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Gauss multiplication formula for extended psi function

Mitsuhiko Kohno

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Abstract. In the previous paper the author defined an extension of the psi function or the gamma function, which seems to be useful in some applications of mathematical science. In order that it plays a role as a special function, the properties of such a function must be investigated in detail.

In this short paper we shall verify the multiplication formula for the extended gamma function like the well-known *Gauss multiplication formula*

$$\Gamma(nz) = \frac{n^{nz-\frac{1}{2}}}{(\sqrt{2\pi})^{n-1}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right)$$

for the classical gamma function $\Gamma(z)$.

1 Gauss multiplication formula

As an extension of the classical psi function, we have defined a series of solutions of the linear difference equation

$$(1) \quad \Delta\Phi_p(z) = \Phi_p(z+1) - \Phi_p(z) = \frac{1}{p!}\, z^p \log z - \gamma_p\, z^p \qquad (p = 0,\, 1,\, 2,\, \ldots\,),$$

where the constant γ_p satisfies the relation

$$p\,\gamma_p - \gamma_{p-1} = \frac{1}{p!}, \qquad \gamma_0 = 0,$$

that is, γ_p is given by

$$\gamma_p = \frac{1}{p!}\left(1 + \frac{1}{2} + \cdots + \frac{1}{p}\right) \qquad (p = 0,\, 1,\, 2,\, \ldots\,).$$

For $p = 0$, the difference equation becomes

$$\Phi_0(z+1) - \Phi_0(z) = \log z$$

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and hence we see that

$$\Phi_0(z) = \log\Gamma(z), \qquad \Phi_0'(z) = \frac{\Gamma'(z)}{\Gamma(z)} \equiv \Psi(z).$$

It is also easily seen that there hold

$$\frac{d^k}{dz^k}\Phi_p(z) \equiv \Phi_p^{(k)}(z) = \Phi_{p-k}(z) \qquad (k = 1, 2, \dots, p).$$

In this paper we shall study the Gauss multiplication formulas for such extended psi functions $\Phi_p(z)$ $(p = 0,\ 1,\ 2,\ \dots)$. To this end, we first introduce some properties needed later for Bernoulli polynomials. (See [5] and [6].)

The Bernoulli polynomials $B_k(z)$ satisfy the linear difference equations

$$\Delta B_k(z) \equiv B_k(z+1) - B_k(z) = k\, z^{k-1} \qquad (k = 0,\ 1,\ 2, \dots)$$

and they can be expressed in terms of the binomial expansion

$$B_k(z) = (z + B)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} B^\ell z^{k-\ell},$$

where the B^ℓ are interpreted as

$$B^\ell \equiv B_\ell \quad (\text{Bernoulli number}).$$

Now we consider the multiplication formula for $\Phi_p(z)$.

Put

$$g(z) = a\left[\sum_{k=0}^{n-1} \Phi_p\left(\frac{z+k}{n}\right)\right] + b\, B_{p+1}(z)$$

and then obtain

$$\begin{aligned}
\Delta g(z) &= a\sum_{k=0}^{n-1}\left[\Phi_p\left(\frac{z+k+1}{n}\right) - \Phi_p\left(\frac{z+k}{n}\right)\right] + b\,\Delta B_{p+1}(z) \\
&= a\left[\Phi_p\left(\frac{z}{n}+1\right) - \Phi_p\left(\frac{z}{n}\right)\right] + b\,(p+1)\, z^p \\
&= a\left[\frac{1}{p!}\left(\frac{z}{n}\right)^p \log\left(\frac{z}{n}\right) - \gamma_p\left(\frac{z}{n}\right)^p\right] + b\,(p+1)\, z^p \\
&= \frac{a}{n^p}\left[\frac{1}{p!} z^p \log z - \gamma_p z^p\right] + \left(b\,(p+1) - \frac{a}{n^p}\frac{1}{p!}\log n\right) z^p.
\end{aligned}$$

Here we take

$$a = n^p, \qquad b = \frac{1}{(p+1)!}\log n.$$

Then, we have

$$\Delta\, g(z) \;=\; \frac{1}{p!}\, z^p \log z \;-\; \gamma_p\, z^p.$$

This implies that $g(z)$ is a solution of the linear difference equation satisfied by $\Phi_p(z)$. So, we have

$$\Phi_p(z) \;=\; n^p \left[\sum_{k=0}^{n-1} \Phi_p\left(\frac{z+k}{n}\right)\right] \;+\; \frac{B_{p+1}(z)}{(p+1)\,!}\log n \;+\; c, \tag{2}$$

where c is generally a periodic function with period 1.

Taking account of the fact that

$$B_1(z) \;=\; z \;-\; \frac{1}{2},$$

we immediately see that the formula (2) for $p = 0$ exactly corresponds to the original Gauss multiplication formula for $\Phi_0(z) = \log\Gamma(z)$:

$$\log\Gamma(z) \;=\; \sum_{k=0}^{n-1} \log\Gamma\left(\frac{z+k}{n}\right) \;+\; \left(z \;-\; \frac{1}{2}\right)\log n \;-\; \frac{(n-1)}{2}\log 2\pi.$$

So, the formula (2) may be considered to be the extension of the Gauss multiplication formula.

2 Asymptotic expansion

In order to determine c in the formula (2), we shall here reconsider the asymptotic expansion of $\Phi_p(z)$.

In the paper [1], we have derived the asymptotic expansion of $\Phi_p(z)$ in the right half plane as follows :

$$\Phi_p(z) \;\sim\; \left(\frac{1}{(p+1)\,!}\log z \;-\; \gamma_{p+1}\right) B_{p+1}(z) \;+\; \sum_{\ell=0}^{p-1} \overline{\alpha}_\ell\, B_{\ell+1}(z) \tag{3}$$

$$+\; \overline{\lambda}\log z \;+\; \sum_{m=1}^{\infty} \frac{d_m}{z^m} \qquad (\operatorname{Re} z > 0,\;\; z \to \infty).$$

In fact, we put

$$\Phi_p(z) \;=\; \frac{1}{(p+1)\,!}\, B_{p+1}(z)\log z \;-\; \frac{\gamma_p}{p+1}\, B_{p+1}(z) \;+\; \psi(z)$$

and then it is easy to see from (1) that $\psi(z)$ must satisfy the linear difference equation

$$\Delta\,\psi(z) \;=\; -\,\frac{1}{(p+1)\,!}\, B_{p+1}(z+1)\,\Delta\,\log z \;\equiv\; a(z).$$

Putting

$$B_{p+1}(z+1) = \sum_{m=0}^{p+1} C_m\, z^m,$$

we calculate the right hand side

$$\begin{aligned}
a(z) &= -\frac{1}{(p+1)\,!}\left(\sum_{m=0}^{p+1} C_m\, z^m\right)\left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}\,\frac{1}{z^m}\right) \\
&= \frac{1}{(p+1)\,!}\sum_{\ell=0}^{p}\left(\sum_{m=1}^{p+1-\ell} C_{m+\ell}\,\frac{(-1)^m}{m}\right) z^\ell \\
&+ \frac{1}{(p+1)\,!}\left\{\sum_{m=1}^{p+1} C_m\,\frac{(-1)^{m+1}}{m+1} - B_{p+1}\right\}\frac{1}{z} + O\left(\frac{1}{z^2}\right) \\
&\equiv \frac{1}{(p+1)\,!}\left\{\sum_{\ell=0}^{p} a_\ell\, z^\ell + \frac{\lambda}{z}\right\} + O\left(\frac{1}{z^2}\right),
\end{aligned}$$

whence we obtain the principal part of the asymptotic expansion as the summation (as for the following notation, see [3])

$$\psi(z) = \mathbf{S}\, a(z) = \frac{1}{(p+1)\,!}\left\{\sum_{\ell=0}^{p}\frac{a_\ell}{\ell+1} B_{\ell+1}(z) + \lambda \log z\right\} + O\left(\frac{1}{z}\right).$$

So, putting

$$\alpha_\ell = \frac{1}{\ell+1}\left(\sum_{m=1}^{p+1-\ell} C_{m+\ell}\,\frac{(-1)^m}{m}\right), \qquad \overline{\alpha}_\ell = \frac{\alpha_\ell}{(p+1)\,!},$$

$$\lambda = \left\{\sum_{m=1}^{p+1} C_m\,\frac{(-1)^{m+1}}{m+1} - B_{p+1}\right\}, \qquad \overline{\lambda} = \frac{\lambda}{(p+1)\,!},$$

we obtain the principal part of the asymptotic expansion of $\Phi_p(z)$

$$\begin{aligned}
\Phi_p(z) \sim\ & \frac{1}{(p+1)\,!} B_{p+1}(z)\log z - \frac{\gamma_p}{p+1} B_{p+1}(z) \\
&+ \sum_{\ell=0}^{p} \overline{\alpha}_\ell\, B_{\ell+1}(z) + \overline{\lambda}\log z + O\left(\frac{1}{z}\right).
\end{aligned}$$

Since

$$\alpha_p = -\frac{1}{p+1}, \qquad \gamma_p + \frac{1}{(p+1)\,!} = (p+1)\,\gamma_{p+1},$$

we therefore derive the asymptotic expansion of the form (3) described above.

We here make a remark on the coefficient of the asymptotic expansion (3). From the property of the Bernoulli polynomial, we have

$$\begin{aligned} B_{p+1}(z+1) &= \sum_{m=0}^{p+1} C_m\, z^m = (z+1+B)^{p+1} \\ &= \sum_{m=0}^{p+1} \binom{p+1}{m} (B+1)^{p+1-m}\, z^m \end{aligned}$$

and hence we can write the coefficient C_m by

$$C_m = \binom{p+1}{m} (B+1)^{p+1-m} \qquad (m = 0, 1, \ldots, p+1). \tag{4}$$

Using this, we can prove that for $p \geq 0$ the constant $\overline{\lambda}$ is vanishing. In the formula

$$\begin{aligned} \lambda &= \left\{ \sum_{m=1}^{p+1} C_m \frac{(-1)^{m+1}}{m+1} - B_{p+1} \right\} \qquad (C_0 = B_{p+1}) \\ &= \sum_{m=0}^{p+1} C_m \frac{(-1)^{m+1}}{m+1}, \end{aligned}$$

we substitute the above expression (4) and then calculate as follows :

$$\begin{aligned} \lambda &= \sum_{m=0}^{p+1} \frac{(p+1)!}{(m+1)!(p+1-m)!} (B+1)^{p+1-m} (-1)^{m+1} \\ &= \frac{1}{p+2} \sum_{m=0}^{p+1} \frac{(p+2)!}{(m+1)!(p+2-(m+1))!} (B+1)^{p+2-(m+1)} (-1)^{m+1} \\ &= \frac{1}{p+2} \sum_{m=1}^{p+2} \binom{p+2}{m} (B+1)^{p+2-m} (-1)^m \\ &= \frac{1}{p+2} \left\{ \sum_{m=0}^{p+2} \binom{p+2}{m} (B+1)^{p+2-m} (-1)^m - (B+1)^{p+2} \right\} \\ &= \frac{1}{p+2} \{ (B+1-1)^{p+2} - (B+1)^{p+2} \} \\ &= \frac{1}{p+2} \{ B_{p+2}(0) - B_{p+2}(1) \} = 0 \qquad (p \geq 0). \end{aligned}$$

3 Determination of c

Taking account of the asymptotic expansion (3) of $\Phi_p(z)$, we calculate the principal part of the asymptotic expansion of the function in the right hand side of (2) :

$$g(z) = n^p \left[\sum_{k=0}^{n-1} \Phi_p \left(\frac{z+k}{n} \right) \right] + \frac{B_{p+1}(z)}{(p+1)!} \log n + c.$$

We begin with the calculation of the sum

$$\begin{aligned}
\left[\sum_{k=0}^{n-1} \Phi_p \left(\frac{z+k}{n} \right) \right] &\sim \sum_{k=0}^{n-1} \left(\frac{1}{(p+1)!} \log \left(\frac{z+k}{n} \right) - \gamma_{p+1} \right) B_{p+1} \left(\frac{z+k}{n} \right) \\
&+ \sum_{k=0}^{n-1} \sum_{\ell=0}^{p-1} \overline{\alpha}_\ell B_{\ell+1} \left(\frac{z+k}{n} \right) + O\left(\frac{1}{z} \right) \\
&= \sum_{k=0}^{n-1} \left(\frac{1}{(p+1)!} \log \left(\frac{z}{n} \right) - \gamma_{p+1} \right) B_{p+1} \left(\frac{z+k}{n} \right) \\
&+ \frac{1}{(p+1)!} \sum_{k=0}^{n-1} \log \left(1 + \frac{k}{z} \right) B_{p+1} \left(\frac{z+k}{n} \right) \\
&+ \sum_{k=0}^{n-1} \sum_{\ell=0}^{p-1} \overline{\alpha}_\ell B_{\ell+1} \left(\frac{z+k}{n} \right) + O\left(\frac{1}{z} \right) \\
&= \left(\frac{1}{(p+1)!} \log \left(\frac{z}{n} \right) - \gamma_{p+1} \right) B_{p+1}(z) n^{-p} \\
&+ \frac{1}{(p+1)!} \sum_{k=0}^{n-1} \log \left(1 + \frac{k}{z} \right) B_{p+1} \left(\frac{z+k}{n} \right) \\
&+ \sum_{\ell=0}^{p-1} \overline{\alpha}_\ell B_{\ell+1}(z) n^{-\ell} + O\left(\frac{1}{z} \right),
\end{aligned}$$

in which we have used the multiplication formula for the Bernoulli polynomial

$$B_m(z) = \left[\sum_{k=0}^{n-1} B_m \left(\frac{z+k}{n} \right) \right] n^{m-1}.$$

We consequently obtain

$$\begin{aligned} g(z) &\sim \left(\frac{1}{(p+1)!}\log z - \gamma_{p+1}\right) B_{p+1}(z) \\ &+ \frac{n^p}{(p+1)!}\sum_{k=0}^{n-1}\log\left(1+\frac{k}{z}\right) B_{p+1}\left(\frac{z+k}{n}\right) \\ &+ \sum_{\ell=0}^{p-1}\overline{\alpha}_\ell\, B_{\ell+1}(z)\, n^{p-\ell} + O\left(\frac{1}{z}\right) + c. \end{aligned}$$

Combining this asymptotic expansion with that of $\Phi_p(z)$, we have

$$\begin{aligned} g(z) - \Phi_p(z) &\sim \frac{n^p}{(p+1)!}\sum_{k=0}^{n-1}\log\left(1+\frac{k}{z}\right) B_{p+1}\left(\frac{z+k}{n}\right) \\ &+ \sum_{\ell=0}^{p-1}\overline{\alpha}_\ell\,(n^{p-\ell}-1)\, B_{\ell+1}(z) + c + O\left(\frac{1}{z}\right) \end{aligned} \tag{5}$$

for sufficiently large values of z in the right half plane.

Let us put

$$\begin{aligned} R(z) &\equiv n^p\sum_{k=0}^{n-1}\log\left(1+\frac{k}{z}\right) B_{p+1}\left(\frac{z+k}{n}\right) \\ &+ \sum_{\ell=0}^{p-1}\alpha_\ell\,(n^{p-\ell}-1)\, B_{\ell+1}(z). \end{aligned}$$

If we can show that the principal part of $R(z)$ includes no terms of z^k ($k = 1, 2, \ldots, p$), then, according to the asymptotic relation (5), we can conclude that c must be a constant, which is determined by the constant term of $R(z)$. Instead of calculating the exact values of coefficients of z^k and verifying that they are all zero, we consider the difference $\Delta R(z)$:

$$n^p\sum_{k=0}^{n-1}\Delta\left[\log\left(1+\frac{k}{z}\right) B_{p+1}\left(\frac{z+k}{n}\right)\right] + \sum_{\ell=0}^{p-1}\alpha_\ell\,(n^{p-\ell}-1)\,\Delta[B_{\ell+1}(z)].$$

Rewriting

$$\begin{aligned}
&\Delta\left[\log\left(1+\frac{k}{z}\right) B_{p+1}\left(\frac{z+k}{n}\right)\right] \\
&= B_{p+1}\left(\frac{z+1+k}{n}\right)\log\left(1+\frac{k+1}{z}\right) - B_{p+1}\left(\frac{z+k}{n}\right)\log\left(1+\frac{k}{z}\right) \\
&+ B_{p+1}\left(\frac{z+1+k}{n}\right)\left[\log\left(1+\frac{k}{z+1}\right) - \log\left(1+\frac{k+1}{z}\right)\right] \\
&= B_{p+1}\left(\frac{z+1+k}{n}\right)\log\left(1+\frac{k+1}{z}\right) - B_{p+1}\left(\frac{z+k}{n}\right)\log\left(1+\frac{k}{z}\right) \\
&- B_{p+1}\left(\frac{z+1+k}{n}\right)\log\left(1+\frac{1}{z}\right),
\end{aligned}$$

we can obtain

$$\begin{aligned}
&\sum_{k=0}^{n-1}\Delta\left[\log\left(1+\frac{k}{z}\right) B_{p+1}\left(\frac{z+k}{n}\right)\right] \\
&= B_{p+1}\left(\frac{z+n}{n}\right)\log\left(1+\frac{n}{z}\right) - \left[\sum_{k=0}^{n-1} B_{p+1}\left(\frac{z+1+k}{n}\right)\right]\log\left(1+\frac{1}{z}\right) \\
&= B_{p+1}\left(\frac{z+n}{n}\right)\log\left(1+\frac{n}{z}\right) - n^{-p} B_{p+1}(z+1)\log\left(1+\frac{1}{z}\right)
\end{aligned}$$

from the multiplication formula for the Bernoulli polynomial.

Moreover, taking account of

$$\Delta B_m(z) = m\, z^{m-1} \qquad (m = 0, 1, 2, \dots),$$

we consequently see that $\Delta R(z)$ is equal to

$$\begin{aligned}
&n^p\left\{B_{p+1}\left(1+\frac{z}{n}\right)\log\left(1+\frac{n}{z}\right) - n^{-p} B_{p+1}(1+z)\log\left(1+\frac{1}{z}\right)\right\} \\
&+ \sum_{\ell=0}^{p-1}\alpha_\ell\,(n^{p-\ell}-1)\,(\ell+1)\,z^\ell.
\end{aligned}$$

As we have already seen, we obtain

$$
\begin{aligned}
&B_{p+1}\left(1+\frac{z}{n}\right)\log\left(1+\frac{n}{z}\right)\\
&\quad = \sum_{\ell=0}^{p}\left(\sum_{m=1}^{p+1-\ell} C_{m+\ell}\frac{(-1)^{m-1}}{m}\right)\left(\frac{z}{n}\right)^{\ell} + O\left(\frac{1}{z^2}\right)\\
&\quad = -\sum_{\ell=0}^{p}\alpha_\ell\,(\ell+1)\left(\frac{z}{n}\right)^{\ell} + O\left(\frac{1}{z^2}\right),\\
&B_{p+1}(1+z)\log\left(1+\frac{1}{z}\right)\\
&\quad = -\sum_{\ell=0}^{p}\alpha_\ell\,(\ell+1)\,z^{\ell} + O\left(\frac{1}{z^2}\right).
\end{aligned}
$$

Substituting these formulas into the above relation, we have thus proved that

$$\Delta R(z) \;=\; O\left(\frac{1}{z^2}\right).$$

This implies that

$$R(z) \;=\; \text{a constant } R_0 \;+\; O\left(\frac{1}{z}\right).$$

Now, we shall calculate the constant term in $R(z)$. For that purpose, we first consider the Taylor expansion of the function

$$f(\xi) \;=\; (1+\xi)^m \log(1+\xi), \qquad f(0) \;=\; 0.$$

We easily obtain

$$f(\xi) \;=\; \frac{1}{1!}f'(0)\,\xi + \frac{1}{2!}f''(0)\,\xi^2 + \cdots + \frac{1}{\ell!}f^{(\ell)}(0)\,\xi^{\ell} + \cdots,$$

where

$$f^{(\ell)}(0) = \sum_{j=1}^{\ell}\binom{\ell}{j}\frac{m!\,(-1)^{j-1}(j-1)!}{(m-\ell+j)!}.$$

In particular, we have

$$\frac{1}{m!}\,f^{(m)}(0) = \sum_{j=1}^{m}\binom{m}{j}\frac{(-1)^{j-1}}{j}.$$

Taking account of the above formula, we calculate the constant in

$$B_{p+1}\left(\frac{z+k}{n}\right)\log\left(1+\frac{k}{z}\right)$$

$$=\sum_{h=0}^{p+1}\binom{p+1}{h}B_h\left(\frac{z}{n}\right)^{p+1-h}\left(1+\frac{k}{z}\right)^{p+1-h}\log\left(1+\frac{k}{z}\right)$$

$$=\sum_{h=0}^{p+1}\binom{p+1}{h}B_h\left[\left(\frac{z}{n}\right)^{p+1-h}f\left(\frac{k}{z}\right)\right].$$

It is not difficult to see that the constant term is given by

$$e_k=\sum_{h=0}^{p}\binom{p+1}{h}B_h\left[\sum_{j=1}^{p+1-h}\binom{p+1-h}{j}\frac{(-1)^{j-1}}{j}\right]\left(\frac{k}{n}\right)^{p+1-h}.$$

Hence, the constant term R_0 in $R(z)$ is equal to

$$R_0=n^p\sum_{k=0}^{n-1}e_k+\sum_{\ell=0}^{p-1}\alpha_\ell\,(n^{p-\ell}-1)\,B_{\ell+1}(0).$$

Since

$$0=g(z)-\Phi_p(z)\sim\frac{1}{(p+1)!}R(z)+c+O\left(\frac{1}{z}\right),$$

the constant term in the right hand side must be zero. We consequently obtain

$$c=\frac{1}{(p+1)!}\left\{\sum_{h=0}^{p}\binom{p+1}{h}B_h\left[\sum_{j=1}^{p+1-h}\binom{p+1-h}{j}\frac{(-1)^j}{j}\right]\right.$$
$$\left.\times\, n^{h-1}\left(\frac{B_{p+2-h}(n)-B_{p+2-h}(0)}{p+2-h}\right)-\sum_{\ell=0}^{p-1}\alpha_\ell\,(n^{p-\ell}-1)\,B_{\ell+1}\right\},\tag{6}$$

where we have used Jakob Bernoulli's formula

$$1^m+2^m+\cdots+(n-1)^m=\int_0^n B_m(z)\,dz=\frac{B_{m+1}(n)-B_{m+1}(0)}{m+1}.$$

For $p=0$, the constant c is given as

$$c=-\frac{n-1}{2}.$$

Theorem *There holds the multiplication formula for the extended psi function* $\Phi_p(z)$:

$$\Phi_p(nz)=n^p\left[\sum_{k=0}^{n-1}\Phi_p\left(z+\frac{k}{n}\right)\right]+\frac{B_{p+1}(nz)}{(p+1)!}\log n+c,$$

where the constant c is given by (6) .

It is remarked that if $\Phi_p(z)$ includes some constant b_p in the asymptotic expansion, then $-(n^{p+1}-1)b_p$ will be added to the constant c.

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Mitsuhiko Kohno
Department of Mathematics
Kumamoto University
Kumamoto 860-8555, Japan