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STABILITY OF LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND IMPULSE EFFECT

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1. Introduction

The investigations of systems of differential equations with variable structure mark their beginning with the works of T. Vogel [1]-[4]. This theory is further developed in the works of A. Myshkis, A. Hohryakov [6] and A. Myshkis, N. Parshikova [7].

The first publications on the theory of systems with impulse effect without variable structure were by V. Mil'man, A. Myshkis [8], [9], A. Samoilenko [10] and A. Samoilenko, N. Perestyuk [11].

The investigation of systems of differential equations with variable structure and impulse effect begins with the works of D. D. Bainov and S. D. Milusheva [12] and A. B. Dishliev and D. D. Bainov [13].

2. Statement of the problem

Let $t_0 < t_1 < t_2 < \dots, \lim_{i\to+\infty} t_i = +\infty$, be a given sequence of real numbers. Linear systems with variable structure and impulse effect in fixed moments of time have the form

$$\left| \begin{array}{ll} \dfrac{dy}{dt} = A_k(t)y + f_k(t), & t \in [t_k, t_{k+1}), \quad k = 0,1,2,\dots \\ y_k^+ = \varphi_k(y_k^-) + \alpha_k & k = 1,2,\dots \end{array} \right. \tag{1}$$

where $A_k(t)$ is a continuous $(n \times n)$-matrix for $t \in [t_k, t_{k+1}]$, $f_k(t)$ is a continuous vector-valued function for $t \in [t_k, t_{k+1}]$, $\varphi_k : \mathbf{R}^n \to \mathbf{R}^n$ is a linear mapping, α_k is an n-dimensional constant vector, $y_k^+ = y(t_k + 0) = \lim_{t\to t_k+0} y(t)$, $y_k^- = y(t_k - 0) = \lim_{t\to t_k-0} y(t)$.

We consider as well the respective homogeneous system

$$\left| \begin{array}{l} \dfrac{dx}{dt} = A_k(t)x \\ x_k^+ = \varphi_k(x_k^-) \end{array} \right. \tag{2}$$

The solutions of systems (1) and (2) are piecewise continuous functions in the interval $[t_o, +\infty]$ with discontinuities of first type in the points t_k, $k = 1, 2, \ldots$

Remark 1. By $|x|$ we shall denote the norm of the vector $x \in \mathbf{R}^n$. We should note that theorems 1 and 2 are valid for an arbitrary vector norm and theorems 3 and 4 only for the Euclidean norm.

Definition 1. The solution $\eta(t)$ of system (1) is called *stable* (for $t = t_o$) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every solution $y(t)$ for which $|y(t_o) - \eta(t_o)| < \delta$ satisfies the inequality $|y(t) - \eta(t)| < \varepsilon$ for $t \in [t_o, +\infty)$.

Otherwise the solution $\eta(t)$ is called *unstable.*

Definition 2. The solution $\eta(t)$ of system (1) is called *globally asymptotically stable* if $\eta(t)$ is stable and if, moreover, each solution $y(t)$ satisfies the condition

$$\lim_{t \to +\infty} |y(t) - \eta(t)| = 0.$$

Definition 3. Linear system (1) is called *stable (globally asymptotically stable)* if all its solutions are stable (globally asymptotically stable).

3. Main results

Theorem 1. *Non-homogeneous system* (1) *is stable if and only if the trivial solution of homogeneous system* (1) *is stable.*

Proof. If $\eta(t)$ is a solution of (1), then all solutions of non-homogeneous system (1) have the form $y(t) = \eta(t) + x(t)$ where $x(t)$ runs over all solutions of homogeneous system (2) and vice versa.

Let $\eta(t)$ be a stable solution of (1). By definition 1 for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each solution $y(t)$ such that $|y(t_o) - \eta(t_o)| < \delta$ the inequality $|y(t) - \eta(t)| < \varepsilon$ holds for $t \in [t_o, +\infty)$. But $x(t) = y(t) - \eta(t)$ is a solution of homogeneous system (2). Hence for any $\varepsilon > 0$ there exists $\delta > 0$ such that each solution $x(t)$ of homogeneous system (2) for which $|x(t_o)| < \delta$ satisfies the inequality $|x(t)| < \varepsilon$ for $t \in [t_o, +\infty)$, i. e. the zero solution of homogeneous system (2) is satble. Conversely, let the trivial solution of homogeneous system (2) be stable, i. e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that each solution $x(t)$ for which $|x(t_o)| < \delta$ satisfies the condition $|x(t)| < \varepsilon$ for $t \in [t_o, +\infty)$. Let $\eta(t)$ be a solution of non-homogeneous system (1) and $y(t)$ be an arbitrary solution of (1) for which $|y(t_o) - \eta(t_o)| < \delta$. Then $|y(t) - \eta(t)| < \varepsilon$ for $t \in [t_o, +\infty)$, i. e. the solution $\eta(t)$ is stable.

Corollary 1. *System* (1) *or* (2) *is stable if and only if at least one of its solutions is stable.*

Corollary 2. *System* (1) *(and* (2) *in particular) is globally asymptotically stable if and only if the trivial solution of homogeneous system* (2) *is globally asymptotically stable.*

Theorem 2. *Homogeneous system* (2) *is stable if and only if any of its solutions is bounded for* $t \in [t_0, +\infty)$.

Proof. Let $z(t)$ be an unbounded solution such that $z(t_0) \neq 0$. For any $\delta > 0$ we construct the solution

$$x(t) = \frac{z(t)}{z(t_0)} \cdot \frac{\delta}{2}$$

Obviously $|x(t_0)| = \frac{\delta}{2} < \delta$ but since the solution $z(t)$ is unbounded, there exists t_1 such that $x(t_1)$ is greater than any ε chosen previously. Hence the trivial solution is unstable.

Now let each solution of homogeneous system (2) be bounded. Denote by $e_j(t)$, $j = 1,\dots, n$ the solutions obtained when $x(t_0)$ runs through the basis vectors $(1, 0,\dots, 0)'$, $(0, 1, 0,\dots, 0)',\dots, (0,\dots, 0,1)'$ where by $(,,,)'$ we have denoted the transposed vector. Since each solution of homogeneous system (2) is bounded, then $|e_j(t)| \leq C$ where C is a positive constant. Then the solution $x(t)$ with initial condition $x(t_0) = (\lambda_1,\dots, \lambda_n)$ has the form $x(t) = \lambda_1 e_1(t) + \cdots + \lambda_n e_n(t)$. If we choose $|\lambda_j| < \delta$ where $\delta = \delta/Cn$, $j = 1,\dots, n$, then we obtain

$$|x(t)| \leq |\lambda_1|\,|e_1(t)| + \cdots + |\lambda_n|\,|e_n(t)| < \varepsilon,$$

i. e. the trivial solution of homogeneous system (2) is stable and by theorem 1 each solution of homogeneous system (2) is stable.

Consider the system with constant coefficients

$$\left|\begin{array}{l} \frac{dx}{dt} = A_k x,\ t \in [t_k, t_{k+1}) \\ x_k^+ = \varphi_k(x_k^-) \end{array}\right.$$

where A_k are constant $(n \times n)$-matrices and by λ_k we shall denote the greatest eigenvalue of the Hermitian-symmetrized matrix $1/2(A_k + A_k^*)$; $\varphi_k : \mathbf{R}^n \to \mathbf{R}^n$ are maps satisfying the conditions $|\varphi_k(x) - \varphi_k(y)| \leq a_k |x - y|$ where $|\cdot|$ is the Euclidean vector norm.

Introduce the following conditions:

H1. $\qquad \overline{\lim_{k\to+\infty}} \prod_{i=1}^{k} a_i < +\infty.$

H2. $\qquad \overline{\lim_{k\to+\infty}} \sum_{i=0}^{k} \lambda_i (t_{i+1} - t_i) < +\infty.$

H3. $\qquad \sum_{i=0}^{\infty} \lambda_i (t_{i+1} - t_i) = -\infty.$

Theorem 3. *If conditions* (H1) *and* (H2) *hold, then system* (3) *is stable and if conditions* (H1) *and* (H3) *hold, then system* (3) *is globally anymptotically stable.*

Proof. Let $\eta(t)$ and $x(t)$ be two solutions of system (3). By the inequality of Wazewskij we have

$$|x(t)-\eta(t)| \le |x_k^+ - \eta_k^+| \, exp[(t-t_k)\lambda_k] \quad \text{for } t \in [t_k, t_{k+1}). \qquad \text{But } |x_k^+ - \eta_k^+|$$

$$= |\varphi_k(x_k^-) - \varphi_k(\eta_k^-)| \le a_k \, |x_k^- - \eta_k^-| \le a_k \, |x_{k-1}^+ - \eta_{k-1}^+| \, exp[\lambda_{k-1}(t_k - t_{k-1})]$$

$$\le a_k \ldots a_1 \, |x_0 - \eta_0| \, exp[\lambda_{k-1}(t_k - t_{k-1}) + \ldots + \lambda_0(t_1 - t_0)],$$

i. e. $|x(t)-\eta(t)| \le a_k \ldots a_1 \, |x_0 - \eta_0| \, exp[\lambda_k(t-t_k) + \lambda_{k-1}(t_k - t_{k-1}) + \ldots + \lambda_0(t_1 - t_0)]$

$$\le a_k \ldots a_1 \, |x_0 - \eta_0| \, exp[\delta\lambda_k(t_{k+1} - t_k) + \lambda_{k-1}(t_k - t_{k-1}) + \cdots + \lambda_0(t_1 - t_0)],$$

$$\text{where} \quad \delta = \begin{cases} 1 & \text{for } \lambda_k \ge 0 \\ 0 & \text{for } \lambda_k < 0 . \end{cases}$$

Hence if conditions (H1) and (H2) hold, then

$$|x(t)-\eta(t)| \le C \, |x_0 - \eta_0|, \qquad C = const.,$$

and system (3) will be stable.

If conditions (H1) and (H3) hold, i. e. if for each $M > 0$ there exists ν such that for $k > \nu$

$$\sum_{i=0}^{k} \lambda_i (t_{i+1} - t_i) < -M$$

then for $t \in [t_k, t_{k+1})$ we have

$$|x(t)-\eta(t)| \le C_1 \, |x_0 - \eta_0| e^{-M}, \quad C_1 = const.,$$

which implies that system (3) is globally asymptotically stable.

Remark 2. If $\overline{\lim_{k\to+\infty}} \sum_{i=0}^{k} \lambda_i (t_{i+1} - t_i) = +\infty$, then system (3) can be stable as well as unstable. As an illustration of this we shall consider two examples.

Example 1. Consider the linear system without impulses, i. e. $\varphi_k(x) = x$:

$$\left| \begin{array}{ll} \dfrac{dx}{dt} = a_k x + b_k y, & t \in [k, k+1), (t_k = k) \\ \dfrac{dy}{dt} = a_k y. & \end{array} \right. \tag{4}$$

Let $a_k = -1$ and $b_k = 4$ for all k, i. e. consider a system of ordinary differential equations as a particular case of a system with variable structure. According to the classical theory it is stable since its characteristic roots are negative. The Hermitian-symmetrized matrix has eigenvalues -3 and 1, i. e.

$$\overline{\lim_{k\to+\infty}} \sum_{i=0}^{k} \lambda_i (t_{i+1} - t_i) = \overline{\lim_{k\to+\infty}} \sum_{i=0}^{k} 1 = +\infty$$

Hence this is an example of a stable system which does not satisfy condition (H2).

Example 2. Now let for the linear system (4) $a_k = -1$ and $b_k = (k+1)e^{k+1}$. Let $x(0) = 0$, $y(0) = y_0 \neq 0$. By straightforward computation we obtain

$$\left| \begin{array}{l} x_k = y_0(b_0 + \dots + b_{k-1})e^{-k} \\ y_k = y_0 e^{-k} \end{array} \right.$$

where $x_k = x(k)$, $y_k = y(k)$. Hence

$$\sqrt{x_k^2 + y_k^2} = |y_0|\, e^{-k}\sqrt{(b_0 + \cdots + b_{k+1})^2 + 1} \geq |y_0|\, e^{-k}\, |b_0 + \cdots + b_{k-1}| =$$
$$= |y_0|\, e^{-k}\, |e + \cdots + ke^k| \geq |y_0|\, k,$$

i. e. the zero solution is not stable.

The eigenvalues of the Hermitian-symmetrized matrix of the system are

$$-1 \pm \frac{k+1}{2} e^{k+1},\ k = 0, 1, \dots, \text{ i. e.}$$

$$\overline{\lim_{k\to+\infty}} \sum_{i=0}^{k} \lambda_i (t_{i+1} - t_i) = \sum_{k=0}^{\infty} \left(-1 \pm \frac{k+1}{2} e^{k+1}\right) = +\infty.$$

Hence this is an example of an unstable system which does not satisfy condition (H2).

Remark 3. In the classical case a system with constant coefficients is stable if the eigenvalues of its matrix have negative real parts. Example 2 shows that for systems with variable structure such an assertion is not valid. In relation to this we shall note that if $Re\lambda$ is the real part of one of the eigenvalues of the matrix A and $M_1 \leq M_2$ are respectively the smallest and the greatest eigenvalues of the Hermitian-symmetrized matrix $B = 1/2(A + A^*)$, then $M_1 \leq Re\lambda \leq M_2$. This follows from the extremal property of Rayleigh's relation

$$\max_{x\neq 0} \frac{\langle x, Bx\rangle}{\langle x, x\rangle} = \max_{\langle x,x\rangle = 1} \langle x, Bx\rangle = M_2, \quad \min_{\langle x,x\rangle = 1} \langle x, Bx\rangle = M_1$$

where by $\langle x, y\rangle$ we have denoted the scalar product of the vectors $x, y \in \mathbf{C}^n$ (see [5]).

Lemma 1. *Let A be a constant matrix and λ be the greatest eigenvalue of the Hermitian-symmetrized matrix* $1/2(A + A^*)$. *Then the inequality*

$$\|e^A\| \leq e^\lambda \tag{5}$$

holds where by $\|\cdot\|$ the spectral norm of the matrix is meant induced by the Euclidean vector norm, i. e.

$$\|B\| = \max_{|x|=1} |Bx|$$

For normal matrices $(AA^* = A^*A)$ *inequality* (5) *turns into an equality.*

Proof. Consider the system $dx/dt = Ax$. Its solution $x(t) = e^{At}x(0)$ satisfies the inequality of Ważewskij $|x(t)| \leq |x(0)|\, e^{\lambda t}$ and for $t=1$ we obtain $|e^{A}x(0)| \leq |x(0)|\, e^{\lambda}$. But for the induced norm there exists a vector x_0, $|x_0| = 1$, depending on A and such that

$$\|e^A\| = |e^A x_0|.$$

Choose the initial condition $x(0) = x_0$. Then

$$\|e^A\| = |e^A x_0| \leq |x_0|\, e^{\lambda} = e^{\lambda}.$$

For normal matrices inequality (5) turns into an equality since each normal matrix is unitary-similar to a diagonal matrix and the unitary-similar matrices have equal spectral norms (see [5]).

Finally consider an analogue of the classical problem for stability of a system of ordinary differential equations with almost constant coefficients:

$$\left|\begin{array}{l} \dfrac{dx}{dt} = (A_k + B_k(t))x, \quad t \in [t_k, t_{k+1}) \\ x_k^- = \varphi_k(x_k^-) \end{array}\right. \tag{6}$$

where A_k are constant $(n \times n)$-matrices and by λ_k we shall denote the greatest eigenvalue of the Hermitian-symmetrized matrix $1/2(A_k + A_k^*)$; $\varphi_k : \mathbf{R}^n \to \mathbf{R}^n$ are linear maps satisfying the conditions $|\varphi_k(x)| \leq a_k\, |x|$ and $B_k(t)$ are continuous matrix-values functions for $t \in [t_k, t_{k+1}]$. We introduce as well the piecewise continuous function $B(t) = B_k(t)$ for $t \in [t_k, t_{k+1})$.

Theorem 4. *If conditions* (H1), (H2) *are satisfied and* $\int_{t_0}^{\infty} \|B(\tau)\|\, d\tau < \infty$, *then system* (6) *is stable and if condition* (H1) *holds and*

$$\lim_{k\to+\infty} \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} (\lambda_i + \|B_i(\tau)\|)\, d\tau = -\infty \tag{7}$$

then system (6) *is globally asymptotically stable.*

Remark 3. If we introduce the step-function $\lambda(t) = \lambda_k$, $t \in [t_k, t_{k+1})$, then condition (7) can be written in the form

$$\int_{t_0}^{\infty} (\lambda(\tau) + \|B(\tau)\|)\, d\tau = -\infty \tag{8}$$

Proof of theorem 4. The solution of system (6) is written in the form

$$x(t) = e^{A_k(t-t_k)}x_k^+ + \int_{t_k}^{t} e^{A_k(t-\tau)}B_k(\tau)x(\tau)\,d\tau,$$

when $t \in [t_k, t_{k+1})$, i. e.

$$|x(t)| \le \|e^{A_k(t-t_k)}\|\,|x_k^+| + \int_{t_k}^{t} \|e^{A_k(t-\tau)}\|\,\|B_k(\tau)\|\,|x(\tau)|\,d\tau$$

and by lemma 1

$$|x(t)| \le e^{\lambda_k(t-t_k)}\,|x_k^+| + \int_{t_k}^{t} e^{\lambda_k(t-\tau)}\,\|B_k(\tau)\|\,|x(\tau)|\,d\tau$$

which implies the inequality

$$|e^{-\lambda_k t}x(t)| \le e^{-\lambda_k t_k}\,|x_k^+| + \int_{t_k}^{t} \|B_k(\tau)\|\,|e^{-\lambda_k\tau}x(\tau)|\,d\tau.$$

We apply the lemma of Gronwall-Bellman and obtain

$$|e^{-\lambda_k t}x(t)| \le e^{-\lambda_k t_k}\,|x_k^+|\,exp[\int_{t_0}^{t} \|B_k(\tau)\|\,d\tau]$$

i. e. $|x(t)| \le |x_k^+|\,exp[\lambda_k(t-t_k) + \int_{t_k}^{t} \|B_k(\tau)\|\,d\tau]$.

In particular, $|x_{m+1}^+| = |\varphi_{m+1}(x_{m+1}^-)| \le a_{m+1}\,|x_{m+1}^-| \le \le a_{m+1}\,|x_m^+|\,exp[\lambda_m(t_{m+1}-t_m) + \int_{t_m}^{t_{m+1}} \|B_m(\tau)\|\,d\tau]$.

We apply this inequality for $m = 0, 1, \ldots, k-1$ and obtain for $t \in [t_k, t_{k+1})$ the inequality

$$|x(t)| \le a_k \ldots a_1\,|y_0|\,exp[\lambda_0(t_1-t_0) + \cdots + \lambda_{k-1}(t_k-t_{k-1}) + \lambda_k(t-t_k) + \int_{t_0}^{t_1} \|B_0(\tau)\|\,d\tau + \cdots + \int_{t_{k-1}}^{t_k} \|B_{k-1}(\tau)\|\,d\tau + \int_{t_k}^{t} \|B_k(\tau)\|\,d\tau], \tag{9}$$

i. e.

$$|x(t)| \le a_k \ldots a_1\,|y_0|\,exp[\lambda_0(t_1-t_0) + \cdots + \lambda_{k-1}(t_k-t_{k-1}) + \delta\lambda_k(t-t_k) + \int_{t_0}^{t_{k+1}} \|B(\tau)\|\,d\tau]$$

where $\delta = \begin{cases} 0 & \lambda_k \le 0 \\ 1 & \lambda_k > 0 \end{cases}$

Hence if conditions (H1), (H2) hold and $\int_{t_0}^{\infty} \|B(\tau)\|\,d\tau < \infty$, then $|x(t)| \le C = \text{const}$, whence it follows that system (6) is stable.

Inequality (9) can be written in the form

$$|x(t)| \le a_k \ldots a_1\,|y_0|\,exp[\int_{t_0}^{t} (\lambda(\tau) + \|B(\tau)\|)\,d\tau].$$

Hence if conditions (H1) and (8) hold, then $\lim_{t\to+\infty} |x(t)| = 0$, i. e. system (6) is globally asymptotically stable.

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