Kumamoto J. Math.
Vol.20 (2007), 33–50

A quantitative characterization of the linear group $L_{p+1}(2)$ where p is a prime number

To the memory of Walter Feit

Mohammad R. Darafsheh and A. Mahmiani

(Received July 21, 2006)
(Accepted August 25, 2006)

Abstract. Let G be a finite group. Regarding the Gruenberg-Kegel graph or the prime graph of G, denoted by $GK(G)$, the primes dividing the order of G are divided into the sets $\pi_1, \pi_2, ..., \pi_{s(G)}$, where $s(G)$ is the number of components of $GK(G)$. Therefore the order of G is divided into a product of co-prime positive integers $m_1, m_2, ..., m_{s(G)}$, where m_i is the product of primes in π_i. These integers are called order components of G and $OC(G) = \{m_1, m_2, ..., m_{s(G)}\}$ is called the set of order components of G. In this paper we will prove that the linear group $L_{p+1}(2)$, where p is a prime number, is characterized by its set of order components. More precisely we will prove that if G is a finite group, then $OC(G) = OC(L_{p+1}(2))$ if and only if $G \cong L_{p+1}(2)$.

1 Introduction

For a positive integer n, let $\pi(n)$ be the set of all prime divisors of n. If G is a finite group, we set $\pi(G) = \pi(|G|)$. The Gruenberg-Kegel graph of G, or the prime graph of G, is denoted by $GK(G)$ and is defined as follows. The vertex set of $GK(G)$ is the set $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G contains an element of order pq. We denote the connected components of $GK(G)$ by $\pi_1, \pi_2, ..., \pi_{s(G)}$, where $s(G)$ denotes the number of

Mathematical Subject Classification (2000): 20D06, 20D60.
Key words: Prime graph, order component, linear group.

connected components of $GK(G)$. If the order of G is even, the notation is chosen so that $2 \in \pi_1$. It is clear that the order of G can be expressed as the product of the numbers $m_1, m_2, ..., m_{s(G)}$, where $\pi(m_i) = \pi_i$, $1 \leq i \leq s(G)$. If the order of G is even and $s(G) \geq 2$, according to our notation $m_2, ..., m_s(G)$ are odd. The positive integers $m_1, m_2, ..., m_{s(G)}$ are called the order components of G and $OC(G) = \{m_1, m_2, ..., m_{s(G)}\}$ is called the set of order components of G. It is a natural question to ask: If the finite groups G and H have the same order components does it follow G is isomorphic to H? For many simple groups H with the number of order components $s(H)$ at least 2, the answer to the above question is affirmative. However if $s(H) = 1$ the answer is negative. However the simple groups $B_n(q)$ and $C_n(q)$ where $n = 2^m \geq 4$ and q is odd, have the same order components but they are not isomorphic. Hence it is natural to adopt the following definition.

Definition 1 *Let G be a finite group. The number of non-isomorphic finite groups with the same order components as G is denoted by $h(G)$ and is called the h-function of G. For any natural number k we say the finite group G is k-recognizable by its set of order components if $h(G) = k$. If $h(G) = 1$ we say that G is characterizable by its set of order components or briefly G is a characterizable group. In this case G is uniquely determined by the set of its order components.*

Obviously for any finite groups G we have $h(G) \geq 1$. The components of the Gruenberg-Kegel graph $GK(P)$ of any non -abelian finite simple group P with $GK(P)$ disconnected are found in [17] from which we can deduce the component orders of P. These information which will be used in proving our main result are listed in Tables 1, 2 and 3.

Table 1. The order components of finite simple groups P with $s(P) = 2$

P	Restrictions on P	m_1	m_2
$\mathbb{A}_n$	$6 < n = p, p+1, p+2$; one of $n, n-2$ is not a prime	$\frac{n!}{2p}$	p
$A_{p-1}(q)$	$(p,q) \neq (3,2),(3,4)$	$q^{\binom{p}{2}} \prod_{i=1}^{p-1}(q^i - 1)$	$\frac{(q^p-1)}{(q-1)(p,q-1)}$
$A_p(q)$	$(q-1) \mid (p+1)$	$q^{\binom{p+1}{2}}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-1)$	$\frac{(q^p-1)}{(q-1)}$
$^2A_{p-1}(q)$		$q^{\binom{p}{2}} \prod_{i=1}^{p-1}(q^i - (-1)^i)$	$\frac{(q^p+1)}{(q+1)(p,q+1)}$
$^2A_p(q)$	$(q+1) \mid (p+1)$, $(p,q) \neq (3,3),(5,2)$	$q^{\binom{p+1}{2}}(q^{p+1}-1)$ $\prod_{i=1}^{p-1}(q^i - (-1)^i)$	$\frac{(q^p+1)}{(q+1)}$
$^2A_3(2)$		$2^6.3^4$	5
$B_n(q)$	$n = 2^m \geq 4$, q odd	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i-1})$	$\frac{(q^n+1)}{2}$
$B_p(3)$		$3^{p^2}(3^p+1)\prod_{i=1}^{p-1}(3^{2i-1})$	$\frac{(3^p-1)}{2}$
$C_n(q)$	$n = 2^m \geq 2$	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i-1})$	$\frac{(q^n+1)}{(2,q-1)}$
$C_p(q)$	$q = 2, 3$	$q^{p^2}(q^p+1)\prod_{i=1}^{p-1}(q^{2i-1})$	$\frac{(q^p-1)}{(2,q-1)}$
$D_p(q)$	$p \geq 5$, $q = 2,3,5$	$q^{p(p-1)}\prod_{i=1}^{p-1}(q^{2i-1})$ $(q^5-1)(q^3-1)(q^2-1)$	$\frac{(q^p-1)}{(q-1)}$
$D_{p+1}(q)$	$q = 2, 3$	$\frac{1}{(2,q-1)}q^{p(p+1)}(q^p+1)$ $(q^{p+1}-1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{(q^p-1)}{(2,q-1)}$
$^2D_n(q)$	$n = 2^m \geq 4$	$q^{n(n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{(q^n+1)}{(2,q+1)}$
$^2D_n(2)$	$n = 2^m + 1 \geq 5$	$2^{n(n-1)}(2^n+1)(2^{n-1}-1)$ $\prod_{i=1}^{n-2}(2^{2i}-1)$	$2^{n-1}+1$

Table 1. (Continued)

${}^2D_p(3)$	$5 \leq p \neq 2^m + 1$	$3^{p(p-1)} \prod_{i=1}^{p-1} (3^{2i} - 1)$	$\frac{(3^p+1)}{4}$
${}^2D_n(3)$	$9 \leq n = 2^m + 1 \neq p$	$\frac{1}{2} 3^{n(n-1)} (3^n + 1)(3^{n-1} - 1) \prod_{i=1}^{n-2} (3^{2i} - 1)$	$\frac{(3^{n-1}+1)}{2}$
$G_2(q)$	$2 < q \equiv \epsilon (\mathrm{mod}\, 3),\ \epsilon = \pm 1$	$q^6(q^3 - \epsilon)(q^2 - 1)(q + \epsilon)$	$q^2 - \epsilon q + 1$
${}^3D_4(q)$		$q^{12}(q^6 - 1)(q^2 - 1)(q^4 + q^2 + 1)$	$q^4 - q^2 + 1$
$F_4(q)$	q odd	$q^{24}(q^8 - 1)(q^6 - 1)^2 (q^4 - 1)$	$q^4 - q^2 + 1$
${}^2F_4(2)'$		$2^{11}.3^3.5^2$	13
$E_6(q)$		$q^{36}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^3 - 1)(q^2 - 1)$	$\frac{(q^6+q^3+1)}{(3,q-1)}$
${}^2E_6(q)$	$q > 2$	$q^{36}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^5 + 1)(q^3 + 1)(q^2 - 1)$	$\frac{(q^6-q^3+1)}{(3,q+1)}$
M_{12}		$2^6.3^3.5$	11
J_2		$2^7.3^3.5^2$	7
Ru		$2^{14}.3^3.5^3.7.13$	29
He		$2^{10}.3^3.5^2.7^3$	17
McL		$2^7.3^6.5^3.7$	11
Co_1		$2^{21}.3^9.5^4.7^2.11.13$	23
Co_3		$2^{10}.3^7.5^3.7.11$	23
Fi_{22}		$2^{17}.3^9.5^2.7.11$	13
HN		$2^{14}.3^6.5^6.7.11$	19

Table 2. The order components of finite simple groups P with $s(P)=3$ (p an odd prime)

P	Restrictions on P	m_1	m_2	m_3
A_n	$n>6,\ n=p,\ p-2$ are primes	$\frac{n!}{2n(n-2)}$	p	$p-2$
$A_1(q)$	$3<q\equiv\epsilon(\bmod 4)$, $\epsilon=\pm1$	$q-\epsilon$	q	$\frac{(q+\epsilon)}{2}$
$A_1(q)$	$q>2$, q even	q	$q-1$	$q+1$
$A_2(2)$		8	3	7
${}^2A_5(2)$		$2^{15}.3^6.5$	7	11
${}^2D_p(3)$	$p=2^m+1\geq5$	$2.3^{p(p-1)}(3^{p-1}-1)\prod_{i=1}^{p-2}(3^{2i}-1)$	$\frac{(3^{p-1}+1)}{2}$	$\frac{(3^p+1)}{4}$
${}^2D_{p+1}(2)$	$p=2^n-1,\ n\geq2$	$2^{p(p+1)}(2^p-1)\prod_{i=1}^{p-1}(2^{2i}-1)$	2^p+1	$2^{p+1}+1$
$G_2(q)$	$q\equiv0(\bmod 3)$	$q^6(q^2-1)^3$	q^2-q+1	q^2+q+1
${}^2G_2(q)$	$q=3^{2m+1}>3$	$q^3(q^2-1)$	$q-\sqrt{3q}+1$	$q+\sqrt{3q}+1$
$F_4(q)$	q even	$q^{24}(q^6-1)^2(q^4-1)^2$	q^4+1	q^4-q^2+1
${}^2F_4(q)$	$q=2^{2m+1}>2$	$q^{12}(q^4-1)(q^3+1)$	$q^2-\sqrt{2q^3}+q-\sqrt{2q}+1$	$q^2+\sqrt{2q^3}+q+\sqrt{2q}+1$
$E_7(2)$		$2^{63}.3^{11}.5^2.7^3.11.13.17.19.31.43$	73	127
$E_7(3)$		$2^{23}.3^{63}.5^2.7^3.11^2.13^2.19.37.41.61.73.547$	757	1093
M_{11}		$2^4.3^2$	5	11
M_{23}		$2^7.3^2.5.7$	11	23
M_{24}		$2^{10}.3^3.5.7$	11	23
J_3		$2^7.3^5.5$	17	19
HiS		$2^9.3^2.5^3$	7	11

Table 2. (Continued)

P	Restrictions on P	m_1	m_2	m_3
Suz		$2^{13}.3^7.5^2.7$	11	13
Co_2		$2^{18}.3^6.5^3.7$	11	23
Fi_{23}		$2^{18}.3^{13}.5^2.7.11.13$	17	23
F_3		$2^{15}.3^{10}.5^3.7^2.13$	19	31
F_2		$2^{41}.3^{13}.5^6.7^2.11.13.$ 17.19.23	31	47

Table 3. The order components of finite simple groups P with $s(P) > 3$

P	Restrictions on P	m_1	m_2	m_3	m_4	m_5	m_6
$A_2(4)$		2^6	3	5	7	-	-
${}^2B_2(q)$	$q = 2^{2m+1} > 2$	q^2	$q-1$	$q-\sqrt{2q}$ $+1$	$q+\sqrt{2q}$ $+1$	-	-
${}^2E_6(2)$		$2^{36}.3^9.5^2.7^2.11$	13	17	19	-	-
$E_8(q)$	$q \equiv 2,3$ (mod 5)	$q^{120}(q^{20}-1)$ $(q^{18}-1)$ $(q^{14}-1)$ $(q^{12}-1)$ $(q^{10}-1)$ (q^8-1) (q^4+1) (q^4+q^2+1)	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4 $+1$	-	-
M_{22}		$2^7.3^2$	5	7	11	-	-
J_1		$2^3.3.5$	7	11	19	-	-
$O'N$		$2^9.3^4.5.7^3$	11	19	31	-	-
LyS		$2^8.3^7.5^6.7.11$	31	37	67	-	-

Table 3. (Continued)

Fi'_{24}		$2^{21}.3^{16}.5^2.7^3.11.13$	17	23	29	-	-
F_1		$2^{46}.3^{20}.5^9.7^6.11^2.13^3.$ $17.19.23.29.31.47$	41	59	71	-	-
$E_8(q)$	$q \equiv 0,1,4$ (mod 5)	$q^{120}(q^{18}-1)(q^{14}-1)$ $(q^{12}-1)^2(q^{10}-1)^2$ $(q^8-1)^2(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4 $+1$	$\frac{q^{10}+1}{q^2+1}$	-
J_4		$2^{21}.3^3.5.7.11^3$	23	29	31	37	43

In [15] and [16] it is proved that if $n = 2^m \geq 4$, then $h(B_n(q)) = h(C_n(q)) = 2$ for q odd and $h(B_n(q)) = h(C_n(q)) = 1$ for q even. Apart from the families $B_n(q)$ and $C_n(q)$, $n = 2^m \geq 4$, q odd. The following groups have been proved to be characterizable by their order components by various authors: All the sporadic simple groups [2], $PSL_2(q)$, ${}^2D_n(3)$ where $9 \leq n = 2^m+1$ is not a prime, ${}^2D_{p+1}(2)$ in [3], [4] and [18], respectively. Some projective special linear (unitary) groups have been characterized in a series of articles in [10], [11], [12] and [13]. A few of the alternating or symmetric groups are proved to be characterizable by their order components in [1] and [14]. Based on these results we put forward the following conjecture.

Conjecture 1 *Let P be a non-abelian finite simple group with $s(P) \geq 2$. If G is a finite group and $OC(G) = OC(P)$, then either $G \cong P$ or $G \cong B_n(q)$ or $C_n(q)$ where $n = 2^m \geq 4$ and q is an odd number or $G \cong B_p(3)$ or $C_p(3)$ where p is an odd prime number.*

A motivation for characterizing finite groups by the set of their order components is the following conjecture due to J. G. Thompson.

Conjecture 2 *(Thompson) For a finite group G let $N(G) = \{n \in \mathbb{N} \mid G$ has a conjugacy class of size $n\}$. Let $Z(G) = 1$ and M be a non-abelian finite simple group satisfying $N(G) = N(M)$. Is it true that $G \cong M$?*

In [5] it is proved that if $s(M) \geq 3$, then the above conjecture holds. Also in [5] it is proved that if G and M are finite groups with $s(M) \geq 2$, $Z(G) = 1$, $N(G) = N(M)$, then $|G| = |M|$, in particular $s(M) = s(G)$ and $OC(G) = OC(M)$. Therefore if the simple group M is characterizable by the set of its order components, then the Thompson's conjecture holds for M.

There is another conjecture due to W. Shi and J. Bi which states:

Conjecture 3 *Let G be a group and M a finite simple group. Then $G \cong M$ if and only if*

(a) $|G| = |M|$ and

(b) $\pi_e(G) = \pi_e(M)$ where $\pi_e(G)$ denotes the set of orders of elements of G.

Clearly conditions (a) and (b) above imply $OC(G) = OC(M)$. Therefore if the group G is characterizable by its order components, then we will deduce $G \cong M$ and conjecture 2 is true for M. According to the main theorem of this paper which is stated below, conjectures 2 and 3 are true for the simple groups $L_{p+1}(2)$ where p is a prime number.

In this paper we consider the projective spacial linear group $PSL_{p+1}(2)$, p a prime number, and prove that this group is characterizable by its order components. Another names for this group are $L_{p+1}(2)$ And $A_p(2)$ in the Lie notation. More precisely we will prove:.

Main Theorem If a finite group G has the same set of order components as $L_{p+1}(2)$, then $G \cong L_{p+1}(2)$.

2 Preliminary results

The structure of finite groups with disconnected Gruenberg-Kegel graph follows from Theorem A of [19] which will be stated below:

Lemma 1 *Let G be a finite group with $s(G) \geq 2$. Then one of the following holds:*

(1) G is either a Frobenius or 2-Frobenius group.

(2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H is a nilpotent π_1-group, $\frac{K}{H}$ is a non-abelian simple group, $\frac{G}{K}$ is a π_1-group, $\left|\frac{G}{K}\right|$ divides $\left|Out(\frac{K}{H})\right|$ and any odd order component of G is equal to one of the odd order components of $\frac{K}{H}$.

To deal with the first case in the above Lemma we need the following results which are taken from [6] and [2], respectively.

Lemma 2 *(a) Let G be a Frobenius group of even order with kernel and complements K and H, respectively. Then $s(G) = 2$ and the prime graph components of G are $\pi(H)$ and $\pi(K)$.*

(b) Let G be a 2-Frobenius group of even order. Then $s(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $|\frac{K}{H}| = m_2$, $|H||\frac{G}{K}| = m_1$ and $|\frac{G}{K}|$ divides $|\frac{K}{H}| - 1$ and H is a nilpotent π_1-group.

Lemma 3 *Let G be a finite group with $s(G) \geq 2$. If $H \trianglelefteq G$ is a π_i-group, then $(\prod_{j=1,\ j\neq i}^{s(G)} m_j) \mid (|H| - 1)$.*

The following result of Zsigmondy [20] is important in some number theoretical considerations.

Lemma 4 *Let n and a be integers greater than 1. There exists a prime divisor p of $a^n - 1$ such that p does not divide $a^i - 1$ for all $1 \leq i < n$, except in the following cases.*

(1) $n = 2$, $a = 2^k - 1$, where $k \geq 2$,

(2) $n = 6$, $a = 2$.

The prime p in Lemma 4 is called a Zsigmondy prime for $a^n - 1$.

Remark 1 *If p is a Zsigmondy prime for $a^n - 1$, then $p > n$. Because if $p \leq n$, then $n = kp+r$, $0 \leq r < p$, and we can write $a^n - 1 = a^r(a^{kp} - a^k) + a^{k+r} - 1$. Since $(p, a) = 1$ we have $a^p \equiv a(\mathrm{mod}\, p)$, hence $a^{kp} \equiv a^k(\mathrm{mod}\, p)$, therefore $p \mid a^{k+r} - 1$. By assumption about p we must have $k+r \geq n$ which implies $k \geq kp$, hence $k = 0$. Therefore $n = r < p$ contradicting $p \leq n$.*

Next we consider the linear group $L_{p+1}(2)$ where p is a prime number. By [9] and [17], for $n \in \mathbb{N}$ we have

$$s(L_n(2)) = \begin{cases} 1 \text{ if } n \neq p,\ p+1 \\ 2 \text{ if } n = p \text{ or } p+1 \end{cases}$$

where $p \geq 3$ is a prime number. Therefore if $p \geq 3$ is a prime number, $L_{p+1}(2)$ has two order components which can be seen from Table 1 to be: $m_1 = 2^{\binom{p+1}{2}}(2^{p+1} - 1)\prod_{i=2}^{p-1}(2^i - 1)$ and $m_2 = 2^p - 1$. The components of the graph $GK(L_{p+1}(2))$ are $\pi_1 = \pi(2(2^{p+1} - 1)\prod_{i=2}^{p-1}(2^i - 1))$ and $\pi_2 = \pi(2^p - 1)$. By [11] the group $L_3(2)$ is characterizable by its order components. Therefore throughout the rest of this paper we assume p is an odd prime number.

3 Proof of the main theorem

We assume G is a finite group with $OC(G) = \{m_1, m_2\}$, where m_1 and m_2 are the order components of the group $L_{p+1}(2)$, and use Lemma 1. First we will prove the following Lemma.

Lemma 5 *If G is a finite group with $OC(G) = \{m_1, m_2\}$, then G is neither a Frobenius nor a 2-Frobenius group.*

Proof. First suppose in the contrary G is a Frobenius group with complement H and kernel K. By Lemma 2 we have $OC(G) = \{|H|, |K|\}$. Since $|H| \mid |K| - 1$ we must have $|H| < |K|$, hence $|K| = m_1 = 2^{\binom{p+1}{2}}(2^{p+1} - 1)\prod_{i=2}^{p-1}(2^i - 1)$, $|H| = m_2 = 2^p - 1$. Since K is a nilpotent group, it is a direct product of its Sylow subgroups. Therefore each Sylow subgroup of K is normal in G. If $p = 5$, then K has a Sylow 5−subgroup of order 5, hence by Lemma 3 we have $5 \equiv 1 (\bmod\, m_2) \equiv 1 (\bmod\, 31)$ a contradiction. Hence we assume $p \neq 5$. Let r be a Zsigmondy prime for $2^{p+1} - 1$ which exists by Lemma 4. From the order of K we see that a Sylow r−subgroup S of K has order a divisor of $2^{p+1} - 1$, hence by Lemma 3 we have $m_2 \mid |S| - 1$. Therefore $|S| - 1 = m_2 k$ for some $k \in \mathbb{N}$. From $r \leq 2^{p+1} - 1$ and the last equality we obtain $k = 1$ or 2. If $k = 1$, then $r = 2^p$ and if $k = 2$, then $r = 2^{p+1} - 1$ and both of them are not primes. This contradiction shows that G can not be a Frobenius group.

Next assume that G is a 2-Frobenius group. By Lemma 2 (b) there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ for G such that H is a nilpotent π_1−group, $|\frac{K}{H}| = m_2$ and $|\frac{G}{K}| \mid (|\frac{K}{H}| - 1) = 2^p - 2 = 2(2^{p-1} - 1)$. Since $p \neq 6$, there is a Zsigmondy prime r for $2^p - 1$. Therefore $r \nmid |\frac{G}{K}|$ from which it follows that $r \mid |H|$. Since the order of the Sylow r−subgroup of H is a divisor of $2^p - 1$ and H is nilpotent, from the Lemma 3 we deduce $m_2 \mid |S| - 1$. But then $|S| - 1 \geq m_2$, hence $|S| \geq 2^p$ contradicting $|S| \mid 2^p - 1$. Finally G is not a 2−Frobenius group and the Lemma is proved.

The following Lemma is useful in our further investigations. Note that for a prime r and a positive integer n, n_r denotes the r−part of n, i. e. $n = mn_r$ where $(m, r) = 1$.

Lemma 6 *Let r be a prime divisor of $(2^{p+1} - 1)\prod_{i=1}^{p-1}(2^i - 1)$. Then for any positive integer k with $r^k \mid |L_{p+1}(2)|$ we have $r^k \not\equiv \pm 1 (\bmod\, m_2)$.*

Proof. First we assume $r = 3$. In this case only $2^i - 1$ with i even is divisible by 3. From $2^{2i}-1 = 4^i - 1 = (4-1)(4^{i-1}+\cdots+4+1)$ we obtain $(2^{2i}-1)_3 \mid 3(i)_3$. Therefore the 3−part of $(2^{p+1}-1)\prod_{i=1}^{p-1}(2^i-1)$ divides $((\frac{p+1}{2})!)_3 3^{\frac{p+1}{2}}$. But for any $n \in \mathbb{N}$ and any prime number t we have $(n!)_t \mid t^{[\frac{n-1}{t-1}]}$. This is because if $t^u \le n < t^{u+1}$, then $(n!)_t = t^k$ where $k = [\frac{n}{t}]+[\frac{n}{t^2}]+\cdots+[\frac{n}{t^u}] \le \frac{n}{t}+\frac{n}{t^2}+\cdots+\frac{n}{t^u} = \frac{n}{t}\frac{1-\frac{u}{t}}{1-\frac{1}{t}} \le \frac{n(1-\frac{1}{n})}{t(1-\frac{1}{t})} = \frac{n-1}{t-1}$. Therefore $((\frac{p+1}{2})!)_3$ divides $3^{[\frac{p-1}{4}]}$. Hence the largest positive integer k for which $3^k \mid |L_{p+1}(2)|$ is $\frac{p+1}{2}+[\frac{p-1}{4}]$. Now examination of different positive integers $t \le k$ reveals that the congruence $r^k \equiv \pm 1 (\mathrm{mod}\, m_2)$ does not hold with $r = 3$.

Next assume $r > 3$. Let s be the least positive integer for which $r \mid 2^s - 1$. We have $3 \le s \le p-1$ or $s = p+1$. Clearly if $r \mid 2^t - 1$, then $s \mid t$. If $t = ks$, $k \in \mathbb{N}$, then $2^t - 1 = 2^{ks} - 1 = (2^s-1)(2^{s(k-1)} + 2^{s(k-2)} + \cdots + 1)$. Therefore $(2^t-1)_r \mid (2^s-1)_r(k)_r = (2^s-1)_r(\frac{t}{s})_r$. It follows that the largest power k of r such that r^k is a divisor of $(2^{p+1}-1)\prod_{i=1}^{p-1}(2^i-1)$ is at most $k = [\frac{p+1}{s}]+[\frac{p-1}{s(r-1)}]$. But from $s > 2$ it follows that $[\frac{p+1}{s}]-[\frac{p-1}{s}] = 0$ or 1.

If $[\frac{p+1}{s}]-[\frac{p-1}{s}] = 0$, then $k = [\frac{p-1}{s}]+[\frac{p-1}{s(r-1)}]$ and since from $r^k \mid r^{[\frac{p-1}{s(r-1)}]}(2^s - 1)_r^{\frac{p-1}{s}}$, it follows that $r^k < m_2 - 1$, we cannot have $r^k \equiv \pm 1 (\mathrm{mod}\, m_2)$. If $[\frac{p+1}{s}] - [\frac{p-1}{s}] = 1$, then $k = [\frac{p-1}{s}] + [\frac{p-1}{s(r-1)}] + 1$. Suppose $r^k = \pm 1 + l m_2$ where $l > 0$. Then $r^k = r\, r^{[\frac{p-1}{s}]+[\frac{p-1}{s(r-1)}]} \le r m_2$ which implies $l m_2 \pm 1 \le r m_2$, again we obtain a contradiction.

By the Lemma 1 and 4, if G is a finite group with $OC(G) = OC(L_{p+1}(2))$, then there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ for G such that $\frac{K}{H}$ is a non-abelian simple group, H and $\frac{G}{K}$ are π_1−group and H is nilpotent. Moreover $|\frac{G}{K}|$ divides $|Out(\frac{K}{H})|$ and the odd order component of G is one of the odd order components of $\frac{K}{H}$ and $s(\frac{K}{H}) \ge 2$.

Since $P = \frac{K}{H}$ is a non-abelian simple group with $s(P) \ge 2$, according to the classification of finite simple groups we have one of the possibilities of Tables 1,2 or 3 for P. We distinguish several cases.

Case 1 $P \cong {}^2A_3(2)$, ${}^2F_4(2)'$, $A_2(2)$, $A_2(4)$, ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, ${}^2E_6(2)$, or one of the 26 sporadic simple groups listed in Tables 1,2 or 3. The odd order component of G is $m_2 = 2^p - 1$ and it must be one of the odd order components of the groups listed above. But by Tables 1,2 and 3 we have the following possibilities for P, m_2 and the prime p. The order of outer automorphism group, $Out(P)$, of P is taken from [7].

Table 4

P	m_2	p	$\|Out(P)\|$
J_2	7	3	2
$A_2(2)$	3 or 7	2 or 3	2
$2A_5(2)$	7	3	3
$E_7(2)$	127	7	1
M_{22}	7	3	2
J_1	7	3	1
J_4	31	5	1
HS	7	3	2
$O'N$	31	5	2
Ly	31	5	1
F_2	31	5	1
F_3	31	5	1

Therefore we can set $|\frac{G}{K}| = t$ where $t = 1, 2$ or 3, from which it follows that $t\,|H| = \frac{|G|}{|P|}$. But $|G| = |L_{p+1}(2)|$ and the orders of relevant P is given in [7] and using Table 4 we can inspect those p for which $|P| \mid |G| = |L_{p+1}(2)|$ and find out that only $A_2(2)$ with $p = 2$ or 3 satisfies this condition. If $p = 3$, then $t\,|H| = 2^3.3.5$ implying $5 \mid H$. Since H is a nilpotent normal subgroup of G, its Sylow 5−subgroup which is cyclic of order 5 must be normal in G and by Lemma 3 we deduce $m_2 = 7 \mid 5 - 1 = 4$, a contradiction. Therefore $p = 2$, which implies $G \cong P = L_3(2)$. This is one of our conclusions with $p = 2$.

Case 2 $P \cong \mathbb{A}_n$ and either $n = p',\ p' + 1,\ p' + 2$, one of n or $n - 2$ is not prime; or $n = p',\ p' - 2$ are both prime, where $p' > 6$ is a prime.

By Tables 1 and 2, the odd order components of $\mathbb{A}_n$ are p' and(or) $p' - 2$. If $p' - 2 = 2^p - 1$, then $p' = 2^p + 1$ is not a prime number. Hence $2^p - 1$ can only be equal to the odd order component p'. In this case if we let $t = |Out(P)|$, then we obtain $\frac{|G|}{|P|} = t\,|H|$. Since $p' > 6$, it is well-known that $|Out(P)| = 2$, hence $t = 1$ or 2.

The largest power of 2 dividing $p' = 2^p - 1$ is $[\frac{p'}{2}] + [\frac{p'}{4}] + \cdots = 2^p - 1$, hence a Sylow 2−subgroup of $\mathbb{A}_{p'}$ has order $2^{2^p - 2}$. But it is easy to prove $2^p - 2 > \frac{p(p+1)}{2}$ holds for all primes $p > 3$. Since $\mathbb{A}_{p'} \leq \mathbb{A}_n$, this implies that for $p > 3$ we have $|P| \nmid |G|$. If $p = 3$, then $p' = 7$ and $t\,|H| = 8$. If $t = 2$, then $|H| = 4$ and by the Lemma 3, $7 \mid |H| - 1 = 3$, a contradiction. Therefore $t = 1$, $|H| = 8$, and H is an elementary abelian 2−group of order 8. In this case $C_G(H) = H$ and hence $\frac{G}{H}$ is isomorphic to a subgroup of $Aut(H) = L_3(2)$. But $\frac{G}{H} \cong \mathbb{A}_7$ is not a subgroup of $L_3(2)$. This final contradiction proves that $P \cong \mathbb{A}_n$ cannot happen.

Case 3 P is a simple group with $s(P) = 2$.

In this case P can be any of the groups listed in Table 1, but we consider those which are not covered in case 1 and 2.

(a) $P \cong F_4(q)$ or ${}^3D_4(q)$

In this case we have $m_2 = q^4 - q^2 + 1 = 2^p - 1$. Therefore $q^2(q^2-1) = 2^p - 2 = 2(2^{p-1} - 1)$. If q is odd, then $q^2 - 1$ is a multiple of 8 and the equality does not hold with any value of p. If q is even, then q^2 is a multiple of 4 and the equality fails. Therefore $P \cong F_q(q)$ or ${}^3D_4(q)$ are ruled out.

(b) $P \cong G_2(q)$, $2 < q \equiv \epsilon (\bmod 3)$, $\epsilon = \pm 1$.

By Table 1 we have $q^2 - \epsilon q + 1 = 2^p - 1$ and $|G_2(q)| = q^6(q^6-1)(q^2-1)$. It is easy to check that for the special case $p \leq 7$ we get a contradiction with regard to the divisibility $|P| \mid |G|$. Therefore we assume $p > 7$. Now calculation shows that $(q^2 - \epsilon q)^2 \mid |G|$, hence from $|P| \mid |G|$ we obtain $2^{p-1} - 1 \mid (2^{p+1}-1)(2^p-1)\prod_{i=1}^{p-2}(2^i-1)$. Let r be a Zsigmondy prime for $2^{p-1} - 1$. Then $r \mid (2^{p+1} - 1)(2^p - 1)$ from which it follows $r \mid 2^{p+1} - 1$ or $r \mid 2^p - 1$. Since we also have $r \mid 2^{p-1} - 1$, a contradiction is derived in this case.

(c) $P \cong E_6(q)$ or ${}^2E_6(q)$.

By Table 1 we have $\frac{q^6 \pm q^3 + 1}{(3, q \mp 1)} = m_2$, from which it follows that $q^9 \pm 1 \equiv 0 (\bmod m_2)$. If q is odd, then by Lemma 6 the above congruence equation has no solutions. Suppose q is even. From the above equality we obtain $q^3(q^3 \pm 1) = 2(2^{p-1} - 1)$ or $4(3 \times 2^{p-2} - 1)$ according to $(3, q \mp 1) = 1$ or 3, respectively which cannot happen because q is a power of 2.

(d) $P \cong {}^2D_n(3)$, $n = 2^m + 1 \geq 9$ not a prime number.

In this case the odd order component is $\frac{3^{n-1}+1}{2} = 2^p - 1$ implying $3^{n-1} - 2^{p+1} = -3$, a contradiction.

(e) $P \cong {}^2D_{p'}(3)$, $p' \neq 2^m + 1$, $p' \geq 5$ is a prime number.

In this case we have $\frac{3^{p'}+1}{4} = 2^p - 1$ implying $3^{p'} + 1 = 4m_2$. Since 3 is an odd π_1–prime, by the Lemma 6 we obtain a contradiction.

(f) $P \cong {}^2D_n(2)$, $n = 2^m + 1 \geq 5$. We must have $2^{n-1} + 1 = 2^p - 1$ implying $2^{n-1} = 2(2^{p-1} - 1)$, a contradiction.

(g) $P \cong {}^2D_n(q)$, $n = 2^m \geq 4$. In this case $\frac{q^n+1}{(2,q+1)} = 2^p - 1$, hence $q^n = 2^p$ or $q^n = 2^{p+1} - 3$ according to $(2, q+1) = 1$ or 2, respectively. If $(2, q+1) = 2$, then

q is an odd prime power and by the Lemma 6 we get a contradiction with respect to $q^n+1=2m_2$. If $q^n=2^p$, then $q=2$ and $p=n=2^m\geq 4$, a contradiction.

(h) $P\cong D_{p'+1}(q)$, $q=2,3$; $D_{p'}(q)$, $p'\geq 5$, $q=2,3$; $C_{p'}(q)$, $q=2,3$; $C_n(q)$, $n=2^m\geq 2$; $B_p(3)$ or $B_n(q)$, $n=2^m\geq 4$, q odd.

In all of these cases repeated use of the Lemma 6 yields contradictions and we do not present the details here.

(i) $P\cong {}^2A_{p'}(q)$, $q+1\mid p'+1$, $(p',q)\neq(3,3),(5,2)$.

In this case we have $\frac{q^{p'}+1}{q+1}=2^p-1$. If q is odd, then we get a contradiction by the Lemma 6. If q is even, then $q^{p'-1}-q^{p'-2}+\cdots-q=2(2^{p-1}-1)$. Therefore $2^{p'}=3.2^p-4=4(3.2^{p-2}-1)$, a contradiction because $p'>3$.

(j) $P\cong {}^2A_{p'-1}(q)$ or $A_{p'-1}(q)$, $(p',q)\neq(3,2),(3,4)$.

In this case we have $m_2=\frac{q^{p'}+1}{(q+1)(p',q+1)}$ or $\frac{q^{p'}-1}{(q-1)(p',q-1)}$, respectively for ${}^2A_{p'-1}(q)$ or $A_{p'-1}(q)$. In the following we investigate the case $m_2=\frac{q^{p'}+1}{(q+1)(p',q+1)}$. The case $m_2=\frac{q^{p'}-1}{(q-1)(p',q-1)}$ can be handled similarly. The cases $p'=3$ and 5 can be verified separately to lead to contradiction. Therefore we assume $p'\geq 5$ and in which follows Zsigmondy primes exist for the numbers that we dealing with.

If q is odd, then by the Lemma 6, from $q^{p'}+1\equiv 0 (\mathrm{mod}\, m_2)$ a contradiction is obtained. Therefore we assume $q=2^f$, $f\geq 1$. If $(p',q+1)=1$, then $q^{p'-1}-q^{p'-2}+\cdots-q+1=2^p-1$ from which we obtain $q^{p'-1}-q^{p'-2}+\cdots-q=2(2^{p-1}-1)$. Hence $q=2$ and consequently $2^{p'}=4(3\times 2^{p-2}-1)$, a contradiction because $p'\geq 3$. Therefore we assume $(p',q+1)=p'$, hence $p'\mid q+1$ and since $q=2^f$, we can write

$$2^{fp'}+1=(2^f+1)p'(2^p-1) \quad (1)$$

Now let r be a Zsigmondy prime for $2^{2fp'}-1$. Hence $r\mid 2^{fp'}+1$, and consequently by (1) $r\mid 2^p-1$ because $r\nmid q+1$ and $r\nmid p'$. Since r is assumed to be a Zsigmondy prime for $2^{2fp'}-1$, we obtain $p\geq 2fp'>f$.

Next we consider the order of $P\cong {}^2A_{p'-1}(q)$. We have $|P|=m_2q^{\frac{p'(p'-1)}{2}}\prod_{i=1}^{p'-1}(q^i-(-1)^i)$ where $q=2^f$. Let l be a Zsigmondy prime for $q^{2(p'-2)}-1=2^{2f(p'-2)}-1$. Since $l\mid q^{p'-2}+1\mid |P|$, from $|P|\mid|G|$ we obtain $l\mid 2^{\binom{p+1}{2}}\prod_{i=1}^{p}(2^i-1)$. Hence $l\mid 2^j-1$ for some $j\leq p+1$. We deduce that $j\geq 2f(p'-2)$, hence $p+1\geq 2f(p'-2)$.

Now we set $|\frac{G}{K}|=t$ which must be a divisor of $|Out(P)|=2(p',q+1)f=2p'f$. From $\frac{K}{H}\cong P$ we obtain $t|H||P|=|G|$, hence using (1) we obtain the following identity:

$$t\,|H|\,q^{p'(p'-1)/2}\prod_{i=1}^{p'-1}(q^i-(-1)^i)=2^{\binom{p+1}{2}}(2^{p+1}-1)\prod_{i=1}^{p-1}(2^i-1) \qquad (2)$$

Note that in (2) we have $q=2^f$, a power of 2.

Finally assume k is a Zsigmondy prime for $2^{p+1}-1$. Then from (2) it follows that $k \mid t\,|H|\prod_{i=1}^{p'-1}(q^i-(-1)^i)$. If $k \mid t$, then $k \mid f$, hence $k \le f$. But k is a Zsigmondy prime for $2^{p+1}-1$ and by the Remark 1, $k \ge p+1 > p > f$, contradicting $k \le f$. If $k \mid q^{(p'-j)}-1 = 2^{f(p'-j)}-1$, for some $1 \le j \le p'-1$, then $f(p'-j) \ge p+1$ and since $f(p'-1) \ge f(p'-j) \ge p+1$ we obtain $p+1 \le f(p'-1)$ contradicting $p+1 \ge 2f(p'-2)$ which was obtained earlier. Therefore $k \mid |H|$. But in this case a Sylow k−subgroup of H has order k and from the Lemma 3 it follows that $m_2 = 2^p-1 \mid k-1$ from which we get a contradiction with respect to $k \mid 2^{p+1}-1$. This final contradiction rules out the possibility of $P \cong {}^2A_{p'-1}(q)$. The case $P \cong A_{p'-1}(q)$ is treated similarly.

(k) $P \cong A_{p'}(q)$, $q-1 \mid p+1$. In this case we have $\frac{q^{p'}-1}{q-1} = 2^p-1$. If q is odd then by the Lemma 5 we obtain a contradiction. We assume q is even and write $q^{p'-1}+q^{p'-2}+\cdots+q = 2(2^{p-1}-1)$ from which it follows $q=2$ and $p'=p$. But This will imply $P \cong A_p(2) = L_{p+1}(2)$, hence $|H|=1$ and $G=K \simeq L_{p+1}(2)$. This is what we aim to prove.

Up to present all simple groups P with $s(P)=2$ have been considered. Next we consider simple groups P with $s(P)=3$ tabulated in Table 2.

Case 4 P is a simple group with $s(P)=3$. Since in case 2 we considered the alternating group $\mathbb{A}_{p'}$ with both p' and $p'-2$ prime numbers, hence we start with the next group in Table 2.

(a) $P \cong A_1(q)$, $3 < q \equiv \epsilon (\operatorname{mod} 4)$, $\epsilon = \pm 1$. The odd order components are q and $\frac{q+\epsilon}{2}$.

First we assume $q = 2^p-1$. If $q=r^m$ is a power of a prime r, then $2^p-r^m=1$ and by [8] there is no solution for this equation with $m>1$. Hence $m=1$ and $q=2^p-1$ is a prime number. Therefore $|Out(P)|=2$ and $|\frac{G}{K}|=t$, where $t=1$ or 2. From $\frac{K}{H} \cong P \cong A_1(q)$ and $|\frac{G}{K}|=t$ we obtain $t\,|H| = 2^{p(p-1)/2}(2^{p+1}-1)(2^{p-2}-1)\cdots(2^2-1)$. Clearly $p \ge 3$. Assume $p \ne 5$ and let p' be a Zsigmondy prime for $2^{p+1}-1$. Then $p' \mid |H|$ and a Sylow p'−subgroup of H has order p'. By the Lemma 3, $m_2 = 2^p-1 \mid p'-1$. Hence $p'-1 = u(2^p-1)$, and since $p' \le 2^{p+1}-1$, we obtain $p'-1 \le 2(2^p-1)$ implying $u=1$ or 2, and in both cases a contradiction with the assumption on p' is obtained. If $p=5$, then $t\,|H| = 2^{10}.3^4.5.7^2$ and a similar argument with a Sylow 5−subgroup of H results a contradiction.

Next we assume $\frac{q+\epsilon}{2} = 2^p - 1$. If $\epsilon = -1$, then $q = 2^{p+1} - 1 = 3^f$, because $2^{p+1} - 1$ is divisible by 3. But by [8] there is no solution to $2^{p+1} - 3^f = 1$. Therefore we assume $\epsilon = 1$. Hence $q = 2^{p+1} - 3 = r^f$, power of a prime r. Clearly $f < p+1$. It is well-known that $|Out(P)| = 2f$, hence $|G| = t\,|H|\,|P|$, where $t \mid 2f$. Substituting for $|G|$ and $|P|$ we obtain $2t\,|H|\,(2^{p+1} - 3)(2^{p-1} - 1)(2^{p+1} - 5) = 2^{\binom{p+1}{2}} \prod_{i=1}^{p+1}(2^i - 1)$. Assume $p \neq 5$ and let p' be a Zsigmondy prime for $2^{p+1} - 1$. We have $p' \mid 2t\,|H|\,(2^{p+1}-3)(2^{p-1}-1)(2^{p+1}-5)$. Then by the Remark 1, $p' > p+1 > f$, so $p' \nmid t$ because p' is odd. Since $2^{p+1} - 3$ and $2^{p+1} - 5$ are prime to $2^{p+1} - 1$ hence $p' \mid |H|$. Therefore the order of a Sylow p'-subgroup of H is a divisor of $2^{p+1} - 1$ and since H is a nilpotent normal subgroup of G, we deduce by the Lemma 3 that $m_2 = 2^{p+1} - 1 \leq |S| - 1$, a contradiction.

(b) $P \cong A_1(q)$, $q > 2$, q even.

In this case the odd order components are $q-1$ and $q+1$. Therefore $q \pm 1 = 2^p - 1$ and similar to (a) above, we reach a contradiction.

(c) $P \cong {}^2D_{p'}(3)$, $p' = 2^m + 1 \geq 5$. In this case we have $\frac{3^{p'-1}+1}{2}$ or $\frac{3^{p'}+1}{4} = 2^p - 1$ implying $3^{p'-1} + 1 \equiv 0 (\mathrm{mod}\, m_2)$ or $3^{p'} + 1 \equiv 0 (\mathrm{mod}\, m_2)$, which are impossible by the Lemma 6.

(d) $P \cong {}^2D_{p'+1}(2)$, $p' = 2^n - 1$, $n \geq 2$. In this case we have $2^{p'+1} + 1 = 2^p - 1$ implying $2^{p'+1} = 2(2^{p-1} - 1)$, a contradiction. If $2^{p'} + 1 = 2^p - 1$, then $2^{p'} + 2 = 2^p$ which is obviously impossible.

(e) $P \cong G_2(q)$, $3 \mid q$; ${}^2G_2(q)$, $q = 3^{2m+1} > 3$; $F_4(q)$, q even, ${}^2F_4(q)$, $q = 2^{2m+1} > 2$. In all of these cases equating m_2 with one of the odd order components of the group concerned results a contradiction in an easy step.

Case 5 In this last case we consider simple groups P with $s(P) \geq 4$. According to Table 3 there remain only the groups ${}^2B_2(q)$, $q = 2^{2m+1} > 2$; and $E_8(q)$ to be considered.

First suppose $P \cong {}^2B_2(q)$, $q = 2^{2m+1} > 2$. The odd order components are $q - 1$, $q - \sqrt{2q} + 1$ and $q + \sqrt{2q} + 1$. If $q \pm \sqrt{2q} + 1 = 2^p - 1$, then $2^{2m+1} \pm 2^{m+1} = 2(2^{p-1} - 1)$ which is a contradiction because $m > 0$. If $q - 1 = 2^p - 1$, then $q = 2^p$, hence $|{}^2B_2(q)| = 2^{2p}(2^p - 1)(2^{2p} + 1)$ and we must have $2^{2p}(2^p - 1)(2^{2p} + 1) \mid |L_{p+1}(2)|$. If p' be a Zsigmondy prime for $2^{4p} - 1$, then $p' > 4p$ and $p' \mid 2^{2p} + 1$, then $p' \mid |L_{p+1}(2)|$, a contradiction.

Next let $P \cong E_8(q)$. According to Table 3 if we equate the odd order component of P with m_2 we obtain equations of the form $q^k \pm 1 \equiv 0 (\mathrm{mod}\, m_2)$ where $k = 10, 12, 15$, which is a contradiction by the Lemma 6 if q is odd. Therefore we assume q is even. By the Table 3 all the odd order components of P are of the form $qf(q)+1$, where $f(q)$ is a polynomial in q. If $qf(q)+1 = 2^p - 1$, then we must have $q = 2$ and examination of each case results a contradiction.

Since we have considered all the cases for the simple group P, by case 1 and case 3 (k) we deduce that $G \cong L_{p+1}(2)$ and the main theorem is proved.

References

[1] S. H. Alavi and A. Daneshkhah, *A new characterization of alternating and symmetric group*, J. Appl. Math. Comput. **17** (1) (2005), 245-258.

[2] G. Y. Chen, *A new characterization of sporadic simple groups*, Algebra Colloq. **3** (1) (1996), 49-58.

[3] G. Y. Chen, *A new characterization of $PSL_2(q)$*, Southeast Asian Bull. Math. **22** (1998), 257-263.

[4] G. Y. Chen, *${}^2D_n(3)$ ($9 \leq n = 2^m + 1$ not a prime) can be characterized by its order components*, J. Appl. Math. Comput. **19** (1-2) (2005), 353-362.

[5] G. Y. Chen, *On Thompson's conjecture.* J. Algebra **185** (1996), 184-193.

[6] G. Y. Chen, *On Frobenius and 2−Frobenius groups*, J. Southwest China Normal Univ. **20** (5) (1995), 485-487.

[7] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[8] P. Crescenzo, *A diophantine equation which arises in the theory of finite groups*, Adv. Math. **17** (1975), 25-29.

[9] N. Iiyori and H. Yamaki, *Prime graph components of the simple groups of Lie type over the field of even characteristic*, Proc. Japan Acad. (Ser. A), **67** (1991), 82-83.

[10] A. Iranmanesh, S. H. Alavi and B. Khosravi, *A characterization of $PSL(3,q)$ where q is an odd prime power*, J. Pure. Appl. Algebra **170** (2-3) (2002), 243-254.

[11] A. Iranmanesh, S. H. Alavi and B. Khosravi, *A characterization of $PSL(3,q)$ for $q = 2^n$*, Acta Math. Sinica **18** (3) (2002), 463-472.

[12] A. Iranmanesh, B. Khosravi and S. H. Alavi, *A characterization of $PSU(3,q)$ for $q > 5$*, Southeast Asian Bull. Math. **26** (2) (2002), 33-44.

[13] A. Khosravi and B. Khosravi, *A new characterization of $PSL(p,q)$*, Comm. Algebra **32** (2004), 2325-2339.

[14] A. Khosravi and B. Khosravi, *A new characterization of some alternating and symmetric groups*, Int. J. Math. Math. Sci. **45** (2003), 2863-2872.

[15] Behrooz Khosravi, Behnam Khosravi and Bahman Khosravi, *The number of isomorphism classes of finite groups with the set of order components of $C_4(q)$*, Appl. Algebra Engrg. Comm. Comput. **15** (2005), 349-359.

[16] A. Khosravi and B. Khosravi, *$r-$recognizability of $B_n(q)$ and $C_n(q)$ where $n = 2^m \geq 4$*, J. Pure Appl. Algebra **199** (2005), 149-165.

[17] A. S. Kondratev, *On prime graph components of finite simple groups*, Math. USSR Sbornik, **190** (6) (1989), 787-797.

[18] H. Shi and G. Y. Chen, *${}^2D_{p+1}(2)$ $(5 < p = 2^m - 1)$ can be characterized by its order components*, Kumamoto J. Math. **18** (2005), 1-8.

[19] J. S. Williams, *Prime graph components of finite groups*, J. Algebra **69** (1981), 487-513.

[20] K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. Phys. **3** (1892), 256-284.

M. R. Darafsheh
Department of Mathematics, Statistics and Computer Science,
Faculty of Science, University of Tehran, Tehran, Iran
e-mail: darafsheh@ut.ac.ir

A. Mahmiani
University of Payame Noor, Gonbade Kavoos, Iran