STABILITY OF LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS 
WITH VARIABLE STRUCTURE AND IMPULSE EFFECT

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1. Introduction

The investigations of systems of differential equations with variable structure mark their beginning with the works of T. Vogel [1]–[4]. This theory is further developed in the works of A. Myshkis, A. Hohryakov [6] and A. Myshkis, N. Parshikova [7].

The first publications on the theory of systems with impulse effect without variable structure were by V. Mil’man, A. Myshkis [8], [9], A. Samoilenko [10] and A. Samoilenko, N. Perestyuk [11].


2. Statement of the problem

Let \( t_0 < t_1 < t_2 < \ldots, \lim_{i \to +\infty} t_i = +\infty \), be a given sequence of real numbers. Linear systems with variable structure and impulse effect in fixed moments of time have the form

\[
\begin{align*}
\frac{dy}{dt} &= A_k(t) y + f_k(t), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots \\
y_k(t) &= \varphi_k(y_{k+1}) + a_k, \quad k = 1, 2, \ldots
\end{align*}
\]

where \( A_k(t) \) is a continuous \((n \times n)\)-matrix for \( t \in [t_k, t_{k+1}] \), \( f_k(t) \) is a continuous vector-valued function for \( t \in [t_k, t_{k+1}] \), \( \varphi_k : \mathbb{R}^n \to \mathbb{R}^n \) is a linear mapping, \( a_k \) is an \( n \)-dimensional constant vector, \( y_k = y(t_k + 0) = \lim_{t \to t_k^+} y(t), y_k = y(t_k - 0) = \lim_{t \to t_k^-} y(t) \).

We consider as well the respective homogeneous system

\[
\begin{align*}
\frac{dx}{dt} &= A_k(t)x \\
x_k &= \varphi_k(x_{k+1})
\end{align*}
\]
The solutions of systems (1) and (2) are piecewise continuous functions in the interval 
\([t_0, +\infty)\) with discontinuities of first type in the points \(t_k, k = 1, 2,\ldots\).

Remark 1. By \(|x|\) we shall denote the norm of the vector \(x \in \mathbb{R}^n\). We should note 
that theorems 1 and 2 are valid for an arbitrary vector norm and theorems 3 and 4 only 
for the Euclidean norm.

Definition 1. The solution \(\eta(t)\) of system (1) is called stable (for \(t = t_0\)) if for any 
\(\varepsilon > 0\) there exists \(\delta > 0\) such that every solution \(y(t)\) for which 
\(|y(t_0) - \eta(t_0)| < \delta\) satisfies the inequality \(|y(t) - \eta(t)| < \varepsilon\) for \(t \in [t_0, +\infty)\).

Otherwise the solution \(\eta(t)\) is called unstable.

Definition 2. The solution \(\eta(t)\) of system (1) is called globally asymptotically stable if 
\(\eta(t)\) is stable and if, moreover, each solution \(y(t)\) satisfies the condition

\[
\lim_{t \to +\infty} |y(t) - \eta(t)| = 0.
\]

Definition 3. Linear system (1) is called stable (globally asymptotically stable) if all its
solutions are stable (globally asymptotically stable).

3. Main results

Theorem 1. Non-homogeneous system (1) is stable if and only if the trivial solution of 
homogeneous system (1) is stable.

Proof. If \(\eta(t)\) is a solution of (1), then all solutions of non-homogeneous system (1) 
have the form \(y(t) = \eta(t) + x(t)\) where \(x(t)\) runs over all solutions of homogeneous 
system (2) and vice versa.

Let \(\eta(t)\) be a stable solution of (1). By definition 1 for any \(\varepsilon > 0\) there exists \(\delta > 0\) 
such that for each solution \(y(t)\) such that \(|y(t_0) - \eta(t_0)| < \delta\) the inequality 
\(|y(t) - \eta(t)| < \varepsilon\) holds for \(t \in [t_0, +\infty)\). But \(x(t) = y(t) - \eta(t)\) is a solution of homoge-
neous system (2). Hence for any \(\varepsilon > 0\) there exists \(\delta > 0\) such that each solution \(x(t)\) 
of homogeneous system (2) for which \(|x(t_0)| < \delta\) satisfies the inequality \(|x(t)| < \varepsilon\) for 
\(t \in [t_0, +\infty)\), i.e. the zero solution of homogeneous system (2) is stable. Conversely, let 
the trivial solution of homogeneous system (2) be stable, i.e. for any \(\varepsilon > 0\) there exists 
\(\delta > 0\) such that each solution \(x(t)\) for which \(|x(t_0)| < \delta\) satisfies the condition 
\(|x(t)| < \varepsilon\) for \(t \in [t_0, +\infty)\). Let \(\eta(t)\) be a solution of non-homogeneous system (1) and 
\(y(t)\) be an arbitrary solution of (1) for which \(|y(t_0) - \eta(t_0)| < \delta\). Then 
\(|y(t) - \eta(t)| < \varepsilon\) for \(t \in [t_0, +\infty)\), i.e. the solution \(\eta(t)\) is stable.

Corollary 1. System (1) or (2) is stable if and only if at least one of its solutions is 
stable.

Corollary 2. System (1) (and (2) in particular) is globally asymptotically stable if and 
only if the trivial solution of homogeneous system (2) is globally asymptotically stable.
Theorem 2. Homogeneous system (2) is stable if and only if any of its solutions is bounded for $t \in [t_0, + \infty)$.

Proof. Let $\mathbf{z}(t)$ be an unbounded solution such that $\mathbf{z}(t_0) \neq 0$. For any $\delta > 0$ we construct the solution

$$
\mathbf{x}(t) = \frac{\mathbf{z}(t)}{\mathbf{z}(t_0)} \cdot \frac{\delta}{2}
$$

Obviously $|\mathbf{x}(t_0)| = \frac{\delta}{2} < \delta$ but since the solution $\mathbf{z}(t)$ is unbounded, there exists $t_i$ such that $\mathbf{x}(t_i)$ is greater than any $\varepsilon$ chosen previously. Hence the trivial solution is unstable.

Now let each solution of homogeneous system (2) be bounded. Denote by $\mathbf{e}_j(t), j = 1, \ldots, n$ the solutions obtained when $\mathbf{x}(t_0)$ runs through the basis vectors $(1, 0, \ldots, 0)'$, $(0, 1, 0, \ldots, 0)'$, $(0, 0, 1, 0, \ldots, 0)'$ where by $(\ldots)'$ we have denoted the transposed vector. Since each solution of homogeneous system (2) is bounded, then $|\mathbf{e}_j(t)| \leq C$ where $C$ is a positive constant. Then the solution $\mathbf{x}(t)$ with initial condition $\mathbf{x}(t_0) = (\lambda_1, \ldots, \lambda_n)$ has the form $\mathbf{x}(t) = \lambda_1 \mathbf{e}_1(t) + \cdots + \lambda_n \mathbf{e}_n(t)$. If we choose $|\lambda_1| < \delta$ where $\delta = \frac{\delta}{Cn}, j = 1, \ldots, n$, then we obtain

$$
|\mathbf{x}(t)| \leq |\lambda_1| |\mathbf{e}_1(t)| + \cdots + |\lambda_n| |\mathbf{e}_n(t)| < \varepsilon,
$$

i.e. the trivial solution of homogeneous system (2) is stable and by theorem 1 each solution of homogeneous system (2) is stable.

Consider the system with constant coefficients

$$
\begin{align*}
\frac{d\mathbf{x}}{dt} &= A_k \mathbf{x}, \quad t \in [t_k, t_{k+1}) \\
\mathbf{x}_{t_k} &= \varphi_k(\mathbf{x}_{t_k})
\end{align*}
$$

where $A_k$ are constant ($n \times n$)-matrices and by $\lambda_k$ we shall denote the greatest eigenvalue of the Hermitian-symmetrized matrix $1/2 \left( A_k + A_k^* \right); \varphi_k : \mathbb{R}^n \to \mathbb{R}^n$ are maps satisfying the conditions $|\varphi_k(x) - \varphi_k(y)| \leq a_k |x - y|$ where $|\cdot|$ is the Euclidean vector norm.

Introduce the following conditions:

H1. $\lim_{k \to \infty} \prod_{i=1}^{k} a_i < + \infty$.

H2. $\lim_{k \to \infty} \sum_{i=0}^{k} \lambda_i (t_{i+1} - t_i) < + \infty$.

H3. $\sum_{i=0}^{m} \lambda_i (t_{i+1} - t_i) = - \infty$.

Theorem 3. If conditions (H1) and (H2) hold, then system (3) is stable and if conditions (H1) and (H3) hold, then system (3) is globally asymptotically stable.
Proof. Let \( \eta(t) \) and \( x(t) \) be two solutions of system (3). By the inequality of Wazewskij we have

\[
|x(t) - \eta(t)| \leq |x_k - \eta_k| \exp\left[(t - t_k) \lambda_k\right] \text{ for } t \in [t_k, t_{k+1}).
\]

But \( |x_k - \eta_k| = |\varphi_k(x_k) - \varphi_k(\eta_k)| \leq a_k |x_k - \eta_k| \leq a_k |x_{k-1} - \eta_{k-1}| \exp[\lambda_{k-1}(t_k - t_{k-1})] \]

\[
\leq a_k \ldots a_1 |x_0 - \eta_0| \exp[\lambda_{k-1}(t_k - t_{k-1}) + \ldots + \lambda_0(t_k - t_0)],
\]

i.e. \( |x(t) - \eta(t)| \leq a_k \ldots a_1 |x_0 - \eta_0| \exp[\lambda_k(t_k - t_0) + \lambda_{k-1}(t_k - t_{k-1}) + \ldots + \lambda_0(t_k - t_0)] \)

\[
\leq a_k \ldots a_1 |x_0 - \eta_0| \exp[\delta \lambda_{k+1}(t_k - t) + \lambda_{k-1}(t_k - t_{k-1}) + \ldots + \lambda_0(t_k - t_0)],
\]

where \( \delta = \begin{cases} 1 & \text{for } \lambda_k \geq 0 \\ 0 & \text{for } \lambda_k < 0 \end{cases} \).

Hence if conditions (H1) and (H2) hold, then

\[
|x(t) - \eta(t)| \leq C |x_0 - \eta_0|, \quad C = \text{const.,}
\]

and system (3) will be stable.

If conditions (H1) and (H3) hold, i.e. if for each \( M > 0 \) there exists \( \nu \) such that for \( k > \nu \)

\[
\sum_{i=0}^{k} \lambda_i(t_{i+1} - t_i) < -M
\]

then for \( t \in [t_k, t_{k+1}) \) we have

\[
|x(t) - \eta(t)| \leq C_i |x_0 - \eta_0|e^{-M}, \quad C_i = \text{const.,}
\]

which implies that system (3) is globally asymptotically stable.

Remark 2. If \( \lim_{k \to \infty} \sum_{i=0}^{k} \lambda_i(t_{i+1} - t_i) = +\infty \), then system (3) can be stable as well as unstable. As an illustration of this we shall consider two examples.

Example 1. Consider the linear system without impulses, i.e. \( \varphi_k(x) = x \):

\[
\begin{align*}
\frac{dx}{dt} &= a_kx + b_ky, \quad t \in [k, k + 1), (t_k = k) \\
\frac{dy}{dt} &= a_ky.
\end{align*}
\]

Let \( a_k = -1 \) and \( b_k = 4 \) for all \( k \), i.e. consider a system of ordinary differential equations as a particular case of a system with variable structure. According to the classical theory it is stable since its characteristic roots are negative. The Hermitian-symmetr-
ized matrix has eigenvalues $-3$ and $1$, i.e.

$$
\lim_{k \to +\infty} \sum_{i=0}^{k} \lambda_i (t_{i+1} - t_i) = \lim_{k \to +\infty} \sum_{i=0}^{k} 1 = + \infty
$$

Hence this is an example of a stable system which does not satisfy condition (H2).

Example 2. Now let for the linear system (4) $a_k = -1$ and $b_k = (k + 1) e^{k+1}$. Let $x(0) = 0$, $y(0) = y_0 \neq 0$. By straightforward computation we obtain

$$
\begin{align*}
x_k &= x_0(b_0 + \ldots + b_{k-1}) e^{-k} \\
y_k &= y_0 e^{-k}
\end{align*}
$$

where $x_k = x(k)$, $y_k = y(k)$. Hence

$$
\sqrt{x_k^2 + y_k^2} = |y_0| e^{-k} \sqrt{(b_0 + \ldots + b_{k-1})^2 + 1} \geq |y_0| e^{-k} |b_0 + \ldots + b_{k-1}| = \\
= |y_0| e^{-k} |e + \ldots + ke^k| \geq |y_0| k,
$$

i.e. the zero solution is not stable.

The eigenvalues of the Hermitian-symmetrized matrix of the system are

$$
-1 \pm \frac{k + 1}{2} e^{k+1}, \quad k = 0, 1, \ldots, \text{i.e.}
$$

$$
\lim_{k \to +\infty} \sum_{i=0}^{k} \lambda_i (t_{i+1} - t_i) = \sum_{k=0}^{\infty} \left( -1 \pm \frac{k + 1}{2} e^{k+1} \right) = + \infty.
$$

Hence this is an example of an unstable system which does not satisfy condition (H2).

Remark 3. In the classical case a system with constant coefficients is stable if the eigenvalues of its matrix have negative real parts. Example 2 shows that for systems with variable structure such an assertion is not valid. In relation to this we shall note that if $\text{Re} \lambda$ is the real part of one of the eigenvalues of the matrix $A$ and $M_1 \leq M_2$ are respectively the smallest and the greatest eigenvalues of the Hermitian-symmetrized matrix $B = 1/2 (A + A^*)$, then $M_1 \leq \text{Re} \lambda \leq M_2$. This follows from the extremal property of Rayleigh's relation

$$
\max_{x \neq 0} \frac{\langle x, Bx \rangle}{\langle x, x \rangle} = \max_{\langle x, x \rangle = 1} \langle x, Bx \rangle = M_2, \quad \min_{\langle x, x \rangle = 1} \langle x, Bx \rangle = M_1,
$$

where $\langle x, y \rangle$ we have denoted the scalar product of the vectors $x, y \in \mathbb{C}^n$ (see [5]).

Lemma 1. Let $A$ be a constant matrix and $\lambda$ be the greatest eigenvalue of the Hermitian-symmetrized matrix $1/2 (A + A^*)$. Then the inequality

$$
\| e^A \| \leq e^{\lambda}
$$

holds where by $\| \cdot \|$ the spectral norm of the matrix is meant induced by the Euclidean vector norm, i.e.
\[ \| B \| = \max_{|x| = 1} |Bx| \]

For normal matrices \((AA^* = A^*A)\) inequality (5) turns into an equality.

Proof. Consider the system \(dx/dt = Ax\). Its solution \(x(t) = e^{At}x(0)\) satisfies the inequality of Ważewskij \(|x(t)| \leq |x(0)| e^{\|A\|t} \) and for \(t = 1\) we obtain \(|e^A x(0)| \leq |x(0)| e^t\). But for the induced norm there exists a vector \(x_0, |x_0| = 1\), depending on \(A\) and such that

\[ \| e^A \| = |e^A x_0|. \]

Choose the initial condition \(x(0) = x_0\). Then

\[ \| e^A \| = |e^A x_0| \leq |x_0| e^t = e^t. \]

For normal matrices inequality (5) turns into an equality since each normal matrix is unitary similar to a diagonal matrix and the unitary similar matrices have equal spectral norms (see [5]).

Finally consider an analogue of the classical problem for stability of a system of ordinary differential equations with almost constant coefficients:

\[
\frac{dx}{dt} = (A_k + B_k(t))x, \quad t \in [t_k, t_{k+1}) \quad (6)
\]

\[
x_k = \varphi_k(x_k)
\]

where \(A_k\) are constant \((n \times n)\)-matrices and by \(\lambda_k\) we shall denote the greatest eigenvalue of the Hermitian-symmetrized matrix \(1/2(A_k + A_k^*)\); \(\varphi_k : \mathbb{R}^n \to \mathbb{R}^n\) are linear maps satisfying the conditions \(|\varphi_k(x)| \leq \alpha_k |x|\) and \(B_k(t)\) are continuous matrix-values functions for \(t \in [t_k, t_{k+1}]\). We introduce as well the piecewise continuous function \(B(t) = B_k(t)\) for \(t \in [t_k, t_{k+1})\).

Theorem 4. If conditions (H1), (H2) are satisfied and \(\int_{t_0}^\infty \|B(\tau)\| d\tau < \infty\), then system (6) is stable and if condition (H1) holds and

\[
\lim_{k \to \infty} \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} (\lambda_i + \|B_i(\tau)\|) d\tau = -\infty \quad (7)
\]

then system (6) is globally asymptotically stable.

Remark 3. If we introduce the step-function \(\lambda(t) = \lambda_k, t \in [t_k, t_{k+1})\), then condition (7) can be written in the form

\[
\int_{t_0}^\infty (\lambda(\tau) + \|B(\tau)\|) d\tau = -\infty \quad (8)
\]

Proof of theorem 4. The solution of system (6) is written in the form
\[ x(t) = e^{\lambda(t-t_k)}x_k^+ + \int_{t_k}^{t} e^{\lambda(t-\tau)}B_k(\tau)x(\tau)\,d\tau, \]

when \( t \in [t_k, t_{k+1}) \), i.e.
\[
| x(t) | \leq \|e^{\lambda(t-t_k)}\| | x_k^+ | + \int_{t_k}^{t} \|e^{\lambda(t-\tau)}\| \|B_k(\tau)\| | x(\tau) | \,d\tau
\]
and by lemma 1
\[
| x(t) | \leq e^{\lambda(t-t_k)} | x_k^+ | + \int_{t_k}^{t} \|B_k(\tau)\| | x(\tau) | \,d\tau
\]
which implies the inequality
\[
| e^{-\lambda t}x(t) | \leq e^{-\lambda t_k} | x_k^+ | + \int_{t_k}^{t} \|B_k(\tau)\| | e^{-\lambda \tau}x(\tau) | \,d\tau.
\]
We apply the lemma of Gronwall-Bellman and obtain
\[
| e^{-\lambda t}x(t) | \leq e^{-\lambda t_k} | x_k^+ | \exp\left[ \int_{t_k}^{t} \|B_k(\tau)\| \,d\tau \right]
\]
i.e. \( | x(t) | \leq | x_k^+ | \exp[\lambda_k(t-t_k) + \int_{t_k}^{t} \|B_k(\tau)\| \,d\tau]. \)
In particular, \( | x_{m+1}^+ | = | \varphi_{m+1}(x_{m+1}^-) | \leq a_{m+1} | x_m^- | \leq \leq a_{m+1} | x_m^+ | \exp[\lambda_m(t_{m+1} - t_m) + \int_{t_m}^{t_{m+1}} \|B_m(\tau)\| \,d\tau]. \)

We apply this inequality for \( m = 0, 1, \ldots, k - 1 \) and obtain for \( t \in [t_k, t_{k+1}) \) the inequality
\[
| x(t) | \leq a_0 a_1 | y_0 | \exp[\lambda_0(t_1 - t_0) + \cdots + \lambda_{k-1}(t_k - t_{k-1}) + \lambda_k(t - t_k) + \int_{t_0}^{t_1} B_0(\tau)\,d\tau + \cdots + \int_{t_{k-1}}^{t_k} B_{k-1}(\tau)\,d\tau + \int_{t_k}^{t} B_k(\tau)\,d\tau],
\]
i.e.
\[
| x(t) | \leq a_0 a_1 | y_0 | \exp[\lambda_0(t_1 - t_0) + \cdots + \lambda_{k-1}(t_k - t_{k-1}) + \delta \lambda_k(t - t_k) + \int_{t_0}^{t_1} B(\tau)\,d\tau]
\]
where
\[
\delta = \begin{cases} 
0 & \lambda_k \leq 0 \\
1 & \lambda_k > 0 
\end{cases}
\]

Hence if conditions (H1), (H2) hold and \( \int_{t_0}^{\infty} \|B(\tau)\| \,d\tau < \infty \), then \( | x(t) | \leq C = \text{const}, \)
whence it follows that system (6) is stable.

Inequality (9) can be written in the form
\[
| x(t) | \leq a_0 a_1 | y_0 | \exp[\int_{t_0}^{t} (\lambda(\tau) + \|B(\tau)\|)\,d\tau].
\]
Hence if conditions (H1) and (8) hold, then \( \lim_{t \to +\infty} | x(t) | = 0 \), i.e. system (6) is globally asymptotically stable.

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