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Coupled Painlevé IV systems in dimension four

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Abstract. We find and study coupled Painlevé IV systems in dimension four, which are different from the systems of type $A_4^{(1)}$. We also give the augmented phase spaces for these systems.

1 Introduction

The purpose of this paper is to characterize higher order Painlevé equations from the viewpoint of algebraic vector fields with favorable properties in regard to accessible singularities (see Definition 2.1) and local index (see Definition 2.2). As an example of higher order Painlevé equations with favorable properties, we studied the systems of type $A_4^{(1)}$, which can be considered as a generalization of the fourth Painlevé equation P_{IV} to fourth-order. Let us summarize important properties of these systems as follows (see [4, 5]):

Notation.
• $H \in \mathbb{C}[t][x,y,z,w]$, • $deg(H)$: degree with respect to x, y, z, w,
• $\Theta_{\mathbb{P}^4}(-\log \mathcal{H})$: subsheaf of $\Theta_{\mathbb{P}^4}$ whose local section v satisfies $v(f) \in (f)$ for any local equation f of the boundary divisor $\mathcal{H}$ of $\mathbb{P}^4$,
• $H_{IV}(q,p,t;\gamma_1,\gamma_2) = -qp^2 + p(\gamma_1 - tq + q^2) + \gamma_2 q$: the Hamiltonian of P_{IV},
• dim. of sol. : dimension of the parameter space of meromorphic solutions which pass through an accessible singular point.

symmetry	$W(A_4^{(1)})$
Hamiltonian	$H_{IV}(x,y,t) + H_{IV}(z,w,t)$ $-2yzw$
form of equations	coupled Painlevé IV
degree of Hamiltonian H	3
$v \in H^0(\mathbb{P}^4, \Theta_{\mathbb{P}^4}(-\log \mathcal{H})(n\mathcal{H}))$	$n = 1$
number of parameter	4

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type of accessible singularity	point	point	point
type of local index	$(+1,+3,+1,+1)$	$(+1,+3,-1,+3)$	(+1,+3,+5,-3)
dim. of solu.	dim. 3	dim. 2	dim. 2

Unlike the second-order case, in the fourth-order case there exist the following possibilities for types of accessible singularities: (i) point, (ii) curve, (iii) surface.

These properties suggest the possibility that there exist higher order versions of P_{IV} as well, and furthermore, suggest a procedure for searching for such higher order versions with different types of accessible singularities from the systems of type $A_4^{(1)}$. The purpose of this paper is to find a fourth-order version of the Painlevé IV systems other than the systems of type $A_4^{(1)}$. Here, we consider the following problem.

Problem.

Can we classify the coupled Painlevé IV systems in dimension four that are the Hamiltonian systems with Hamiltonian $H \in \mathbb{C}[t][x,y,z,w]$ of $deg(H)=3$, and moreover, have given some accessible singularities?

To answer this, in the present paper, we construct a 3-parameter family of fourth-order algebraic ordinary differential equations that can be considered as coupled Painlevé IV systems in dimension four, which are given as follows:

$$\begin{cases} \dfrac{dx}{dt} = x^2 - 2xy + xz - tx + \alpha_1, \\ \dfrac{dy}{dt} = y^2 - 2xy - yz - zw + ty - \alpha_2, \\ \dfrac{dz}{dt} = z^2 - 2zw + xz - tz + \alpha_3, \\ \dfrac{dw}{dt} = w^2 - 2zw - xy - xw + tw - \alpha_2. \end{cases} \tag{1}$$

Here x, y, z and w denote unknown complex variables and α_1, α_2 and α_3 are complex parameters. This system is equivalent to a Hamiltonian system given by

$$\begin{aligned} H &= H_{IV}(x,y,t;\alpha_1,\alpha_2) + H_{IV}(z,w,t;\alpha_3,\alpha_2) + xyz + xzw \\ &= -xy^2 + y(\alpha_1 - tx + x^2) + \alpha_2 x - zw^2 + w(\alpha_3 - tz + z^2) + \alpha_2 z + xyz + xzw. \end{aligned}$$

From the viewpoint of symmetry, it is worthwhile to point out the following theorem.

Theorem 1.1 *The system* (1) *is invariant under the following transformations:*

$$s_1 : (x,y,z,w,t;\alpha_1,\alpha_2,\alpha_3) \rightarrow (x, y-\frac{\alpha_1}{x}, z, w, t; -\alpha_1, \alpha_2+\alpha_1, \alpha_3),$$

$$s_2 : (x, y, z, w, t; \alpha_1, \alpha_2, \alpha_3) \rightarrow (x, y, z, w - \frac{\alpha_3}{z}, t; \alpha_1, \alpha_2 + \alpha_3, -\alpha_3),$$

$$s_3 : (x, y, z, w, t; \alpha_1, \alpha_2, \alpha_3) \rightarrow$$

$$(\frac{\sqrt{-1}y(xy - \alpha_1)}{xy + zw + \alpha_2}, \frac{\sqrt{-1}(xy + zw + \alpha_2)}{y}, \frac{\sqrt{-1}w(zw - \alpha_3)}{xy + zw + \alpha_2}, \frac{\sqrt{-1}(xy + zw + \alpha_2)}{w},$$

$$\sqrt{-1}t; \alpha_1, \alpha_2 + \alpha_1 + \alpha_3, \alpha_3),$$

$$\pi_1 : (x, y, z, w, t; \alpha_1, \alpha_2, \alpha_3) \rightarrow (z, w, x, y, t; \alpha_3, \alpha_2, \alpha_1),$$

$$\pi_2 : (x, y, z, w, t; \alpha_1, \alpha_2, \alpha_3) \rightarrow$$

$$(\sqrt{-1}x, \sqrt{-1}(x - y + z - t), \sqrt{-1}z, \sqrt{-1}(x - w + z - t), -\sqrt{-1}t; \alpha_1, 1 - \alpha_1 - \alpha_2 - \alpha_3, \alpha_3).$$

For the system (1), K. Kimura showed the following theorem.

Theorem 1.2 *The system* (1) *has the following first integral* I

$$I = \alpha_3 xw - \alpha_1 zw - \alpha_3 xy + \alpha_1 yz - xy^2 z + 2xyzw - xzw^2.$$

Theorems 1.1 and 1.2 can be checked by a direct calculation.

In 1979, K. Okamoto [2] constructed the spaces of initial conditions of Painlevé equations, which can be considered as the parameter spaces of all meromorphic solutions (including holomorphic solutions). We call these spaces *augmented* phase spaces, in accordance with [8]. For the system (1), we can construct the following phase space.

Theorem 1.3 *The augmented phase spaces* $\mathcal{X}$ *over* $B = \mathbb{C}$ *for the system* (1) *is obtained by gluing twelve copies of* $\mathbb{C}^4 \times \mathbb{C}$:

$U_0 \times \mathbb{C} = \mathbb{C}^4 \times \mathbb{C} \ni (x, y, z, w, t)$,

$U_j \times \mathbb{C} = \mathbb{C}^4 \times \mathbb{C} \ni (x_j, y_j, z_j, w_j, t)$ $(j = 1, 2, .., 11)$,

via the following rational and symplectic transformations:

1) $x_1 = \dfrac{1}{x}$, $y_1 = -x(xy + zw + \alpha_2)$, $z_1 = \dfrac{z}{x}$, $w_1 = xw$,

2) $x_2=\frac{x}{z},\ y_2=yz,\ z_2=\frac{1}{z},\ w_2=-z(zw+xy+\alpha_2),$

3) $x_3=-x(xy-\alpha_1),\ y_3=\frac{1}{y},\ z_3=z,\ w_3=w,$

4) $x_4=x,\ y_4=y,\ z_4=-w(zw-\alpha_3),\ w_4=\frac{1}{w},$

5) $x_5=y(1-\alpha_1-\alpha_2-\alpha_3+ty-xy+y^2+zw-2yz),\ y_5=\frac{1}{y},$

$$z_5=-(w-y)y(-\alpha_3+zw-yz),\ w_5=\frac{1}{y(w-y)},$$

6) $x_6=y(1-\alpha_1-\alpha_2-\alpha_3+ty-xy+y^2+zw-2yz),\ y_6=\frac{1}{y},\ z_6=\frac{z}{y},\ w_6=(w-y)y,$

7) $x_7=-y(xy-\alpha_1),\ y_7=\frac{1}{y},\ z_7=-w(zw-\alpha_3),\ w_7=\frac{1}{w},$

8) $x_8=\frac{1}{x},\ y_8=-x(xy+zw+\alpha_2),\ z_8=-xw(zw-\alpha_3),\ w_8=\frac{1}{xw},$

9) $x_9=-zy(xy-\alpha_1),\ y_9=\frac{1}{zy},\ z_9=\frac{1}{z},\ w_9=-z(zw+xy+\alpha_2),$

10) $x_{10}=-(y-w)w(-\alpha_1+xy-wx),\ y_{10}=\frac{1}{w(y-w)},$

$$z_{10}=w(1-\alpha_1-\alpha_2-\alpha_3+tw-zw+w^2+xy-2xw),\ w_{10}=\frac{1}{w},$$

11) $x_{11}=\frac{x}{w},\ y_{11}=(y-w)w,$

$$z_{11}=w(1-\alpha_1-\alpha_2-\alpha_3+tw-zw+w^2+xy-2xw),\ w_{11}=\frac{1}{w}.$$

By taking suitable choices among the data 1) – 11) in Theorem 1.3, we can reconstruct the system (1).

Theorem 1.4 *Let us consider an algebraic and Hamiltonian differential system with Hamiltonian $H \in \mathbb{C}[t][x,y,z,w]$. We assume that*

(A1) $deg(H)=3$ with respect to x,y,z,w.

(A2) This system becomes again a holomorphic differential system in each coordinate system (x_i,y_i,z_i,w_i) $(i=1,2,3,4)$:

$$x_1=\frac{1}{x},\ y_1=-x(xy+zw+\alpha_2),\ z_1=\frac{z}{x},\ w_1=xw,$$

$$x_2=-x(xy-\alpha_1),\ y_2=\frac{1}{y},\ z_2=z,\ w_2=w,$$

$$x_3=x,\ y_3=y,\ z_3=-w(zw-\alpha_3),\ w_3=\frac{1}{w},$$

$$x_4=y(1-\alpha_1-\alpha_2-\alpha_3+ty-xy+y^2+zw-2yz),\ y_4=\frac{1}{y},\ z_4=\frac{z}{y},\ w_4=(w-y)y.$$

Then such a system coincides with the system (1).

Remark 1.1 Each coordinate system given in Theorem 1.4 is a holomorphic coordinate system with a three-parameter family of meromorphic solutions of the system (1) as the initial conditions.

There is the following symplectic transformation in addition to the symplectic transformations given in Theorems 1.1 and 1.3.

Proposition 1.1 *By using the following rational and symplectic transformation φ*

$$\varphi:(X,Y,Z,W)=(x+\frac{zw+\alpha_2}{y},y,yz,\frac{w}{y}),$$

the Hamiltonian H is transformed to the polynomial Hamiltonian $\tilde{H}$ in the coordinate system (X,Y,Z,W)

$$\tilde{H}=X^2Y-XY^2-tXY-\alpha_2X+(\alpha_1+\alpha_2)Y+\alpha_2t+XZ-XZW+YZW-YZW^2+\alpha_3YW,$$

and satisfies the following condition:

$$dx\wedge dy+dz\wedge dw-dH\wedge dt=dX\wedge dY+dZ\wedge dW-d\tilde{H}\wedge dt.$$

Theorem 1.4 and Proposition 0.1 can be checked by a direct calculation.

This paper is organized as follows. In Section 1, the notions of accessible singularity and local index are reviewed. In Section 2, we will prove Theorem 1.3 by an explicit birational transformation for each step.

2 Accessible singularities and local index

Let us review the notion of accessible singularity in accordance with [1, 6]. Let B be a connected open domain in $\mathbb{C}$ and $\pi : \mathcal{W} \longrightarrow B$ a smooth proper holomorphic map. We assume that $\mathcal{H} \subset \mathcal{W}$ is a normal crossing divisor which is flat over B. Let us consider a rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfying the condition

$$\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$

Fixing $t_0 \in B$ and $P \in \mathcal{W}_{t_0}$, we can take a local coordinate system $(x_1, x_2, ..., x_n)$ of $\mathcal{W}_{t_0}$ centered at P such that $\mathcal{H}_{\mathrm{smooth}}$ can be defined by the local equation $x_1 = 0$. Since $\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H}))$, we can write down the vector field $\tilde{v}$ near $P = (0, 0, ..., 0, t_0)$ as follows:

$$\tilde{v} = \frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial x_1} + \frac{a_2}{x_1}\frac{\partial}{\partial x_2} + ... + \frac{a_n}{x_1}\frac{\partial}{\partial x_n}.$$

This vector field defines the following system of differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = a_1(x_1, x_2, ..., x_n, t), \\ \dfrac{dx_2}{dt} = \dfrac{a_2(x_1, x_2, ..., x_n, t)}{x_1}, \\ \quad\cdot \\ \quad\cdot \\ \quad\cdot \\ \dfrac{dx_n}{dt} = \dfrac{a_n(x_1, x_2, ..., x_n, t)}{x_1}. \end{cases} \tag{2}$$

Here $a_i(x_1, x_2, ..., x_n, t)$, $i = 1, 2, ..., n$, are holomorphic functions defined near $P = (0, .., 0, t_0)$.

Definition 2.1 *With the above notation, assume that the rational vector field $\tilde{v}$ on $\mathcal{W}$ satisfies the condition*

$$\tilde{v} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})).$$

We say that $\tilde{v}$ has an accessible singularity at $P = (0, 0, ..., 0, t_0)$ if

$$x_1 = 0 \text{ and } a_i(0, 0, ..., 0, t_0) = 0 \text{ for every } i, \ 2 \leq i \leq n.$$

If $P \in \mathcal{H}_{\mathrm{smooth}}$ is not an accessible singularity, all solutions of the ordinary differential equation passing through P are vertical solutions, that is, the solutions are contained in the fiber $\mathcal{W}_{t_0}$ over $t = t_0$. If $P \in \mathcal{H}_{\mathrm{smooth}}$ is an accessible singularity, there may be a solution of (2) which passes through P and goes into the interior $\mathcal{W} - \mathcal{H}$ of $\mathcal{W}$.

Let us recall the notion of local index. When we construct the phase spaces of the higher order Painlevé equations, an object, called the local index, is the key to determining when we need to make a blowing-up of an accessible singularity or a blowing-down to a minimal phase space. In the case of equations of higher order with favorable properties, for example the systems of type $A_4^{(1)}$ [4], the local index at the accessible singular point corresponds to the set of orders that appears in the free parameters of formal solutions passing through that point [8].

Definition 2.2 *Let v be an algebraic vector field which is given by (2) and (X, Y, Z, W) be a boundary coordinate system in a neighborhood of an accessible singularity $P = (0, 0, 0, 0, t)$. Assume that the system is written as*

$$\begin{cases} \dfrac{dX}{dt} = a + f_1(X, Y, Z, W, t), \\ \dfrac{dY}{dt} = \dfrac{bY + f_2(X, Y, Z, W, t)}{X}, \\ \dfrac{dZ}{dt} = \dfrac{cZ + f_3(X, Y, Z, W, t)}{X}, \\ \dfrac{dW}{dt} = \dfrac{dW + f_4(X, Y, Z, W, t)}{X} \end{cases}$$

near the accessible singularity P, where a, b, c and d are nonzero constants. We say that the vector field v has the local index (a, b, c, d) at P if $f_1(X, Y, Z, W, t)$ is a polynomial which vanishes at $P = (0, 0, 0, 0, t)$ and $f_i(X, Y, Z, W, t)$, $i = 2, 3, 4$, are polynomials of order 2 in X, Y, Z, W. Here $f_i \in \mathbb{C}[X, Y, Z, W, t]$ for $i = 1, 2, 3, 4$.

Remark 2.1 We are interested in the case with local index $(1, b/a, c/a, d/a) \in \mathbb{Z}^4$. If each component of $(1, b/a, c/a, d/a)$ has the same sign, we may resolve the accessible singularity by blowing-up finitely many times. However, when different signs appear, we may need to both blow up and blow down.

3 Resolution of the accessible singularities

To prove Theorem 1.3, the latter part of this paper is devoted to giving an explicit resolution of accessible singularities of the systems (1) and to construct a family of phase spaces for the systems. For second-order Painlevé equations, we can obtain the entire space of initial conditions by adding subvarieties of codimension 1 (equivalently, of dimension 1) to the space of initial conditions of holomorphic solutions (see [2, 3, 7]). However, in the case of fourth-order differential equations, we need to add codimension 2 subvarieties to the space in addition to codimension 1 subvarieties. In order to resolve singularities, we need to both blow up and blow down. Moreover, to obtain a smooth variety by blowing-down, we need to resolve for a pair of singularities.

2.1. Accessible singularities of the system (1)

Let P be an accessible singular point in the boundary divisor $\mathcal{H}$ and (X, Y, Z, W) a coordinate system centered at P, where $\{X = 0\} \subset \mathcal{H}$. Rewriting the systems in the local coordinate system (X, Y, Z, W), the right hand side of each differential equation has poles along $\mathcal{H}$. If we resolve the accessible singularity P and the right hand side of each differential equation becomes holomorphic in the coordinate system $(X', Y', Z', W') \in U \cong \mathbb{C}^4$, then we can use Cauchy's existence and uniqueness theorem of solutions. In order to consider a family of phase spaces for the system (1), let us take the compactification $\mathbb{P}^4 \times B$ of $\mathbb{C}^4 \times B$. Moreover, we denote the boundary divisor in $\mathbb{P}^4$ by $\mathcal{H}$. Fixing the parameter α_i, consider the product $\mathbb{P}^4 \times B$ and extend the regular vector field on $\mathbb{C}^4 \times B$ to a rational vector field $\tilde{v}$ on $\mathbb{P}^4 \times B$. The following lemma shows that this rational vector field $\tilde{v}$ has the following accessible singular loci on the boundary divisor $\mathcal{H} \times \{t\} \subset \mathbb{P}^4 \times \{t\}$ for each $t \in B$.

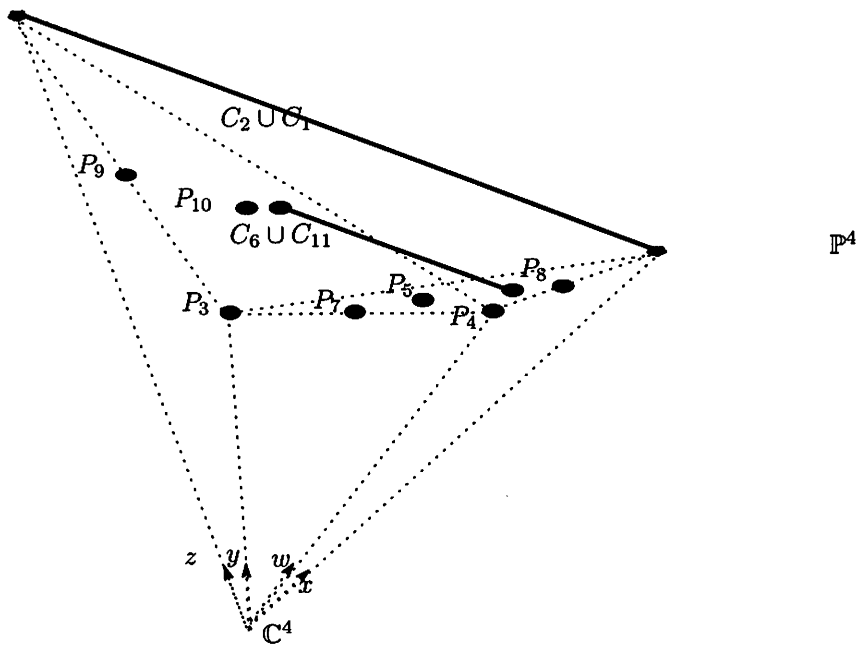

Figure 1: Accessible singularities of the system (1)

Lemma 3.1 *The rational vector field $\tilde{v}$ has the following accessible singular loci* (see Figure 1):

$$\begin{cases} C_1 = \{(X_1,Y_1,Z_1,W_1)|X_1 = Y_1 = W_1 = 0\}, \\ C_2 = \{(X_2,Y_2,Z_2,W_2)|Y_2 = Z_2 = W_2 = 0\}, \\ P_3 = \{(X_3,Y_3,Z_3,W_3)|X_3 = Y_3 = Z_3 = W_3 = 0\}, \\ P_4 = \{(X_4,Y_4,Z_4,W_4)|X_4 = Y_4 = Z_4 = W_4 = 0\}, \\ P_5 = \{(X_3,Y_3,Z_3,W_3)|Y_3 = Z_3 = 0, X_3 = 1, W_3 = -1\}, \\ C_6 = \{(X_3,Y_3,Z_3,W_3)|Y_3 = 0, W_3 = 1, X_3 = 1 - Z_3\}, \\ P_7 = \{(X_3,Y_3,Z_3,W_3)|X_3 = Y_3 = Z_3 = 0, W_3 = 1\}, \\ P_8 = \{(X_1,Y_1,Z_1,W_1)|X_1 = Y_1 = Z_1 = 0, W_1 = 2\}, \\ P_9 = \{(X_2,Y_2,Z_2,W_2)|X_2 = Z_2 = W_2 = 0, Y_2 = 2\}, \\ P_{10} = \{(X_4,Y_4,Z_4,W_4)|X_4 = W_4 = 0, Y_4 = -1, Z_4 = 1\}, \\ C_{11} = \{(X_4,Y_4,Z_4,W_4)|W_4 = 0, Y_4 = 1, Z_4 = 1 - X_4\}. \end{cases}$$

Lemma 3.1 can be checked by a direct calculation.

Remark 3.1 By the symmetry

$$\pi_1 : (x,y,z,w,t;\alpha_1,\alpha_2,\alpha_3) \longrightarrow (z,w,x,y,t;\alpha_3,\alpha_2,\alpha_1),$$

it is easy to see that

$$\pi_1(C_1) = C_2,\ \pi_1(P_3) = P_4,\ \pi_1(P_5) = P_{10},\ \pi_1(P_8) = P_9,\ \pi_1(C_6) = C_{11}.$$

By Remark 3.1, it is easy to see that for the accessible singularities $C_2, P_4, P_9, P_{10}, C_{11}$, each resolution process is the same as the case for the accessible singularities C_1, P_3, P_8, P_5,
C_6 respectively, provided the variables and parameters $x, y, z, w, t, \alpha_1, \alpha_2, \alpha_3$ are replaced by the transformation π_1.

For the case of accessible singular points P_i $(i = 1,3,5,6,7,8)$, we calculate the local index at each point.

Notations

$$P_1 = \{(X_1,Y_1,Z_1,W_1) = (0,0,0,0)\} \in C_1,\ P_6 = \{(X_3,Y_3,Z_3,W_3) = (1,0,0,1)\} \in C_6.$$

Singular point	Type(dim. of sol.)	Type of local index
P_1	dim. 2	$(-1,-3,0,-2)$
P_3	dim. 3	$(-3,-1,-1,-1)$
P_5	dim. 2	$(+3,+1,+4,-2)$
P_6	dim. 2	$(+3,+1,0,+2)$
P_7	dim. 2	$(-3,-1,-3,+1)$
P_8	dim. 2	$(-1,-3,-4,+2)$

In the case of the accessible singular points P_5 and P_6, we give an explicit description of the local index at each point.

By using the coordinate system (u_5, v_5, q_5, p_5) around the point $P_5 = \{(u_5, v_5, q_5, p_5) = (0,0,0,0)\}$, the system (1) is rewritten as follows:

$$\frac{d}{dt}\begin{pmatrix} u_5 \\ v_5 \\ q_5 \\ p_5 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ -2 & 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} u_5 \\ v_5 \\ q_5 \\ p_5 \end{pmatrix} + \cdots .$$

To the above system, we make the linear transformation

$$\begin{pmatrix} U_5 \\ V_5 \\ Q_5 \\ P_5 \end{pmatrix} = \begin{pmatrix} -1/5 & 0 & 1/5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2/5 & 0 & -2/5 & 1 \end{pmatrix} \begin{pmatrix} u_5 \\ v_5 \\ q_5 \\ p_5 \end{pmatrix}$$

to arrive at

$$\frac{d}{dt}\begin{pmatrix} U_5 \\ V_5 \\ Q_5 \\ P_5 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} U_5 \\ V_5 \\ Q_5 \\ P_5 \end{pmatrix} + \cdots ,$$

and we obtain the local index $(+3, +1, +4, -2)$ at P_5.

By using the coordinate system (u_6, v_6, q_6, p_6) around the point $P_6 = \{(u_6, v_6, q_6, p_6) = (0,0,0,0)\}$, the system (1) is rewritten as follows:

$$\frac{d}{dt}\begin{pmatrix} u_6 \\ v_6 \\ q_6 \\ p_6 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} u_6 \\ v_6 \\ q_6 \\ p_6 \end{pmatrix} + \cdots .$$

To the above system, we make the linear transformation

$$\begin{pmatrix} U_6 \\ V_6 \\ Q_6 \\ P_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_6 \\ v_6 \\ q_6 \\ p_6 \end{pmatrix}$$

to arrive at

$$\frac{d}{dt}\begin{pmatrix} U_6 \\ V_6 \\ Q_6 \\ P_6 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} U_6 \\ V_6 \\ Q_6 \\ P_6 \end{pmatrix} + \cdots ,$$

and we obtain the local index $(+3, +1, 0, +2)$ at P_6.

2.2. Summary of resolution process

Let us start by summarizing the steps (see Figure 2,3,4) which are needed to resolve the accessible singularities of $\tilde{v}$:

1. **Step 1**: We blow up at two points P_3 and P_4.
2. **Step 2**: We blow up along the curve L_1.
3. **Step 3**: We blow down the 3-fold V_1.
4. **Step 4**: We blow up along the curve $L_2 = C_1 \cup C_2$.
5. **Step 5**: We blow up along the surfaces F_1 and F_2.
6. **Step 6**: We blow down the 3-folds V_2 and V_3.
7. **Step 7**: We blow up along the curve $L_3 = C_1{}' \cup C_2{}'$.
8. **Step 8**: We blow up along the surfaces F_3 and F_4.
9. **Step 9**: We blow down the 3-folds V_4 and V_5.
10. **Step 10**: We blow up along the curve $L_4 = C_6 \cup C_{11}$.
11. **Step 11**: We blow up along the surfaces F_5 and F_6.
12. **Step 12**: We blow down the 3-folds V_6 and V_7.
13. **Step 13**: We blow up along the curve $L_5 = C_6{}' \cup C_{11}{}'$.
14. **Step 14**: We blow up along the surfaces F_7 and F_8.
15. **Step 15**: We blow down the 3-folds V_8 and V_9.
16. **Step 16**: We blow up along the surfaces F_9 and F_{10}.
17. **Step 17**: We blow up along the surfaces F_{11} and F_{12}.
18. **Step 18**: We blow up along the surface F_{13}.
19. **Step 19**: We blow up along the surface F_{14}.

Now we are ready to prove Theorem 1.3.

2.3. Resolution of the accessible singular locus C_1

By the following 3 steps, we can resolve the accessible singular locus C_1.

Step 0: We take the coordinate system (X_1, Y_1, Z_1, W_1) centered at P_1, where $(X_1, Y_1, Z_1, W_1) = (\frac{1}{x}, \frac{y}{x}, \frac{z}{x}, \frac{w}{x})$, and (x, y, z, w) is the original coordinate system of $\mathbb{C}^4$.

Step 2: We blow up along the curve $C_1 = \{(X_1, Y_1, Z_1, W_1) | X_1 = Y_1 = W_1 = 0\}$

$$X_1{}' = X_1 \ , \quad Y_1{}' = \frac{Y_1}{X_1} \ , \quad Z_1{}' = Z_1 \ , \quad W_1{}' = \frac{W_1}{X_1}.$$

Step 7: We blow up along the curve $C_1{}' = \{(X_1', Y_1', Z_1', W_1') | X_1{}' = Y_1{}' = W_1{}' = 0\}$

$$X_1{}'' = X_1' \ , \quad Y_1{}'' = \frac{Y_1'}{X_1'} \ , \quad Z_1{}'' = Z_1' \ , \quad W_1{}'' = \frac{W_1'}{X_1'}.$$

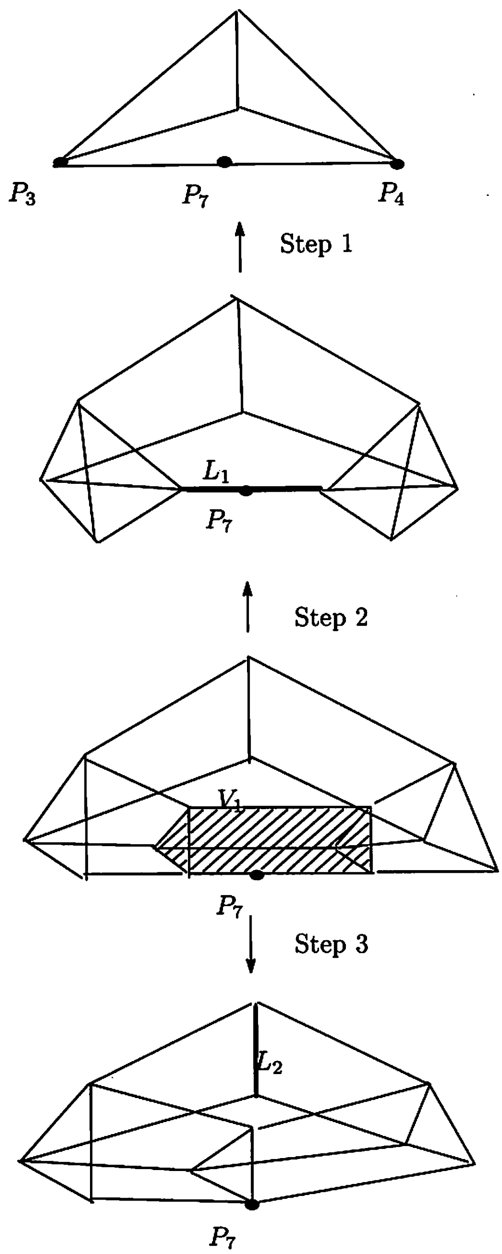

Figure 2:

Step 19: We blow up along the surface $F_{14} = \{(X_1'', Y_1'', Z_1'', W_1'') | X_1'' = Y_1'' + Z_1''W_1'' + \alpha_2 = 0\}$

$$X_1''' = X_1'', \quad Y_1''' = \frac{Y_1'' + Z_1''W_1'' + \alpha_2}{X_1''}, \quad Z_1''' = Z_1'', \quad W_1''' = W_1''.$$

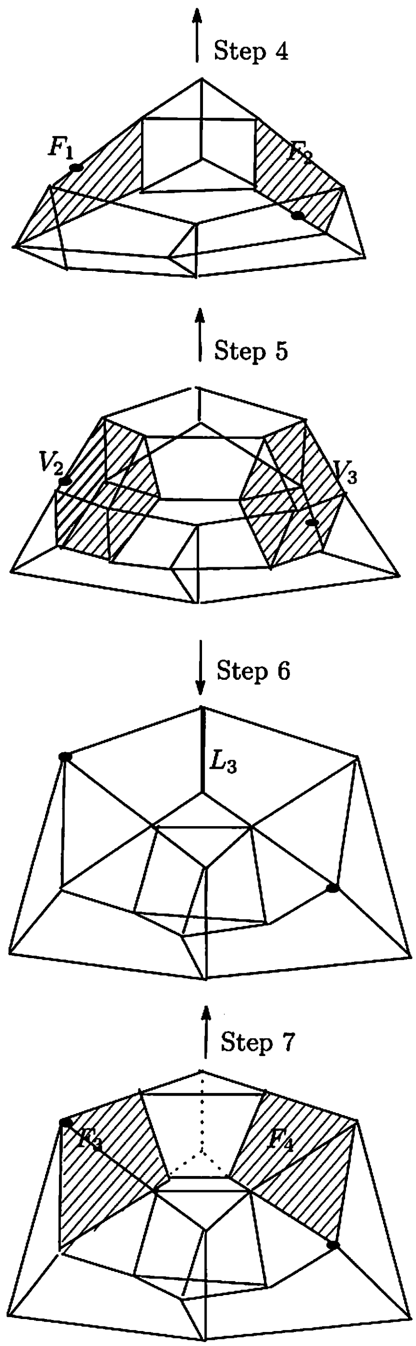

Figure 3:

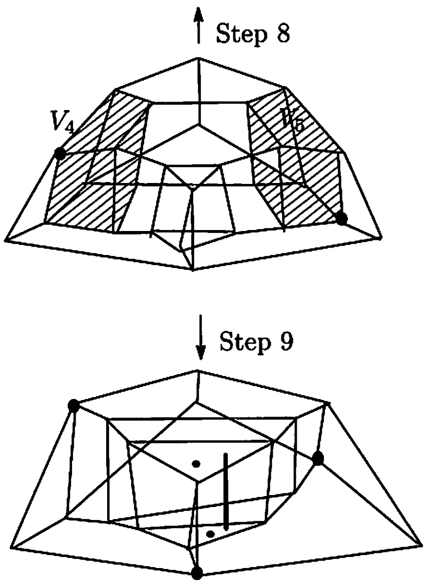

Figure 4:

We have resolved the accessible singular locus C_1. By choosing a new coordinate system as

$$(x_1, y_1, z_1, w_1) = (X_1''', -Y_1''', Z_1''', W_1'''),$$

we can obtain the coordinate system (x_1, y_1, z_1, w_1) in the description of $\mathcal{X}$ given in Theorem 1.3.

2.4. Resolution of the accessible singularity P_3

By the following 3 steps, we can resolve the accessible singularity P_3.

Step 0: We take the coordinate system (X_3, Y_3, Z_3, W_3) centered at P_3, where $(X_3, Y_3, Z_3, W_3) = (\frac{x}{y}, \frac{1}{y}, \frac{z}{y}, \frac{w}{y})$.

Step 1: We blow up at the point P_3

$$X_3' = \frac{X_3}{Y_3}, \quad Y_3' = Y_3, \quad Z_3' = \frac{Z_3}{Y_3}, \quad W_3' = \frac{W_3}{Y_3}.$$

Step 16: We blow up along the surface $F_9 = \{(X_3', Y_3', Z_3', W_3') | X_3{}' = Y_3' = 0\}$

$$X_3'' = \frac{X_3'}{Y_3'}, \quad Y_3'' = Y_3', \quad Z_3'' = Z_3', \quad W_3'' = W_3'.$$

Step 17: We blow up along the surface $F_{11} = \{(X_3'', Y_3'', Z_3'', W_3{}'') | X_3'' - \alpha_1 = Y_3'' = 0\}$

$$X_3{}''' = \frac{X_3'' - \alpha_1}{Y_3''}, \quad Y_3{}''' = Y_3'', \quad Z_3{}''' = Z_3'', \quad W_3{}''' = W_3''.$$

We have resolved the accessible singularity P_3. By choosing a new coordinate system as

$$(x_3, y_3, z_3, w_3) = (-X_3''', Y_3''', Z_3''', W_3'''),$$

we can obtain the coordinate system (x_3, y_3, z_3, w_3) in the description of $\mathcal{X}$ given in Theorem 1.3.

2.5. Resolution of the accessible singularity P_7

By the following steps, we can resolve the accessible singularity P_7.

Step 0: We take the coordinate system centered at P_7

$$P_1 = \frac{x}{y}, \quad Q_1 = \frac{1}{y}, \quad R_1 = \frac{z}{y}, \quad S_1 = \frac{w}{y} - 1.$$

Step 2: We blow up along the curve $L_1 \cong \mathbb{P}^1$ (see Figure 2)

$$P_2 = \frac{P_1}{Q_1}, \quad Q_2 = Q_1, \quad R_2 = \frac{R_1}{Q_1}, \quad S_2 = S_1.$$

By Step 2, each point on the curve L_1 is transformed to $\mathbb{P}^2$.

Step 3: We blow down the 3-fold $V_1 \cong \mathbb{P}^2 \times \mathbb{P}^1$ (see Figure 2)

$$P_3 = P_2, \quad Q_3 = Q_2, \quad R_3 = R_2, \quad S_3 = \frac{Q_2}{S_2 + 1}.$$

The resolution process from Step 2 to Step 3 is well-known as $\mathbb{P}^2$-flop. In order to resolve the accessible singularity P_7 and obtain a holomorphic coordinate system, we need to blow down the 3-fold $V_1 \cong \mathbb{P}^2 \times \mathbb{P}^1$ along the $\mathbb{P}^1$-fiber. After we blow down the 3-fold V_1, we can resolve the accessible singularity P_7 by only blowing-ups.

Step 16: We blow up along the surfaces F_9 and F_{10}

$$P_4 = \frac{P_3}{Q_3}, \quad Q_4 = Q_3, \quad R_4 = \frac{R_3}{S_3}, \quad S_4 = S_3.$$

By Step 16, each point on each surface F_i $(i = 9, 10)$ is transformed to $\mathbb{P}^1$.

Step 17: We blow up along the surfaces F_{11} and F_{12}

$$P_5 = \frac{P_4 - \alpha_1}{Q_4}, \quad Q_5 = Q_4, \quad R_5 = \frac{R_4 - \alpha_3}{S_4}, \quad S_5 = S_4.$$

We have resolved the accessible singularity P_7. By choosing a new coordinate system as

$$(x_7, y_7, z_7, w_7) = (-P_5, Q_5, -R_5, S_5),$$

we can obtain the coordinate system (x_7, y_7, z_7, w_7) in the description of $\mathcal{X}$ given in Theorem 1.3.

2.6. Resolution of the accessible singularity P_8

By the following steps, we can resolve the accessible singularity P_8.

Step 0: We take the coordinate system centered at P_8

$$p_1 = \frac{1}{x}\,, \quad q_1 = \frac{y}{x}\,, \quad r_1 = \frac{z}{x}\,, \quad s_1 = \frac{w}{x} - 2.$$

Step 5: We blow up along the surface $F_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ (see Figure 3)

$$p_2 = p_1\,, \quad q_2 = \frac{q_1}{p_1}\,, \quad r_2 = r_1\,, \quad s_2 = s_1.$$

Step 6: We blow down the 3-fold $V_3 \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see Figure 3)

$$p_3 = p_2\,, \quad q_3 = q_2\,, \quad r_3 = r_2\,, \quad s_3 = \frac{p_2}{s_2 + 2}.$$

The resolution process from Step 5 to Step 6 is well-known as $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-flop. The surface F_2 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. By Step 5, each point on F_2 is transformed to $\mathbb{P}^1$. The 3-fold V_3 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In order to resolve the accessible singularities P_8, P_9 and obtain holomorphic coordinate systems, we need to blow down the 3-fold V_3 along another $\mathbb{P}^1$-fiber. The surface F_4 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Step 8: We blow up along the surface $F_4 \cong \mathbb{P}^1 \times \mathbb{P}^1$ (see Figure 4)

$$p_4 = p_3\,, \quad q_4 = \frac{q_3}{p_3}\,, \quad r_4 = r_3\,, \quad s_4 = s_3.$$

Step 9: We blow down the 3-fold $V_5 \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see Figure 4)

$$p_5 = p_4\,, \quad q_5 = q_4\,, \quad r_5 = r_4\,, \quad s_5 = p_4 s_4.$$

The 3-fold V_5 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In order to resolve the accessible singularity P_8 and obtain a holomorphic coordinate system, we need to blow down the 3-folds $V_i (i = 2, 3)$ along another $\mathbb{P}^1$-fiber.

Step 16: We blow up along the surface F_{10}

$$p_6 = p_5\,, \quad q_6 = q_5\,, \quad r_6 = \frac{r_5}{s_5}\,, \quad s_6 = s_5.$$

Step 17: We blow up along the surface F_{12}

$$p_7 = p_6\,, \quad q_7 = q_6\,, \quad r_7 = \frac{r_6 - \alpha_3}{s_6}\,, \quad s_7 = s_6.$$

Step 19: We blow up along the surface F_{14}

$$p_8 = p_7\,, \quad q_8 = \frac{p_7 + r_7 s_7 + \alpha_2 + \alpha_3}{p_7}\,, \quad r_8 = r_7\,, \quad s_8 = s_7.$$

We have resolved the accessible singularity P_8. By choosing a new coordinate system as

$$(x_8, y_8, z_8, w_8) = (p_8, -q_8, r_8, s_8),$$

we can obtain the coordinate system (x_8, y_8, z_8, w_8) in the description of $\mathcal{X}$ given in Theorem 1.3.

2.7. Resolution of the accessible singular locus C_6

By the following steps, we can resolve the accessible singular locus C_6.

Step 0: We take the coordinate system centered at $\{(X_2, Y_2, Z_2, W_2)|X_2 = W_2 = 1, Y_2 = Z_2 = 0\}$

$$f_1 = \frac{x}{y} - 1 , \quad g_1 = \frac{1}{y} , \quad h_1 = \frac{z}{y} , \quad j_1 = \frac{w}{y} - 1.$$

Step 10: We blow up along the curve $L_4 = C_6 \cup C_{11}$

$$f_2 = \frac{f_1 + h_1}{g_1} , \quad g_2 = g_1 , \quad h_2 = h_1 , \quad j_2 = \frac{j_1}{g_1}.$$

Step 13: We blow up along the curve $L_5 = C_6' \cup C_{11}'$

$$f_3 = \frac{f_2 - t}{g_2} , \quad g_3 = g_2 , \quad h_3 = h_2 , \quad j_3 = \frac{j_2}{g_2}.$$

Step18: We blow up along the surface F_{13}

$$f_4 = \frac{f_3 - h_3 j_3 - 1 + \alpha_1 + \alpha_2 + \alpha_3}{g_3} , \quad g_4 = g_3 , \quad h_4 = h_3 , \quad j_4 = j_3.$$

We have resolved the accessible singular locus C_6. By choosing a new coordinate system as

$$(x_6, y_6, z_6, w_6) = (-f_4, g_4, h_4, j_4),$$

we can obtain the coordinate system (x_6, y_6, z_6, w_6) in the description of $\mathcal{X}$ given in Theorem 1.3.

2.8. Resolution of the accessible singularity P_5

By the following steps, we can resolve the accessible singularity P_5.

Step 0: We take the coordinate system centered at P_5

$$F_1 = \frac{x}{y} - 1 , \quad G_1 = \frac{1}{y} , \quad H_1 = \frac{z}{y} , \quad J_1 = \frac{w}{y} + 1.$$

Step 11: We blow up along the surface $F_5 \cong \mathbb{P}^1 \times \mathbb{P}^1$

$$F_2 = \frac{F_1 + H_1}{G_1} , \quad G_2 = G_1 , \quad H_2 = H_1 , \quad J_2 = J_1.$$

Step 12: We blow down the 3-fold $V_6 \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

$$F_3 = F_2 , \quad G_3 = G_2 , \quad H_3 = H_2 , \quad J_3 = \frac{G_2}{J_2 - 2}.$$

The resolution process from Step 11 to Step 12 is well-known as $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$-flop. The surface F_5 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. By Step 11, each point on F_5 is transformed to $\mathbb{P}^1$. The 3-fold V_6 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In order to resolve the accessible singularity P_5 and obtain a holomorphic coordinate system, we need to blow down the 3-fold V_6 along another $\mathbb{P}^1$-fiber. The surface F_7 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Step 14: We blow up along the surface $F_7 \cong \mathbb{P}^1 \times \mathbb{P}^1$

$$F_4 = \frac{F_3 - t}{G_3}, \quad G_4 = G_3, \quad H_4 = H_3, \quad J_4 = J_3.$$

Step 15: We blow down the 3-fold $V_8 \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

$$F_5 = F_4, \quad G_5 = G_4, \quad H_5 = H_4, \quad J_5 = G_4 J_4.$$

Step 16: We blow up along the surface F_{10}

$$F_6 = F_5, \quad G_6 = G_5, \quad H_6 = \frac{H_5}{J_5}, \quad J_6 = J_5.$$

Step 17: We blow up along the surface F_{12}

$$F_7 = F_6, \quad G_7 = G_6, \quad H_7 = \frac{H_6 - \alpha_3}{J_6}, \quad J_7 = J_6.$$

Step 18: We blow up along the surface F_{13}

$$F_8 = \frac{F_7 - H_7 J_7 - 1 + \alpha_1 + \alpha_2}{G_7}, \quad G_8 = G_7, \quad H_8 = H_7, \quad J_8 = J_7.$$

We have resolved the accessible singularity P_5. By choosing a new coordinate system as

$$(x_5, y_5, z_5, w_5) = (-F_8, G_8, H_8, J_8),$$

we can obtain the coordinate system (x_5, y_5, z_5, w_5) in the description of $\mathcal{X}$ given in Theorem 1.3.

2.9. Resolution of the remaining accessible singularities

Each procedure is the same as that given in the preceding sections 2.3 through 2.8, provided the variables and parameters $x, y, z, w, \alpha_1, \alpha_2, \alpha_3$ are replaced by the transformation

$$\pi_1 : (x, y, z, w; \alpha_1, \alpha_2, \alpha_3) \longmapsto (z, w, x, y; \alpha_3, \alpha_2, \alpha_1).$$

Each coordinate system (x_j, y_j, z_j, w_j) for $j = 2, 4, 9, 10, 11$ is explicitly given as follows:

$$(x_j, y_j, z_j, w_j) = \pi_1(x_k, y_k, z_k, w_k), \quad k = 1, 3, 8, 5, 6, \textit{respectively}.$$

In sections 2.3 through 2.9, we have resolved all the accessible singularities for the system (1), thus completing the proof of Theorem 1.3.

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