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A PROJECTION-ITERATIVE METHOD FOR SOLVING THE PERIODIC PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS WITH IMPULSE EFFECT

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1. Introduction

Systems with impulse effect are an object of active research. This is due to the fact that by means of them numerous processes and phenomena in science and technology are simulated. The beginning of the mathematical theory of systems with impulse effect was marked by the work oh Mil'man and Myshkis [1]. In the subsequent period the number of publications on this subject constantly increases. First of all the qualitative and asymptotic theory of systems with impulse effect developed [2]-[13]. The specific character of systems with impulse effect is such that solving the various problems set for them in a closed form can be realized only in exceptional cases. This requires the elaboration of approximate methods for their solution.

In the present paper a projection-iterative method is proposed for solving the periodic problem for systems of ordinary differential equations with impulse effect.

2. Statement of the problem. Preliminary remarks.

Consider the system of differential eqations with impulse effect.

$$\begin{aligned} &\dot{x}(t) = X(t, x(t)),\ t \neq t_i \\ &\Delta x\,|_{t=t_i} \equiv x(t_i+0) - x(t_i-0) = I_i(x(t_i-0)), \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$, t_i are fixed numbers satisfying the inequalities $t_i < t_{i+1}$ for $i \in \mathbf{Z}$ and $\lim_{i\to\pm\infty} t_i = \pm\infty$.

Denote by S_t the mapping point of the system with impulse effect (1), i. e. the point with current coordinates $(t, x(t))$, where $x(t)$ is the solution of (1). The motion of the point S_t in the $(n+1)$-dimensional space (t, x) can be described as follows. Starting

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from the point (t_o, x_c) the point S_t moves along the integral curve $(t, x(t))$ of the system with impulse effect (1) until the moment $t = t_1$, $t_1 > t_0$, when it is "instantly" transferred from the position $(t_1, x(t_1 - 0))$ to the position $(t_1, x(t_1 + 0))$. Further on the mapping point S_t continues its motion along the integral curve $(t, x(t))$ of the system with impulse effect (1) until the moment $t = t_2$, $t_2 > t_1$, etc.

From the above description it is clear that the solution $x(t)$ of the system with impulse effect (1) is a piecewise continuous functions with points of discontinuity at the points t_i, which for $t \neq t_i$ satisfies the equation $\dot{x}(t) = X(t, x(t))$ and for $t = t_i$ satisfies the condition for a jump

$$x(t_i + 0) = x(t_i - 0) + I_i(x(t_i - 0).$$

Further on, for the sake of definiteness, as a value of a given function $x(t)$ at the point t_i we shall understand $x(t_i - 0)$, i. e. $x(t_i) = x(t_i - 0) = \lim_{t \to t_i - 0} x(t)$ and as a norm of the element $x \in \mathbf{R}^n$-the number $|x| = \max_i |x_i|$.

We shall say that conditions (A) hold if the following conditions are satisfied:

A1. The function $X(t, x)$ is continuous in the domain $G = \{t \in \mathbf{R}, x \in \overline{D}\}$, where $\overline{D}$ is a bounded closed domain in $\mathbf{R}^n$, periodic with respect to the variable t with period 2π and satisfies the inequality $|X(t, x) - X(t, \tilde{x})| \leq \lambda |x - \tilde{x}|$, where $t \in \mathbf{R}^n$, x, $\tilde{x} \in \overline{D}$ and λ is a positive constant.

A2. There exists a positive integer k such that

$$t_{i+k} = t_i + 2\pi, \quad t_i \neq 0, \quad i \in \mathbf{Z}.$$

A3. The functions $I_i(x)$ are continuous in the domain $\overline{D}$ and satisfy the conditions

$$|I_i(x) - I_i(\tilde{x})| \leq J\,|x - \tilde{x}|, \quad J = const. > 0,$$
$$I_{i+k}(x) = I_i(x), x, \tilde{x} \in \overline{D}, \quad i \in \mathbf{Z}.$$

Consider the space $\tilde{C}$ of all n-dimensional piecewise continuous functions $x(t)$ defined in the interval $[0, 2\pi]$ with a finite number of coinciding points of discontinuity of first type contained in the interval $(0, 2\pi)$ and with the norm $\|x(t)\| = \sup_{t \in [0, 2\pi]} |x(t)|$. To each function $x(t) \in \tilde{C}$ we put in correspondence the respective Fourier series

$$x(t) \sim \frac{a_0}{2} + \sum_{q=1}^{\infty} (a_q cosqt + h_q sinqt),$$

where

$$a_q = \frac{1}{\pi} \int_0^{2\pi} x(t) \cos qt \; dt \quad , \qquad b_q = \frac{1}{\pi} \int_0^{2\pi} x(t) \sin qt \; dt.$$

Introduce the operators

$$P_0x(t)=\frac{1}{2\pi}\int_0^{2\pi}x(t)\,dt, \tag{2}$$

$$P_lx(t)=\sum_{q=1}^{l}(a_q\cos qt+b_q\sin qt), \tag{3}$$

$$Q_lx(t)=\sum_{q=l+1}^{\infty}(a_q\cos qt+b_q\sin qt),\quad x(t)\in\tilde{C}. \tag{4}$$

We shall use the following lemma [14].

Lemma 1 [13]. *If $x(t)\in\tilde{C}$, then the following inequalities hold :*

$$\left\|\int_0^t P_lx(\tau)\,d\tau\right\|\le\frac{2\pi}{\sqrt{3}}\|x(t)\|, \tag{5}$$

$$\left\|\int_0^t Q_lx(\tau)\,d\tau\right\|\le 2\sqrt{2}\sigma(l)\,\|x(t)\|, \tag{6}$$

where

$$\sigma^2(l)=\begin{cases}\dfrac{\pi^2}{6}, & l=0\\ \displaystyle\sum_{q=l+1}^{\infty}\frac{1}{q^2}, & l=1,2,\ldots\end{cases}$$

Under the assumption that the system with impulse effect (1) has a unique 2π-periodic solution $x(t)$, which satisfies the condition $x(0)=x_0$, $x_0\in\overline{D}$, we shall construct a sequence of periodic functions converging uniformly to it. We shall also consider the question of existence and uniqueness of periodic solutions of the system with impulse effect (1).

Let $\tilde{C}_D$ be the set of all n-dimensional piecewise continuous functions $x(t)$ defined in the interval $[0,2\pi]$ with points of discontinuity of first type at the points $t_i\in(0,2\pi)$ and satisfying the conditions: $x(t)\in\overline{D}$ for $t\in[0,2\pi]$ and $x(0)=x(2\pi)=x_0$. $\tilde{C}_D$ is a closed set in the complete metric space $\tilde{C}$.

Define the sequence of functions $\{x_j(t)\}$ by the formulae $x_0(t)=x_0$,

$$x_j(t)=\begin{cases}x_0+\int_0^t P_lX(\tau,x_j(\tau))\,d\tau+\int_0^t Q_lX(\tau,x_{j-1}(\tau))\,d\tau+\\ +\sum_{0<t_i<t}I_i(x_{j-1}(t_i))-\dfrac{t}{2\pi}\sum_{0<t_i<2\pi}I_i(x_{j-1}(t_i)), & t\in[0,2\pi),\end{cases} \tag{7}$$

where $l\in\mathbf{N}$ is a fixed number and $j\in\mathbf{N}$.

Denote by $D_{a(l)}$ the set of points of the domain $\overline{D}$ belonging to $\overline{D}$ together with their $a(l)$-neighbourhood, where

$$a(l)=2(\pi/\sqrt{3}+\sqrt{2}\sigma(l)+k)M, \tag{8}$$
$$M=\sup_{\substack{t\in[0,2\pi]\\ x\in\overline{D}}}|X(t,x)|+\max_{1\le i\le k}\|I_i(x)\|$$

We shall say that conditions (B) are satisfied if the following conditions hold:

B1. The set $D_{a(l)}$ is not empty.

B2. The point x_0 belongs to $D_{a(l)}$

B3. The following inequalities hold

$$\frac{2\pi}{\sqrt{3}}\lambda < 1, \quad \frac{2\sqrt{3}(\sqrt{2}\sigma(l)\lambda + KJ)}{\sqrt{3} - 2\pi\lambda} < 1.$$

Lemma 2. *Let conditions* (A) *and* (B) *hold. Then equation* (7) *for given* $x_{j-1}(t) \in \tilde{C}_D$ *has a unique solution* $x_j(t) \in \tilde{C}_D$ $(j \in \mathbf{N})$.

Proof. Define the operator T by the formula

$$T(x,z)(t) = \begin{cases} x_0 + \int_0^t P_l X(\tau, x(\tau))\,d\tau + \int_0^t Q_l X(\tau, z(\tau))\,d\tau + \\ + \sum_{0<t_i<t} \mathrm{I}_i(z(t_i)) - \frac{t}{2\pi} \sum_{0<t_i<2\pi} \mathrm{I}_i(z(t_i)), \qquad t \in [0, 2\pi], \end{cases} \tag{9}$$

where the functions $x(t)$ and $z(t)$ belong to $\tilde{C}_D$ and $z(t)$ is an arbitrarily chosen and fixed function from $\tilde{C}_D$.

For any two functions x and z from $\tilde{C}_D$ the equalities $T(x,z)(0) = T(x,z)(2\pi) = x_0$ hold.

From (5), (6) and (9) we obtain

$$\| T(x,z)(t) - x_0 \| \leq a(l).$$

Hence the operator T for a fixed function $z(t) \in \tilde{C}_D$ transforms the set $\tilde{C}_D$ into the set $T\tilde{C}_D \subset \tilde{C}_D$.

Let the functions $x(t)$ and $\bar{x}(t)$ belong to $\tilde{C}_D$. Then for $z(t) \in \tilde{C}_D$ fixed by lemma 1 and property A1 we obtain

$$\begin{aligned} &\| T(x,z)(t) - T(\bar{x},z)(t) \| \leq \\ &\leq \left\| \int_0^t P_l [X(\tau, x(\tau)) - X(\tau, \bar{x}(\tau))]\,d\tau \right\| \leq \\ &\leq \frac{2\pi}{\sqrt{3}} \| X(t, x(t)) - X(t, \bar{x}(t)) \| \leq \frac{2\pi}{\sqrt{3}}\lambda \| x(t) - \bar{x}(t) \| \end{aligned} \tag{10}$$

From condition B3 and (10) it follows that the operator T which transforms the set $\tilde{C}_D$ into the set $T\tilde{C}_D \subset \tilde{C}_D$ is a contracting operator and by Banach's fixed point theorem the operator equation $x(t) = T(x,z)(t)$ for $z \in \tilde{C}_D$ fixed has a unique solution $x(t) \in \tilde{C}_D$. Hence equation (7) for given $x_{j-1}(t) \in \tilde{C}_D$ has a nuique solution $x_j(t) \in \tilde{C}_D$ $(j \in \mathbf{N})$. This completes the proof of lemma 2.

Lemma 3. *Let conditions* (A) *and* (B) *hold. Then there exists a function* $x^*(t) \in$

$\tilde{C}_D$ such that the sequence $\{x_j(t)\}$ uniformly in the interval $[0,2\pi]$ converges to it.

Proof. From conditions (A) and inequalities (5) and (6) for $j \geq 1$ we obtain

$$\|x_{j+1}(t) - x_j(t)\| \leq \|\int_0^t P_l[X(\tau, x_{j+1}(\tau)) - X(\tau, x_j(\tau))]\,d\tau\| +$$

$$+ \|\int_0^t Q_l[X(\tau, x_j(\tau)) - X(\tau, x_{j-1}(\tau))]\,d\tau\| +$$

$$+ 2\sum_{0<t_i<2\pi} \|I_i(x_j(t_i)) - I_i(x_{j-1}(t_i))\| \leq \tag{11}$$

$$\leq \frac{2\pi}{\sqrt{3}} \|X(t, x_{j+1}(t)) - X(t, x_j(t))\| + 2\sqrt{2}\sigma(l)\, \|X(t, x_j(t)) -$$

$$- X(t, x_{j-1}(t))\| + 2KJ\, \|x_j(t) - x_{j-1}(t))\| \leq$$

$$\leq \frac{2\pi}{\sqrt{3}}\lambda\, \|x_{j+1}(t) - x_j(t)\| + 2(\sqrt{2}\sigma(l)\lambda + KJ)\, \|x_j(t) - x_{j-1}(t)\|.$$

From (11) there follows the inequality

$$\|x_{j+1}(t) - x_j(t)\| \leq \alpha_l\, \|x_j(t) - x_{j-1}(t)\|, \tag{12}$$

where

$$\alpha_l = \frac{2\sqrt{6}\sigma(l)\lambda + 2\sqrt{3}KJ}{\sqrt{3} - 2\pi\lambda}, \qquad 0 < \alpha_l < 1 \tag{13}$$

Using consecutively inequalities analogous to (12) we get

$$\|x_{j+1}(t) - x_j(t)\| \leq \alpha_l^j\, \|x_1(t) - x_0\|. \tag{14}$$

From (14) it follows

$$\|x_{j+p}(t) - x_j(t)\| \leq \frac{\alpha_l^j}{1-\alpha_l}\, a(l). \tag{15}$$

Last inequality shows that the sequence $\{x_j(t)\}$ is uniformly convergent in the interval $[0,2\pi]$. Since $\tilde{C}_D$ is a closed set in the complete metric space $\tilde{C}$, then there exists a function $x^*(t) \in \tilde{C}_D$ such that $\lim_{j\to\infty} x_j(t) = x^*(t)$. This completes the proof of lemma 3.

For $p \to \infty$ from (15) the following inequality is obtained

$$\|x^*(t) - x_j(t)\| \leq \frac{\alpha_l^j}{1-\alpha_l}\, a(l). \tag{16}$$

From the definition and the properties of the functions $x_j(t), j \geq 1$ it follows that the function $x^*(t)$ for $t \in [0,2\pi]$ satisfies the equation

$$x(t) = x_0 + \int_0^t P_l X(\tau, x(\tau))d\tau + \int_0^t Q_l X(\tau, x(\tau))d\tau + \tag{17}$$

$$+ \sum_{0<t_i<t} I_i(x(t_i)) - \frac{t}{2\pi} \sum_{0<t_i<2\pi} I_i(x(t_i)).$$

Denote by $X^*(t)$ the 2π-periodic continuation of the function $x^*(t)$

3. Main results.

Theorem 1. *Let conditions* (A) *and* (B) *hold and let the system with impulse effect* (1) *have a* 2π*-periodic solution* $x(t)$ *such that* $x(0) = x(2\pi) = x_0$ *and the function* $X(t, x(t))$ *can be extended in Fourier series in the interval* $[0, 2\pi]$. *Then* $x(t) \equiv X^*(t)$ *for* $x_0 \in \tilde{C}_D$ *and the following inequality holds*

$$\| x(t) - x_j(t) \| \leq \frac{\alpha_l^j}{1 - \alpha_l} a(l), \quad t \in [0, 2\pi],$$

where $a(l)$ *and* α_l *are defined respectively by* (8) *and* (13).

Proof. The solution $x(t)$ of the system with impulse effect (1) satisfies the equality

$$x(t) = x_0 + \int_0^t X(\tau, x(\tau))d\tau + \sum_{0<t_i<t} I_i(x(t_i))$$

The conditions $x(0) = x(2\pi) = x_0$ implies the equality

$$\int_0^{2\pi} X(\tau, x(\tau))d\tau + \sum_{0<t_i<2\pi} I_i(x(t_i)) = 0. \tag{18}$$

From (2)-(4) and (18) it follows that the function $x(t)$ for $t \in [0, 2\pi]$ satisfies equation (17). But equation (17) under the conditions of lemma 3 has a unique solution in the interval $[0, 2\pi]$. Hence $x(t) \equiv x^*(t)$ for $t \in [0, 2\pi]$ and $x(t) \equiv X^*(t)$ for $t \in \mathbf{R}$. This completes the proof of theorem 1.

Further on we consider the question of existence of a 2π-periodic solution of the system with impulse effect (1). For this purpose we assume that the function $X(t, x^*(t))$ can be extended in Fourier series and set

$$\Delta(x_0) = P_0 X(t, x^*(t)) + \frac{1}{2\pi} \sum_{0<t_i<2\pi} I_i(x^*(t_i)),$$

$$\Delta_j(x_0) = P_0 X(t, x_j(t)) + \frac{1}{2\pi} \sum_{0<t_i<2\pi} I_i(x_j(t_i)).$$

Since the function $x^*(t)$ is a solution of equation (17), then for $\Delta(x_0) = 0$ the function $x^*(t)$ will satisfy the system with impulse effect (1). Hence the question of existence of a 2π-periodic solution of (1) is related to the question of existence of zeroes of the function $\Delta(x_0)$. The points x_0 at which $\Delta(x_0)$ vanishes are singular points of the mapping $\Delta : \overline{D} \to \mathbf{R}^n$. But since only the functions $x_j(t)$ are known, then in order to apply the method proposed the problem of finding the zeroes of $\Delta(x_0)$ must be reduced to the problem of finding the zeroes of the functions $\Delta_j(x_0)$.

Theorem 2. *Let conditions* (A) *and* (B) *hold and the function* $X(t, x^*(t))$ *can be extended in Fourier series in the interval* $[0, 2\pi]$. *Let a convex closed domain* D_l *exist belonging to* $\overline{D}$ *such that for some* $j \geq 1$ *the mapping* $\Delta_j(x_0): \overline{D} \to \mathbf{R}^n$ *has in the domain* D_l *a unique singular point* x^0 *with a nonzero index. Let on the boundary* Γ_{D_l} *of the domain* D_l *the following inequality be fulfilled*

$$\inf_{x \in \Gamma_{D_l}} |\Delta_j(x)| > \frac{\alpha_l^j}{1-\alpha_l}\left(\lambda + \frac{Jk}{2\pi}\right) a(l). \tag{19}$$

Then the system with impulse effect (1) *has a* 2π-*periodic solution* $x(t)$ *such that* $x(0) \in \overline{D}$.

Proof. By definition the index of the isolated singular point x^0 of the continuous mapping $\Delta_j(x_0)$ is equal to the characteristic of the vector field generated by the mapping Δ_j on a sufficiently small sphere S^n with centre at the point x^0. Since x^0 is the unique singular point belonging to D_l and D_l is homeomorphic to the unit ball, then the characteristic of the vector field induced by the mapping $\Delta_j(x_0)$ on the sphere S^n is equal to its characteristic on the boundary Γ_{D_l} of the domain D_l.

We shall show that the vector fields induced by the mappings $\Delta_j(x_0)$ and $\Delta(x_0)$ are homotopic on Γ_{D_l}.

This follows from the fact that the continuously depending on the parameter θ, $0 \leq \theta \leq 1$, family of continuous on Γ_{D_l} vector fields

$$V(\theta, x_0) = \Delta_j(x_0) + \theta(\Delta(x_0) - \Delta_j(x_0)),$$

which connects the vector fields $V(0, x_0) = \Delta_j(x_0)$ and $V(1, x_0) = \Delta(x_0)$ does not vanish on Γ_{D_l}.

In fact, in view of (16) we obtain

$$\begin{aligned}
|\Delta(x_0) - \Delta_j(x_0)| &\leq |P_0[X(t, x^*(t)) - X(t, x_j(t))]| + \\
&+ \frac{1}{2\pi} \sum_{0<t_i<2\pi} |I_i(x^*(t_i)) - I_i(x_j(t_i))| \leq \\
&\leq \left| \frac{1}{2\pi} \int_0^{2\pi} [X(t, x^*(t)) - X(t, x_j(t))]\, dt \right| + \\
&+ \frac{1}{2\pi} \sum_{0<t_i<2\pi} J\, |x^*(t_i) - x_j(t_i)| \leq \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \lambda\, |x^*(t) - x_j(t)|\, dt + \frac{1}{2\pi} Jk\, \|x^*(t) - x_j(t)\| \leq
\end{aligned} \tag{20}$$

$$\leq (\lambda + \frac{Jk}{2\pi}) \| x^*(t) - x_j(t) \| \leq \frac{1}{2\pi}(2\pi\lambda + Jk) \frac{\alpha_l^j}{1 - \alpha_l} a(l)$$

and from (19) and (20) it follows that on Γ_{Dl} the following inequality holds

$$| V(\theta, x_0) | \geq |\Delta_j(x_0)| - |\Delta_j(x_0) - \Delta(x_0)| > 0$$

Since the characteristics of the homotopic on a compact set vector fields are equal, then the characteristic on Γ_{Dl} of the field $\Delta(x_0)$ is equal to the index of the singular point x^0 of the field $\Delta_j(x_0)$. Hence the vector field $\Delta(x_0)$ in the domain D_l has a singular point x^0 for which $\Delta(x^0) = 0$ and system (1) has a 2π-periodic solution.

Finally we shall note that the proof of theorem 2 follows in an ideological aspect the proof of theorem 7.1 form [15].

REFERENCES

[1] Mil'man, V. D., Myshkis, A. D., On the stability of motion in the presence of impulses, Sib. Math. J., 1, No. 2(1960), 223-237 (in Russian).

[2] Bainov, D. D., Milusheva, S. D., Justification of the averaging method for a system of differential equations with fast and slow variables and with impulses, Journal of Applied Mathematics and Physics (ZAMP), Vol. 32(1981), 237-254.

[3] Bainov, D. D., Milusheva, S. D., Application of the averaging method for functional differential equations with impulses, Journ. Math. Anal. Appl., Vol. 95, No. 1 (1983), 85-105.

[4] Bainov, D. D., Milusheva, S. D., Justification of partially-multiplicative averaging for a class of functional differential equations with variable structure and impulses, Tsukuba J. Math., Vol. 9, No. 1 (1985), 1-19.

[5] Dishliev, A. B., Bainov, D. D., Sufficient conditions for absence of "beating" in systems of differential equations with impulses, Applicable Analysis, Vol. 18(1984), 67-73.

[6] Dishliev, A. B., Bainov, D. D., Conditions for the absence of the phenomenon "beating" for systems of impulse differential equations, Bulletin of the Institute of Mathematics Academia Sinica, Vol. 13, No. 3 (1985), 237-256.

[7] Hekimova, M. A., Bainov, D. D., A method for asymptotic integration of a periodic singulary perturbed system with impulses and its application in the theory of optimal control, J. Math. pures et appl., 65(1986), 307-321.

[8] Milusheva, S. D., Bainov, D. D., Justification of the averaging method for a class of functional differential equations with impulses, J. London Math. Soc. (2), 25(1982), 309-331.

[9] Milusheva, S. D., Bainov, D. D., Justification of partially-multiplicative averaging

for a class of functional differential equations with impulses, Publ. RIMS, Kyoto Univ. 19 (1983), 549-576.

[10] Milusheva, S. D., Bainov D. D., Justification of the averaging method for a system of singularly perturbed differential equations with impulses, Journal of Applied Mathematics and Physics (ZAMP), 36 (1985), 293-308.

[11] Pandit, S. G., Differential systems with impulsive perturbations, Pacific J. Math. 86 (1980), 553-560.

[12] Simeonov, P. S., Bainov D. D., Stability under persistent disturbances for systems with impulse effect, Journ. Math. Anal. Appl., Vol. 109, No. 2 (1985), 546-563.

[13] Simeonov, P. S., Bainov D. D., Stability with respect to part of the variables in systems with impulse effect, Journ. Math. Anal. Appl., Vol. 117, No. 1 (1986), 247-263.

[14] Grechko, V. I., On a projection-iterative method of finding periodic solutions of systems of ordinary differential equations, Ukr. Math. J., Vol. 26, No. 4 (1974), 534-539 (in Russian).

[15] Samoilenko, A. M., A numerical-analytic method of investigation of periodic systems of ordinary differential equations, Ukr. Math. J., Vol. 18, No. 2 (1966), 50-59 (in Russian).

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