum_{l=1}^{n} \left(\sum_{r=0}^{ql} a_{l,r} t^r \right) t^{n-l} \frac{d^{n-l} x}{dt^{n-l}} \tag{1.1}$$

to the Birkhoff canonical system

$$t \frac{dY}{dt} = B(t)\, Y = (B_0 + B_1 t + \cdots + B_q t^q)\, Y. \tag{1.2}$$

In (1.1) the characteristic constants ρ_j $(j = 1, 2, \cdots, n)$ of the regular singularity at the origin are given as roots of the equation

$$[\rho]_n = \sum_{l=1}^{n} a_{l,0} [\rho]_{n-l} \qquad ([\rho]_p = \rho(\rho-1)\cdots(\rho-p+1)),$$

and the principal characteristic constants λ_j $(j = 1, 2, \cdots, n)$ of the irregular singularity at infinity as roots of the equation

$$\lambda^n = \sum_{l=1}^{n} a_{l,ql}\, \lambda^{n-l}.$$

We here treat only the case in which no solutions near the regular singularity include logarithmic terms. We can then obtain the following

Theorem 1. *Assume that* ρ_j $(j = 1, 2, \cdots, n)$ *are mutually distinct and*

$$\rho_j \not\equiv \rho_i \pmod 1 \qquad (i \neq j\,;\, i, j = 1, 2, \cdots, n).$$

Then (1.1) *can be reduced to* (1.2) *with*

$$B(t) = \begin{bmatrix} b_{1,1}(t) & t^q & & & 0 \\ b_{1,2}(t) & b_{2,2}(t) & t^q & \ddots & \\ \vdots & \vdots & \ddots & \ddots & t^q \\ b_{n,1}(t) & b_{n,2}(t) & \cdots\cdots & & b_{n,n}(t) \end{bmatrix}, \tag{1.3}$$

$$\begin{cases} b_{j,j}(t) = \rho_j - (j-1)\,q + b^1_{j,j}\, t + \cdots + b^{q-1}_{j,j} t^{q-1} & (j \neq n), \\ b_{i,j}(t) = b^0_{i,j} + b^1_{i,j}\, t + \cdots + b^{q-1}_{i,j} t^{q-1} & (i > j\,;\, i \neq n), \\ b_{n,j}(t) = b^0_{n,j} + b^1_{n,j}\, t + \cdots + b^{q-1}_{n,j} t^{q-1} + a_{n+1-j,q(n+1-j)}\, t^q, & \\ b_{n,n}(t) = \rho_n - (n-1)\,q + b^1_{n,n}\, t + \cdots + b^{q-1}_{n,n} t^{q-1} + a_{1,q} t^q, & \end{cases} \tag{1.4}$$

by means of a linear transformation with polynomials in t^{-1} *as its coefficients.*

From (1.4) we can easily observe that (1.2) has the same characteristic constants as (1.1).

2. In the paper [1] we used the following relations

$$x_p(t) \equiv t^{-(q-1)p} \frac{d^p x}{dt^p}, \tag{2.1}$$

$$t\frac{dx_p}{dt} + (q-1)px_p = t^q x_{p+1} \qquad (p = 0, 1, \cdots, n-1) \tag{2.2}$$

in order to obtain informations on characteristic constants of the irregular singularity and estimates of coefficients of formal solutions.

Now, multiplying both sides of (1.1) by t^{-qn}, we can rewrite (1.1) in the form

$$x_n = \sum_{l=1}^{n} A_l(t)\, x_{n-l}, \tag{2.3}$$

where we put

$$A_l(t) = \sum_{r=0}^{ql} a_{l,r} t^{r-ql} \qquad (l = 1, 2, \cdots, n). \tag{2.4}$$

Then, taking account of (2.2) and (2.3), we have

$$\mathscr{D}X = \mathscr{D}\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} \theta_1 & t^q & & & 0 \\ 0 & \theta_2 & t^q & & \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \cdots & \theta_{n-1} & t^q \\ t^q A_n(t) & t^q A_{n-1}(t) & \cdots & & \theta_n + t^q A_1(t) \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} \tag{2.5}$$

$$= \mathscr{A}(t) X \qquad \left(\mathscr{D} \equiv t\frac{d}{dt}\ ;\ \theta_j = -(j-1)(q-1), \quad j = 1, 2, \cdots, n\right).$$

We here put

$$Y = E(t) X, \qquad E(t) = \begin{bmatrix} 1 & & & & 0 \\ e_{2,1}(t) & 1 & & & \\ e_{3,1}(t) & e_{3,2}(t) & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ e_{n,1}(t) & e_{n,2}(t) & \cdots & e_{n,n-1}(t) & 1 \end{bmatrix}, \tag{2.6}$$

obtaining the relation

$$\begin{aligned} \mathscr{D}Y &= (\mathscr{D}E(t) + E(t)\mathscr{A}(t))E^{-1}(t)\, Y \\ &= B(t)\, Y, \end{aligned} \tag{2.7}$$

which implies that

$$\mathscr{D}E(t) + E(t)\mathscr{A}(t) = B(t) E(t). \tag{2.8}$$

From (2.8) we now have only to determine $e_{i,j}(t)$ $(i > j)$ such that all the elements of the matrix $B(t)$ are polynomials of the required form (1.4). If we put

$$E(t)\mathscr{A}(t)=(\alpha_{j,k}(t)), \qquad B(t)E(t)=(\beta_{j,k}(t)),$$

then we have

$$(2.9) \qquad \begin{cases} \alpha_{j,k}(t)=t^q e_{j,k-1}(t)+\theta_k e_{j,k} & (j=1,2,\cdots,n-1), \\ \alpha_{n,k}(t)=t^q e_{n,k-1}(t)+\theta_k e_{n,k}(t)+t^q A_{n-k+1}(t), \end{cases}$$

and

$$(2.10) \qquad \beta_{j,k}(t)=b_{j,k}(t)+\sum_{l=k+1}^{j} b_{j,l}(t)\,e_{l,k}(t)+t^q e_{j+1,k}(t),$$

where we understand

$$e_{j,j}(t)=1, \qquad e_{j,k}(t)=0 \quad (k>j), \qquad e_{j,0}(t)=0.$$

From (2. 8) it follows that

$$(2.11) \qquad \mathscr{D} e_{j,k}(t)+\alpha_{j,k}(t)=\beta_{j,k}(t) \qquad (k<j+2).$$

For the upper subdiagonal elements these are identical relations since $\alpha_{j,j+1}(t)=t^q$ and $\beta_{j,j+1}(t)=t^q$.

From the relations of the diagonal elements

$$\alpha_{j,j}(t)=\beta_{j,j}(t)$$

we have

$$(2.12) \qquad t^q e_{j,j-1}(t)+\theta_j=b_{j,j}(t)+t^q e_{j+1,j}(t) \qquad (j=1,2,\cdots,n-1),$$

$$(2.13) \qquad t^q e_{n,n-1}(t)+\theta_n+t^q A_1(t)=b_{n,n}(t),$$

and for the k-th lower subdiagonal elements we have

$$(2.14) \qquad \mathscr{D} e_{j,j-k}(t)+t^q e_{j,j-k-1}(t)+\theta_{j-k} e_{j,j-k}(t) = b_{j,j-k}(t)+\sum_{l=0}^{k-1} b_{j,j-l}(t)\,e_{j-l,j-k}(t)+t^q e_{j+1,j-k}(t) \qquad (j=2,3,\cdots,n-1),$$

$$(2.15) \qquad \mathscr{D} e_{n,n-k}(t)+t^q e_{n,n-k-1}(t)+\theta_{n-k} e_{n,n-k}(t)+t^q A_{k+1}(t) = b_{n,n-k}(t)+\sum_{l=0}^{k-1} b_{n,n-l}(t)\,e_{n-l,n-k}(t) \qquad (k=1,2,\cdots,n-1).$$

From these relations (2. 12-15) it will be observed that the subdiagonal elements of $E(t)$ can be determined as polynomials of the same degree. We put

(2.16) $$e_{j,j-k}(t)=t^{-q}e^{1}_{j,j-k}(t)+t^{-2q}e^{2}_{j,j-k}(t)+\cdots+t^{-kq}e^{k}_{j,j-k}(t)$$
$$(j=2,3,\cdots,n\,;\,k=1,2,\cdots,j-1),$$

where $e^{\nu}_{j,j-k}(t)\,(\nu=1,2,\cdots,k)$ are polynomials in t of degree at most $(q-1)$. We also write the coefficients $A_l(t)\,(l=1,2,\cdots,n)$ in the form

(2.17) $$A_l(t)=a_{l,ql}+t^{-q}A^1_l(t)+t^{-2q}A^2_l(t)+\cdots+t^{-lq}A^l_l(t)$$
$$(l=1,2,\cdots,n),$$

where $A^{\nu}_l(t)\,(\nu=1,2,\cdots,l)$ are polynomials of degree $(q-1)$, i. e.,

(2.18) $$A^{\nu}_l(t)=A^{\nu}_l(0)+A^{\nu}_l(1)t+\cdots+A^{\nu}_l(q-1)\,t^{q-1}.$$

In particular, $A^l_l(0)=a_{l,0}\ (l=1,2,\cdots,n)$. We moreover introduce the following notations:

$$\begin{aligned}\mathscr{D}(t^{-\nu q}e^{\nu}(t))&=\mathscr{D}(t^{-\nu q}\sum_{k=0}^{q-1}\xi_k t^k)\\&=t^{-\nu q}(\sum_{k=0}^{q-1}\xi_k(-\nu q+k)\,t^k)\\&\equiv t^{-\nu q}\mathscr{D}_{\nu q}e^{\nu}(t),\end{aligned}$$

$$\begin{aligned}b(t)e(t)&=(\sum_{k=0}^{q}\beta_k t^k)(\sum_{k=0}^{q-1}\xi_k t^k)\\&=t^q\{(\beta_q\xi_0+\beta_{q-1}\xi_1+\cdots+\beta_1\xi_{q-1})+(\beta_q\xi_1+\beta_{q-1}\xi_2\\&\quad+\cdots+\beta_2\xi_{q-1})\,t+\cdots+(\beta_q\xi_{q-1})\,t^{q-1}\}\\&\quad+\{(\beta_0\xi_0)+(\beta_0\xi_1+\beta_1\xi_0)\,t+\cdots\\&\quad+(\beta_0\xi_{q-1}+\beta_1\xi_{q-2}+\cdots+\beta_{q-1}\xi_0)\,t^{q-1}\}\\&\equiv t^q[b(t)\,e(t)]^0+[b(t)\,e(t)]^1.\end{aligned}$$

Substituting (2.16) and (2.17) into (2.14) and (2.5), we have

(2.19) $$\mathscr{D}_{\nu q}e^{\nu}_{j,j-k}+\theta_{j-k}e^{\nu}_{j,j-k}+e^{\nu+1}_{j,j-k-1}=\sum_{l=0}^{k-1}\{[b_{j,j-l}e^{\nu}_{j-l,j-k}]^1+[b_{j,j-l}e^{\nu+1}_{j-l,j-k}]^0\}+e^{\nu+1}_{j+1,j-k},$$

(2.20) $$b_{j,j-k}=e^1_{j,j-k-1}-\sum_{l=0}^{k-1}[b_{j,j-l}e^1_{j-l,j-k}]^0-e^1_{j+1,j-k},$$

(2.21) $$\mathscr{D}_{\nu q}e^{\nu}_{n,n-k}+\theta_{n-k}e^{\nu}_{n,n-k}+e^{\nu-1}_{n,n-k-1}+A^{\nu+1}_{k+1}=\sum_{l=0}^{k-1}\{[b_{n,n-l}e^{\nu}_{n-l,n-k}]^1+[b_{n,n-l}e^{\nu+1}_{n-l,n-k}]^0\},$$

(2.22) $$b_{n,n-k}=e^1_{n,n-k-1}+A^1_{k+1}+a_{k+1,q(k+1)}t^q-\sum_{l=0}^{k-1}[b_{n,n-l}e^1_{n-l,n-k}]^0$$
$$(\nu=1,2,\cdots,k),$$

where we also understand

$$e^{\nu}_{j-l,j-k}=0 \qquad (\nu>k-l).$$

From (2. 22) we immediately see that

$$b_{n,n-k}=a_{k+1,q(k+1)}t^{q}+\cdots \qquad (k=0,1,\cdots,n-1). \tag{2. 23}$$

3. We are now in a position to prove Theorem 1 by determining all coefficients of polynomials $e^{\nu}_{j,k}(t)$ and $b_{j,k}(t)$ uniquely. To prove this, we only use a very simple result which we describe in the form of

LEMMA. *Let $\eta^1_0,\ \eta^2_0,\ \cdots,\ \eta^N_0$ be known constants such that*

$$[\rho]_N+\eta^1_0[\rho]_{N-1}+\cdots+\eta^N_0=\prod_{l=1}^{N}(\rho-\rho_l). \tag{3. 1}$$

Let μ be an unknown variable which satisfies the relations

$$\begin{cases} \xi^1_0=(\mu-(N-1))+\eta^1_0, \\ \xi^k_0=(\mu-(N-k))\xi^{k-1}_0+\eta^k_0 \qquad (k=2,3,\cdots,N-1), \\ 0=\mu\xi^{N-1}_0+\eta^N_0. \end{cases} \tag{3. 2}$$

Then μ is equal to one of ρ_l, i. e., for instance, $\mu=\rho_N$, and there holds

$$\begin{vmatrix} \xi^1_0+\rho-(N-2) & -1 & & 0 \\ \xi^2_0 & \rho-(N-3) & -1 & \\ \vdots & & \ddots & \ddots \quad -1 \\ \xi^{N-1}_0 & 0 & & \rho \end{vmatrix} \tag{3. 3}$$

$$=[\rho]_{N-1}+\xi^1_0[\rho]_{N-2}+\cdots+\xi^{N-1}_0$$

$$=\prod_{l=1}^{N-1}(\rho-\rho_l).$$

PROOF. The first part of the lemma can be easily verily verified. Since

$$\begin{vmatrix} \xi^1_0+x_1 & -1 & & 0 \\ \xi^2_0 & x_2 & -1 & \\ \vdots & & \ddots & \ddots \quad -1 \\ \xi^{N-1}_0 & 0 & & x_{N-1} \end{vmatrix}$$

$$= x_1x_2\cdots x_{N-1} + \xi_0^1 x_2x_3\cdots x_{N-1} + \xi_0^2 x_3x_4\cdots x_{N-1} + \cdots + \xi_0^{N-1},$$

we have, denoting the determinant by $f(\rho)$,

$$f(\rho) = [\rho]_{N-1} + \xi_0^1[\rho]_{N-2} + \cdots + \xi_0^{N-1}$$

and

$$\begin{aligned}(\rho-\rho_N)f(\rho) &= [\rho]_N - (\rho_N - N + 1)[\rho]_{N-1}\\ &\quad + \xi_0^1[\rho]_{N-1} - (\rho_N - N + 2)\xi_0^1[\rho]_{N-2}\\ &\quad +\\ &\quad \vdots\\ &\quad + \xi_0^{N-1}[\rho]_1 - \rho_N\xi_0^{N-1}\\ &= [\rho]_N + \eta_0^1[\rho]_{N-1} + \cdots + \eta_0^N\\ &= \prod_{l=1}^{N}(\rho-\rho_l) = (\rho-\rho_N)\prod_{l=1}^{N-1}(\rho-\rho_l).\end{aligned}$$

Now we shall begin with the determination of $e_{n,n-k}^k(t)\,(k=1,2,\cdots,n-1)$ and $b_{n,n}(t)$. From (2. 13) and (2. 21) we have

$$(3.4)\qquad \begin{cases} e_{n,n-1}^1 + \theta_n + A_1^1 = b_{n,n} - a_{1,q}t^q,\\ \mathscr{D}_{kq}e_{n,n-k}^k + \theta_{n-k}e_{n,n-k}^k + e_{n,n-k-1}^{k+1} + A_{k+1}^{k+1} = [b_{n,n}e_{n,n-k}^k]^1\\ \qquad\qquad (k=1,2,\cdots,n-1).\end{cases}$$

We put

$$\begin{aligned} e_{n,n-k}^k &= (\xi_0^k + \xi_1^k t + \cdots + \xi_{q-1}^k t^{q-1}) \qquad (k=1,2,\cdots,n-1),\\ b_{n,n} - a_{1,q}t^q &= (\beta_0 + \beta_1 t + \cdots + \beta_{q-1}t^{q-1})\end{aligned}$$

and then (3. 4) yields that

$$(3.5)\qquad \xi_0^1 = \beta_0 - \theta_n - a_{1,0} = \beta_0 + (n-1)q - (n-1) - a_{1,0}$$

$$(3.6)\qquad \xi_\nu^1 = \beta_\nu - A_1^1(\nu) \qquad (\nu = 1,2,\cdots,q-1),$$

$$(3.7)\qquad \xi_0^{k+1} = (\beta_0 + kq - \theta_{n-k})\xi_0^k - a_{k+1,0} = (\beta_0 + (n-1)q - (n-k-1))\xi_0^k - a_{k+1,0},$$

$$(3.8)\qquad \begin{aligned}\xi_\nu^{k+1} &= (\beta_0 + (n-1)q - \nu - (n-k-1))\xi_\nu^k + \beta_\nu\xi_0^k\\ &\quad + (\beta_1\xi_{\nu-1}^k + \cdots + \beta_{\nu-1}\xi_1^k) - A_{k+1}^{k+1}(\nu)\\ &\qquad (k=1,2,\cdots,n-1;\ \nu=1,2,\cdots,q-1),\end{aligned}$$

where $\xi_\nu^n = 0$. Applying the lemma to (3. 5) and (3. 7), we immediately see that $\beta_0 + (n-1)q$ is one of roots ρ_j $(j=1,2,\cdots,n)$ of the equation

$$[\rho]_n - \sum_{l=1}^{n} a_{l,0}[\rho]_{n-l} = 0.$$

We put

$$\beta_0 + (n-1)q = \rho_n, \tag{3.9}$$

whence ξ_0^k $(k=1,2,\cdots,n-1)$ can be determined uniquely. Using (3. 6) and (3. 8), we determine ξ_ν^k $(k=1,2,\cdots,n-1)$ and β_ν successively. We have

$$\begin{bmatrix} \xi_0^1+\rho_n-\nu-(n-2) & -1 & & 0 \\ \xi_0^2 & \rho_n-\nu-(n-3) & -1 & \\ \vdots & & \ddots & \ddots \quad -1 \\ \xi_0^{n-1} & 0 & & \rho_n-\nu \end{bmatrix} \begin{bmatrix} \xi_\nu^1 \\ \xi_\nu^2 \\ \vdots \\ \xi_\nu^{n-1} \end{bmatrix}$$

$= \textit{known data.}$

Again, from the lemma we see that the determinant of the matrix in the left hand side of the above formula is equal to

$$[\rho_n-\nu]_{n-1} + \xi_0^1[\rho_n-\nu]_{n-2} + \cdots + \xi_0^{n-1} = \prod_{l=1}^{n-1}(\rho_n-\nu-\rho_l) \tag{3.10}$$

which is non-vanishing from the assumption of the theorem. Thus we have determined all the coefficients of $e_{n,n-k}^k(t)$ and $b_{n,n}(t)$ uniquely.

Next we shall determine $e_{j,j-k}^k(t)(k=1,2,\cdots,j-1)$ and $b_{j,j}(t)$ successively. From (2. 12) and (2. 19) we have

$$\begin{cases} e_{j,j-1}^1 + \theta_j = b_{j,j} + e_{j+1,j}^1, \\ \mathscr{D}_{kq}e_{j,j-k}^k + \theta_{j-k}e_{j,j-k}^k + e_{j,j-k-1}^{k+1} = [b_{j,j}e_{j,j-k}^k]^1 + e_{j+1,j-k}^{k+1} \end{cases} \tag{3.11}$$
$$(k=1,2,\cdots,j-1).$$

We use mathematical induction. Let

$$e_{j+1,j-k}^{k+1}(t) = \eta_0^{k+1} + \eta_1^{k+1}t + \cdots + \eta_{q-1}^{k+1}t^{q-1}$$
$$(k=0,1,\cdots,j-1)$$

be known polynomials and the constants η_0^k $(k=1,2,\cdots,j)$ satisfy the relation

$$[\rho]_j + \eta_0^1[\rho]_{j-1} + \cdots + \eta_0^j = \prod_{l=1}^{j}(\rho-\rho_l). \tag{3.12}$$

Putting

$$e_{j,j-k}^k = (\xi_0^k + \xi_1^k t + \cdots + \xi_{q-1}^k t^{q-1}) \qquad (k=1,2,\cdots,j-1),$$
$$b_{j,j} = (\beta_0 + \beta_1 t + \cdots + \beta_{q-1}t^{q-1}),$$

We have

(3. 13) $$\xi_0^1=\beta_0+(j-1)q-(j-1)+\eta_0^1,$$

(3. 14) $$\xi_\nu^1=\beta_\nu+\eta_\nu^1 \qquad (\nu=1,2,\cdots,q-1),$$

(3. 15) $$\xi_0^{k+1}=(\beta_0+(j-1)q-(j-k-1))\,\xi_0^k+\eta_0^{k+1},$$

(3. 16) $$\xi_\nu^{k+1}=(\beta_0+(j-1)q-\nu-(j-k-1))\,\xi_\nu^k+\beta_\nu\xi_0^k + (\beta_1\xi_{\nu-1}^k+\cdots+\beta_{\nu-1}\xi_1^k)+\eta_\nu^{k+1}$$
$$(k=1,2,\cdots,j-1\,;\,\nu=1,2,\cdots,q-1),$$

where $\xi_\nu^j=0$. Applying the lemma to (3. 13-16), we can immediately obtain

(3. 17) $$\beta_0+(j-1)q=\rho_j$$

and

(3. 18) $$[\rho]_{j-1}+\xi_0^1[\rho]_{j-2}+\cdots+\xi_0^{j-1}=\prod_{l=1}^{j-1}(\rho-\rho_l).$$

Again, under the assumption of the theorem we can determine $e_{j,j-k}^k(t)\,(k=1,2,\cdots,j-1)$ and $b_{j,j}(t)$ uniquely. Thus we have proved by mathematical induction that for all $j\,(j=1,2,\cdots,n)$ $e_{j,j-k}^k(t)\,(k=1,2,\cdots,j-1)$ and $b_{j,j}(t)$ can be determined uniquely. Taking account of (3. 9), (3. 10), (3. 12), (3. 17) and (3. 18), we can also see that the $b_{j,j}(t)$ have the form

$$b_{j,j}(t)=\rho_j-(j-1)q+b_{j,j}^1t+\cdots+b_{j,j}^{q-1}t^{q-1}+b_{j,j}^qt^q$$
$$(j=1,2,\cdots,n),$$

where $b_{j,j}^q=0\;(j\neq n)$ and $b_{n,n}^q=a_{1,q}$.

Lastly we shall show that the sets of polynomials

$$\{b_{n,n-k}(t),\; e_{n,n-k-\nu}^\nu(t)\quad(\nu=1,2,\cdots,n-k-1)\}$$

and

$$\{b_{j,j-k}(t),\; e_{j,j-k-\nu}^\nu(t)\quad(\nu=1,2,\cdots,j-k-1)\}$$
$$(j=n-1,n-2,\cdots,1)$$

can be determined in succession as k takes values $1,2,\cdots,(n-1)$. We prove this by mathematical induction. Let the sets $\{b_{n,n-l},\; e_{n,n-l-\nu}^\nu\}$ and $\{b_{j,j-l},\; e_{j,j-l-\nu}^\nu\}$ $(j=1,2,\cdots,n-1)$ be known for $l=0,1,\cdots,k-1$. Then from (2. 21) we have

(3.19)
$$\mathscr{D}_{\nu q} e^{\nu}_{n,n-k-\nu} + \theta_{n-k-\nu} e^{\nu}_{n,n-k-\nu} + e^{\nu+1}_{n,n-k-\nu-1}$$
$$= [b_{n,n} e^{\nu}_{n,n-k-\nu}]^1 + [b_{n,n-k} e^{\nu}_{n-k,n-k-\nu}]^1 + \sum_{l=1}^{k-1} [b_{n,n-l} e^{\nu}_{n-l,n-k-\nu}]^1 + \sum_{l=0}^{k-1} [b_{n,n-l} e^{\nu+1}_{n-l,n-k-\nu}]^0 - A^{\nu+1}_{k+\nu+1}$$
$$(\nu = 1, 2, \cdots, n-k-1).$$

We here remark that if we put

$$e^{\nu}_{n-k,n-k-\nu} = \eta^{\nu}_0 + \eta^{\nu}_1 t + \cdots + \eta^{\nu}_{q-1} t^{q-1} \qquad (\nu = 1, 2, \cdots, n-k-1),$$

then, taking account of (3.12) and (3.18), we have

(3.20)
$$[\rho]_{n-k-1} + \eta^1_0 [\rho]_{n-k-2} + \cdots + \eta^{n-k-1}_0 = \prod_{j=1}^{n-k-1} (\rho - \rho_j).$$

Putting

$$b_{n,n-k} = (\beta_0 + \beta_1 t + \cdots + \beta_{q-1} t^{q-1} + \beta_q t^q),$$
$$e^{\nu}_{n,n-k-\nu} = (\xi^{\nu}_0 + \xi^{\nu}_1 t + \cdots + \xi^{\nu}_{q-1} t^{q-1}) \qquad (\nu = 1, 2, \cdots, n-k-1),$$

from (2.22) we first obtain the relations

(3.21)
$$\beta_q = a_{k+1,q(k+1)}.$$

(3.22)
$$\beta_l = \xi^1_l + \textit{known value} \qquad (l = 0, 1, \cdots, q-1)$$

and from (3.19) we have

(3.23)
$$(-\nu q + l + \theta_{n-k-\nu}) \xi^{\nu}_l + \xi^{\nu+1}_l = (\alpha_0 \xi^{\nu}_l + \cdots + \alpha_l \xi^{\nu}_0) + (\beta_0 \eta^{\nu}_l + \cdots + \beta_l \eta^{\nu}_0) + \textit{known value}$$
$$(\nu = 1, 2, \cdots, n-k-1 \,;\, l = 0, 1, \cdots, q-1),$$

where we have put

$$b_{n,n} = \alpha_0 + \alpha_1 t + \cdots + \alpha_{q-1} t^{q-1} + \alpha_q t^q \qquad (\alpha_0 = \rho_n - (n-1)q).$$

We can rewrite (3.23) in the form

(3.24)
$$\xi^{\nu+1}_l = (\rho_n - kq - l - (n-k-\nu-1)) \xi^{\nu}_l + \beta_l \eta^{\nu}_0 + (\alpha_1 \xi^{\nu}_{l-1} + \cdots + \alpha_l \xi^{\nu}_0) + (\beta_0 \eta^{\nu}_l + \cdots + \beta_{l-1} \eta^{\nu}_1) + \textit{known value},$$

whence we have only to prove the non-vanishing of the determinants:

$$\begin{vmatrix} \eta_0^1 + \rho_n - kq - l - (n-k-2) & -1 & & 0 \\ \eta_0^2 & \rho_n - kq - l - (n-k-3) & -1 & \\ \vdots & & \ddots & \ddots \\ & & & -1 \\ \eta_0^{n-k-1} & 0 & & \rho_n - kq - l \end{vmatrix}$$

$$= [\rho_n - kq - l]_{n-k-1} + \eta_0^1 [\rho_n - kq - l]_{n-k-2} + \cdots + \eta_0^{n-k-1}$$

$$= \prod_{j=1}^{n-k-1} (\rho_n - kq - l - \rho_j) \neq 0$$

$$(l = 0, 1, \cdots, q-1).$$

We have thus determined the set $\{b_{n,n-k},\ e_{n,n-k-\nu}^{\nu}\}$ uniquely. In order to prove that the sets $\{b_{j,j-l},\ e_{j,j-l-\nu}\}$ $(j = 1, 2, \cdots, n-1)$ can be determined uniquely, we again use mathematical induction. The explanation of their proof will not be needed here since the situations for such sets of polynomials are quite the same as above.

Applying a series of the well-known Hukuhara-Turrittin's transformations to the companion matrix constructed directly from (1. 1), we may probably be able to obtain the required system (1. 2). However, the method of reduction in this paper seems to be a more simple one and be useful for the concrete computation. We shall possibly prove the same reduction theorem with no assumptions on the ρ_j by a slight modification of this method.

4. The method explained so far is also applicable to the reduction of single linear differential equations with many regular singularities and one irregular singularity to systems of linear differential equations of the Schlesinger type. For simplicity, we here consider the reduction of the single linear differential equation

$$\phi^n \frac{d^n x}{dt^n} = \sum_{l=1}^{n} a_l(t)\, \phi^{n-l} \frac{d^{n-l} x}{dt^{n-l}} \qquad (\phi = t(t-1)) \tag{4. 1}$$

with two regular singularities at $t = 0$ and $t = 1$ and an irregular singularity of rank 1 at infinity to the Schlesinger canonical system

$$\frac{dY}{dt} = \left(\frac{C_0}{t} + \frac{C_1}{t-1} + C_\infty\right) Y. \tag{4. 2}$$

In (4. 1) the coefficients $a_l(t)\,(l = 1, 2, \cdots, n)$ are polynomials of degree at most $2l$ which are expressed in the form

$$a_l(t)=\sum_{r=0}^{l-1}\{a_{l,r}^0+a_{l,r}^1\phi'\}\phi^r+a_{l,l}^0\phi^l$$

$$(\phi'=(2t-1)\,;\ l=1,2,\cdots,n).$$

The characteristic constasts ρ_j^0 and ρ_j^1 $(j=1,2,\cdots,n)$ of the regular singularities at $t=0$ and $t=1$ are given by roots of the equations

$$[\rho]_n=\sum_{l=1}^{n}(a_{l,0}^0-a_{l,0}^1)(-1)^l[\rho]_{n-l}\qquad(t=0),$$

$$[\rho]_n=\sum_{l=1}^{n}(a_{l,0}^0-a_{l,0}^1)[\rho]_{n-l}\qquad(t=1),$$

respectively and the characteristic constants λ_j $(j=1,2,\cdots,n)$ of the irregular singularity at $t=\infty$ are given by roots of the equation

$$\lambda^n=\sum_{l=1}^{n}\alpha_l\,\lambda^{n-l}\qquad(\alpha_l=a_{l,l}^0\,;\ l=1,2,\cdots,n).$$

Now, according to (2. 1), where in this case $q=1$, we put

$$x_k=\frac{d^kx}{dt^k}\qquad(k=0,1,\cdots,n-1)$$

and form a system of linear differential equations for the column vector $X(t)=(x_0(t), x_1(t),\cdots,x_{n-1}(t))_*$ as follows:

$$(4.\,3)\qquad \phi\frac{dX}{dt}=\begin{bmatrix}0&\phi&&&\mathbf{0}\\0&0&\phi&&\\\vdots&\vdots&\ddots&\ddots&\\0&0&\cdots&0&\phi\\a_n(t)\phi^{1-n}&a_{n-1}(t)\phi^{2-n}&\cdots&a_2(t)\phi^{-1}&a_1(t)\end{bmatrix}X$$

$$=\left\{\begin{bmatrix}0&1&&&\mathbf{0}\\0&0&1&&\\\vdots&\vdots&\ddots&\ddots&\\0&0&\cdots&0&1\\\alpha_n&\alpha_{n-1}&&\cdots&\alpha_1\end{bmatrix}\phi+\begin{bmatrix}0&0&\cdots&0\\\vdots&\vdots&&\vdots\\0&0&\cdots&0\\A_n^0(\phi)&A_{n-1}^0(\phi)&\cdots&A_1^0(\phi)\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}0&0&\cdots&0\\\vdots&\vdots&&\vdots\\0&0&\cdots&0\\A_n^1(\phi)&A_{n-1}^1(\phi)&\cdots&A_1^1(\phi)\end{bmatrix}\phi'\right\}X$$

$$\equiv\{A_\infty\phi+\mathscr{A}^0(\phi)+\mathscr{A}^1(\phi)\phi'\}X=\mathscr{A}(t)X,$$

where we have put

$$A_l^h(\phi) = \sum_{r=0}^{l-1} a_{l,r}^h \phi^{r-l+1} \qquad (l = 1, 2, \cdots, n\,;\, h = 0, 1).$$

As in Section 2, we next make the following linear transformation with a triangular matrix as its coefficient

$$\begin{aligned} Y &= E(t)X \\ &= (E^0(\phi) + E^1(\phi)\phi')X, \end{aligned}$$

where we write

$$E^h(\phi) = \begin{bmatrix} \delta^h & & & & 0 \\ e_{2,1}^h(\phi) & \delta^h & & & \\ \vdots & & \ddots & \ddots & \\ e_{n,1}^h(\phi) & e_{n,2}^h(\phi) & \cdots & e_{n,n-1}^h(\phi) & \delta^h \end{bmatrix}$$

$$(h = 0, 1\,;\, \delta^0 = 1, \delta^1 = 0).$$

Then our aim is to determine $E(t)$ such that $E(t)$ satisfies the system of linear differential equations

$$\phi \frac{d}{dt} E(t) + E(t)\mathscr{A}(t) = (B_2\phi + B_1\phi' + B_0)E(t) \tag{4.5}$$

$$(C_0 = B_1 - B_0, \quad C_1 = B_1 + B_0, \quad C_\infty = B_2),$$

together with an appropriate choice of the constant matrices B_i $(i = 0, 1, 2)$. In other words, since the differential equation (4. 1) includes n constants α_l $(l = 1, 2, \cdots, n)$ and $n(n+1)$ constants $a_{l,r}^h$ $(l = 1, 2, \cdots, n\,;\, r = 0, 1, \cdots, l-1\,;\, h = 0, 1)$, we may put $B_2 = A_\infty$, the fact of which will be seen later, and may consider B_i $(i = 0, 1)$ as lower triangular matrices, and then we attempt to show that $n(n+1)$ elements of B_i $(i = 0, 1)$ can be determined uniquely by the same number of the constants $a_{l,r}^h$ through the differential equation (4. 5).

Substituting the expressions (4. 3) and (4. 4) for $\mathscr{A}(t)$ and $E(t)$ into (4. 5) and then comparing the expressions attached to ϕ' and those not including ϕ' in both sides, we have

$$\mathscr{D}E^0 + E^1(A_\infty\phi + \mathscr{A}^0) + E^0\mathscr{A}^1 = B_1E^0 + (B_2\phi + B_0)E^1, \tag{4.6}$$

$$\begin{aligned} (1+4\phi)\mathscr{D}E^1 + 2\phi E^1 + E^0(A_\infty\phi + \mathscr{A}^0) + (1+4\phi)E^1\mathscr{A}^1 \\ = (B_2\phi + B_0)E^0 + (1+4\phi)B_1E^1 \end{aligned} \tag{4.7}$$

$$\left(\mathscr{D} \equiv \phi \frac{d}{d\phi}\right).$$

As is also seen in Section 2, taking account of the fact that $\mathscr{A}^h(\phi)$ $(h = 0, 1)$ are matrices of polynomials in ϕ^{-1}, we may take $E^h(\phi)(h=0,1)$ as polynomials in ϕ^{-1} with no constant term. From this and (4. 7) we can easily deduce that B_2 must be equal to A_∞.

Now, denoting

$$B_h = \begin{bmatrix} \beta^h_{1,1} & & & 0 \\ \beta^h_{2,1} & \beta^h_{2,2} & & \\ \vdots & & \ddots & \\ \beta^h_{n,1} & \beta^h_{n,2} & \cdots & \beta^h_{n,n} \end{bmatrix} \qquad (h=0,1)$$

and considering the relations $E^1\mathscr{A}^h = 0$ $(h=0,1)$, we rewrite (4. 6) and (4. 7) in the following elementwise form:

(4. 8) $$\phi e^1_{j,j-1}(\phi) = \beta^1_{j,j} + \phi e^1_{j+1,j}(\phi) \qquad (j=1,2,\cdots,n-1),$$

(4. 9) $$\phi e^1_{n,n-1}(\phi) + A^1_1(\phi) = \beta^1_{n,n},$$

(4. 10) $$\mathscr{D} e^0_{j,j-k}(\phi) + \phi e^1_{j,j-k-1}(\phi) = \sum_{l=j-k}^{j} \beta^1_{j,l}\, e^0_{l,j-k}(\phi) + \sum_{l=j-k+1}^{j} \beta^0_{j,l}\, e^1_{l,j-k}(\phi) + \phi e^1_{j+1,j-k}(\phi)$$
$$(j=2,3,\cdots,n-1\,;\ k=1,2,\cdots,j-1),$$

(4. 11) $$\mathscr{D} e^0_{n,n-k}(\phi) + \phi e^1_{n,n-k-1}(\phi) + A^1_{k+1}(\phi) = \sum_{l=n-k}^{n} \beta^1_{n,l}\, e^0_{l,n-k}(\phi) + \sum_{l=n-k+1}^{n} \beta^0_{n,l}\, e^1_{l,n-k}(\phi) + \phi\Big(\sum_{l=n-k+1}^{n} \alpha_{n+1-l} e^1_{l,n-k}(\phi)\Big)$$
$$(k=1,2,\cdots,n-1),$$

(4. 12) $$\phi e^0_{j,j-1}(\phi) = \beta^0_{j,j} + \phi e^0_{j+1,j}(\phi) \qquad (j=1,2,\cdots,n-1),$$

(4. 13) $$\phi e^0_{n,n-1}(\phi) + A^0_1(\phi) = \beta^0_{n,n},$$

(4. 14) $$(1+4\phi)\,\mathscr{D} e^1_{j,j-k}(\phi) + 2\phi e^1_{j,j-k}(\phi) + \phi e^0_{j,j-k-1}(\phi) = \sum_{l=j-k}^{j} \beta^0_{j,l}\, e^0_{l,j-k}(\phi) + (1+4\phi) \sum_{l=j-k+1}^{j} \beta^1_{j,l}\, e^1_{l,j-k}(\phi) + \phi e^0_{j+1,j-k}(\phi)$$
$$(j=2,3,\cdots,n-1\,;\ k=1,2,\cdots,j-1),$$

(4. 15) $$(1+4\phi)\,\mathscr{D} e^1_{n,n-k}(\phi) + 2\phi e^1_{n,n-k}(\phi) + \phi e^0_{n,n-k-1}(\phi) + A^0_{k+1}(\phi) = \sum_{l=n-k}^{n} \beta^0_{n,l}\, e^0_{l,n-k}(\phi) + (1+4\phi) \sum_{l=n-k+1}^{n} \beta^1_{n,l}\, e^1_{l,n-k}(\phi)$$
$$+ \phi\Big(\sum_{l=n-k+1}^{n} \alpha_{n+1-l} e^0_{l,n-k}(\phi)\Big)$$
$$(k=1,2,\cdots,n-1).$$

In the above we interprete $e^h_{j,k} = 0$ $(k \leqq 0)$. From these relations we can easily observe that the k-th subdiagonal elements $\{e^h_{j,j-k}(\phi)\,;\, h=0,1\}$ are taken as polynomials in ϕ^{-1} of degree k with no constant term.

Let

(4.16) $$e^h_{n,n-k}(\phi)=\xi^h_k\phi^{-k}+\cdots \qquad (k=1,2,\cdots,n-1\,;\,h=0,1).$$

Then from (4.9) and (4.13) we immediately obtain

(4.17) $$\begin{cases} \xi^1_1+a^1_{1,0}=\beta^1_{n,n},\\ \xi^0_1+a^0_{1,0}=\beta^0_{n,n} \end{cases}$$

and, substituting (4.16) into (4.11) and (4.15) and equating coefficients of the power ϕ^{-k} in both sides, we have

(4.18) $$\begin{cases} \xi^1_{k+1}+a^1_{k+1,0}=(\beta^1_{n,n}+k)\,\xi^0_k+\beta^0_{n,n}\,\xi^1_k,\\ \xi^0_{k+1}+a^0_{k+1,0}=\beta^0_{n,n}\xi^0_k+(\beta^1_{n,n}+k)\,\xi^1_k \end{cases}$$
$$(k=1,2,\cdots,n-1),$$

where $\xi^h_n=0$ $(h=0,1)$. We here put

(4.19) $$\begin{cases} \mu^0=\beta^1_{n,n}-\beta^0_{n,n}, & \mu^1=\beta^1_{n,n}+\beta^0_{n,n},\\ \eta^0_k=\xi^1_k-\xi^0_k, & \eta^1_k=\xi^1_k+\xi^0_k \end{cases} \qquad (k=1,2,\cdots,n)$$

and rewrite (4.17) and (4.18) as follows:

(4.20) $$\begin{cases} \eta^0_1=\mu^0+(a^0_{1,0}-a^1_{1,0}),\\ \eta^0_{k+1}=-(\mu^0+k)\,\eta^0_k+(a^0_{k+1,0}-a^1_{k+1,0}) \end{cases}$$
$$(k=1,2,\cdots,n-1),$$

(4.21) $$\begin{cases} \eta^1_1=\mu^1-(a^0_{1,0}+a^1_{1,0}),\\ \eta^1_{k+1}=(\mu^1+k)\,\eta^1_k-(a^0_{k+1,0}+a^0_{k+1,0}) \end{cases}$$
$$(k=1,2,\cdots,n-1).$$

Applying the lemma in the beginning of Section 3 to (4.20) and (4.21), we can see that $\mu^0+(n-1)$ and $\mu^1+(n-1)$ are equal to some of the characteristic constants ρ^0_j and ρ^1_j $(j=1,2,\cdots,n)$, respectively. We put

$$\mu^0=\rho^0_n-(n-1),\quad \mu^1=\rho^1_n-(n-1)$$

and then we can determine all values ξ^h_k in (4.16) uniquely by means of (4.19-21).

Next we set

$$e^h_{n-1,n-1-k}(\phi)=\zeta^h_k\phi^{-k}+\cdots \qquad (k=1,2,\cdots,n-2\,;\,h=0,1)$$

and determine ζ^h_k $(k=1,2,\cdots,n-2)$ and $\beta^h_{n-1,n-1}$. From (4.8) and (4.12) it follows that

$$\begin{cases} \zeta_1^1 = \xi_1^1 + \beta_{n-1,n-1}^1, \\ \zeta_1^0 = \xi_1^0 = \beta_{n-1,n-1}^0. \end{cases}$$

From (4. 10) and (4. 14) the following are obtained:

$$\begin{cases} \zeta_{k+1}^1 = (\beta_{n-1,n-1}^1 + k)\,\zeta_k^0 + \beta_{n-1,n-1}^0\,\zeta_k^1 + \xi_{k+1}^1, \\ \zeta_{k+1}^0 = \beta_{n-1,n-1}^0\zeta_k^0 + (\beta_{n-1,n-1}^1 + k)\,\zeta_k^1 + \xi_{k+1}^0 \end{cases} \qquad (k = 1, 2, \cdots, n-2),$$

where $\zeta_{n-1}^h = 0$ $(h = 0, 1)$. Then, putting again

$$(4.22) \qquad \begin{cases} \bar{\mu}^0 = \beta_{n-1,n-1}^1 - \beta_{n-1,n-1}^0, \qquad \bar{\mu}^1 = \beta_{n-1,n-1}^1 + \beta_{n-1,n-1}^0, \\ \gamma_k^0 = \zeta_k^1 - \zeta_k^0, \qquad \gamma_k^1 = \zeta_k^1 + \zeta_k^0 \qquad (k = 1, 2, \cdots, n-1), \end{cases}$$

we rewrite the above relations as follows:

$$(4.23) \qquad \begin{cases} \gamma_1^0 = \bar{\mu}^0 + \eta_1^0, \\ \gamma_{k+1}^0 = -(\bar{\mu}^0 + k)\,\gamma_k^0 + \eta_{k+1}^0 \qquad (k = 1, 2, \cdots, n-2), \end{cases}$$

$$(4.24) \qquad \begin{cases} \gamma_1^1 = \bar{\mu}^1 + \eta_1^1, \\ \gamma_{k+1}^1 = -(\bar{\mu}^1 + k)\,\gamma_k^1 + \eta_{k+1}^1 \qquad (k = 1, 2, \cdots, n-2). \end{cases}$$

Since the lemma also yields that

$$\begin{cases} [\rho]_{n-1} + \sum_{l=1}^{n-1} (-1)^{l-1}\,\eta_l^0\,[\rho]_{n-1-l} = \prod_{j=1}^{n-1} (\rho - \rho_j^0), \\ [\rho]_{n-1} + \sum_{l=1}^{n-1} \eta_l^1\,[\rho]_{n-1-l} = \prod_{j=1}^{n-1} (\rho - \rho_j^1), \end{cases}$$

we again apply the lemma to (4. 23) and (4. 24) and see that $\bar{\mu}^0 + (n-2)$ and $\bar{\mu}^1 + (n-2)$ are equal to some of ρ_j^0 and ρ_j^1 $(j = 1, 2, \cdots, n-1)$, respectively. We put

$$\bar{\mu}^0 = \rho_{n-1}^0 - (n-2), \quad \bar{\mu}^1 = \rho_{n-1}^1 - (n-2)$$

and then all values ζ_k^h are determined uniquely. Moreover we can proceed to the determination of $\beta_{j,j}^h$ $(h = 0, 1)$ and the coefficients

$$e_{j,j-k}^h = \zeta_k^h \phi^{-k} + \cdots \qquad (k = 1, 2, \cdots, j-1\,;\, h = 0, 1)$$

for $j = n-2, n-3, \cdots, 1$ in exactly the same manner, and we can put

$$\beta_{j,j}^1 - \beta_{j,j}^0 = \rho_j^0 - (j-1), \qquad \beta_{j,j}^1 + \beta_{j,j}^0 = \rho_j^1 - (j-1).$$

We now return to (4. 11) and (4. 15) and determine $\beta_{n,n-1}^h$ $(h = 0, 1)$ and the coefficients χ_k^h:

$$e^h_{n,n-k} = \xi^h_k \phi^{-k} + \chi^h_{k-1}\phi^{-(k-1)} + \cdots \qquad (k=2,3,\cdots,n-1\,;\,h=0,1).$$

We have

$$\begin{cases} \theta^0_1 = \chi^1_1 - \chi^0_1 = \beta^1_{n,n-1} - \beta^0_{n,n-1} + \textit{known value}, \\ \theta^1_1 = \chi^1_1 + \chi^0_1 = \beta^1_{n,n-1} + \beta^0_{n,n-1} + \textit{known value}, \end{cases}$$

and, taking account of these relations, we obtain

$$(4.25) \qquad \begin{cases} \theta^0_k = \chi^1_k - \chi^0_k = -\{\rho^0_n - (n-k)\}\,\theta^0_{k-1} - \theta^0_1\gamma^0_{k-1} + \textit{known value}, \\ \theta^1_k = \chi^1_k + \chi^0_k = \{\rho^0_n - (n-k)\}\,\theta^1_{k-1} - \theta^1_1\gamma^1_{k-1} + \textit{known value}, \end{cases}$$
$$(k=2,3,\cdots,n-1),$$

where the γ^h_k denote the values defined in (4.23-24) and $\theta^h_{n-1}=0$ $(h=0,1)$. We now have only to show that the coefficient matrices of the column vectors $(\theta^0_1, \theta^0_2, \cdots, \theta^0_{n-2})_*$ and $(\theta^1_1, \theta^1_2, \cdots, \theta^1_{n-2})_*$ in (4.25) are non-singular. In proving this, we take into consideration the relations

$$\begin{cases} [\rho]_{n-2} + \sum_{l=1}^{n-2}(-1)^{l-1}\gamma^0_l[\rho]_{n-2-l} = \prod_{j=1}^{n-2}(\rho - \rho^0_j), \\ [\rho]_{n-2} + \sum_{l=1}^{n-2}\gamma^1_l[\rho]_{n-2-l} = \prod_{j=1}^{n-2}(\rho - \rho^1_j), \end{cases}$$

obtaining

$$(4.26) \qquad \begin{vmatrix} \gamma^h_1 + \rho^h_n - (n-2) & \delta^h & & & 0 \\ \gamma^h_2 & \rho^h_n - (n-3) & \delta^h & & \\ \vdots & & \ddots & \ddots & \\ & & & \ddots & \delta^h \\ \gamma^h_{n-2} & 0 & & & \rho^h_n - 1 \end{vmatrix}$$
$$= \prod_{j=1}^{n-2}(\rho^h_n - 1 - \rho^h_j) \qquad (h=0,1\,;\,\delta^0=1, \delta^1=-1).$$

Under the assumption that $\rho^h_j \neq \rho^h_i \pmod 1$ $(i \neq j\,;\,h=0,1)$ the determinants (4.26) are non-vanishing, and hence $\beta^h_{n,n-1}$ $(h=0,1)$ and the coefficients χ^h_k are determined uniquely. Continuing the above procedure in succession, we can dermine all β^h_{jk} and all the coefficients of polynomials $e^h_{j,k}(\phi)$. The complete proof of the validity will be done by mathematical induction as in Section 3. We here omit the details.

We consequently obtain the following

Theorem 2. *Under the assumption that*

$$\rho_j^h \neq \rho_i^h \qquad (\mathrm{mod}\, 1) \qquad (i \neq j\,;\, i, j = 1, 2, \cdots, n\,;\, h = 0, 1)$$

the single linear differential equation (4.1) *can be reduced to the Schlesinger canonical system* (4. 2) *by a linear transformation with rational functions as its coefficients. And that, in* (4. 2)

$$C_\infty = \begin{bmatrix} 0 & 1 & & & & \mathbf{0} \\ 0 & 0 & 1 & & & \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & & \cdots & 0 & 1 \\ a_{n,n}^0 & a_{n-1,n-1}^0 & & \cdots & a_{2,2}^0 & a_{1,1}^0 \end{bmatrix},$$

whose eigenvalues are the characteristic constants λ_j $(j = 1, 2, \cdots, n)$, *and* C_0 *and* C_1 *are triangular matrices, whose diagonal elements are given by* $(\rho_1^0, \rho_2^0 - 1, \cdots, \rho_n^0 - (n-1))$ *and* $(\rho_1^1, \rho_2^1 - 1, \cdots, \rho_n^1 - (n-1))$, *respectively.*

Our method will be able to be applied to the reduction of a single linear differential equation of the more general form

$$\phi^n \frac{d^n x}{dt^n} = \sum_{l=1}^{n} a_l(t) \phi^{n-l} \frac{d^{n-l} x}{dt^{n-l}} \qquad (\phi = \prod_{j=1}^{p} (t - t_j)),$$

where $a_l(t)$ $(l = 1, 2, \cdots, n)$ are polynomials of degree at most $(q + p - 1)l$ to the Schlesinger canonical system

$$\frac{dY}{dt} = (\sum_{j=1}^{p} \frac{C_j}{t - t_j} + \sum_{k=1}^{q} B_k t^{k-1}) Y.$$

Concerning the connection problem for the Schlesinger canonical system (4. 2), the paper [3] is referred to.

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