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New wide classes of complex vector functional equations

Ice B. Risteski

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Abstract. In this present paper three wide classes of complex vector functional equations are considered. First we solve a class of linear functional equations with operation addition between the arguments and after this some functional equations with operation multiplication between the arguments are solved. A second class of quadratic complex vector functional equations with constant complex coefficients is solved by its linearization and use of a matrix method. A third class of functional equations solved in the paper is the class of nonhomogeneous linear hypercomplex vector functional equations with noncommutative properties. All the considered classes of complex vector functional equations appear for the first time in the literature.

1 Introduction

Our focus in the present article will particularly be on the area of complex vector functional equations which are the newest mathematical discipline that has rapid development in the recent years. These kinds of functional equations are very often hard to handle and for many of them mathematics has not yet determined a general solution.

Motivated by this idea we continue to study this topic and this article aims at providing new general classes of complex vector functional equations solved by methods given in [1]–[4]. The results presented here supplement and generalize some of our previous results [5]–[7].

2 Preliminaries

Let A be an $n \times n$ complex matrix. Suppose that by elementary transformations the matrix A is transformed into $A = P_1 D P_2$, where P_1 and P_2 are regular matrices and D is a diagonal matrix with diagonal entries 0 and 1 such that the number

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of the units is equal to the rank of the matrix A. The matrix $B = P_2^{-1}DP_1^{-1}$ satisfies the equality $ABA = A$. This means that the matrix equation $AXA = A$ has at least one solution for X.

If A satisfies the identity

$$A^r + k_1A^{r-1} + \cdots + k_{r-1}A = O,$$

where $k_{r-1} \neq 0$ and O is the zero $n \times n$ matrix, then the matrix

$$X = -\frac{1}{k_{r-1}}(A^{r-2} + k_1A^{r-3} + \cdots + k_{r-2}I),$$

where I is the unit $n \times n$ matrix, is also a solution of the equation $AXA = A$.

Now we recall the following theorem proved in [1].

Theorem 2.1 *If B satisfies the condition $ABA = A$, then*

1° $AX = O \iff X = (I - BA)Q$ *(X and Q are $n \times m$ matrices);*

2° $XA = O \iff X = Q(I - AB)$ *(X and Q are $m \times n$ matrices);*

3° $AXA = A \iff X = B + Q - BAQAB$ *(X and Q are $n \times n$ matrices);*

4° $AX = A \iff X = I + (I - BA)Q$*;*

5° $XA = A \iff X = I + Q(I - AB)$.

Throughout this article, if not stated otherwise, $\mathcal{V}$ is an n-dimensional complex vector space. The vectors from $\mathcal{V}$ will be denoted by $\mathbf{Z}_i = (z_{i1}, \ldots, z_{in})^T$ $(1 \leq i \leq n)$, $\mathbf{O} = (0, 0, \ldots, 0)^T$ is the zero vector in $\mathcal{V}$, $\mathbf{I} = (1, 1, \ldots, 1)^T$. We define multiplication of two arbitrary vectors $\mathbf{U} = (u_1, \ldots, u_n)^T$ and $\mathbf{V} = (v_1, \ldots, v_n)^T$ in $\mathcal{V}$ as $\mathbf{UV} = (u_1v_1, \ldots, u_nv_n)^T$. Further on, for $\alpha \in \mathbb{N}$ we define $\mathbf{U}^\alpha = (u_1^\alpha, \ldots, u_n^\alpha)^T$, and we denote $|\mathbf{U}| = (|u_1|, \ldots, |u_n|)^T$, where $|u_k|$ is the modulus of the complex number u_k.

3 Some complex vector functional equations with operations between the arguments

In this section some simple complex vector functional equations which extend our previous results obtained in [7] will be solved.

Theorem 3.1 *The general continuous solution of the functional equation*

$$f\Big(\sum_{i=1}^{n} \mathbf{Z}_i\Big) = \sum_{i=1}^{n} f_i(\mathbf{Z}_i) \tag{3.1}$$

is given by

$$f(\mathbf{U}) = C\mathbf{U} + \sum_{i=1}^{n} \mathbf{A}_i \tag{3.2}$$

and

$$f_i(\mathbf{U}) = C\mathbf{U} + \mathbf{A}_i \qquad (1 \leq i \leq n), \tag{3.3}$$

where C is a constant complex matrix and $\mathbf{A}_i$ $(1 \leq i \leq n)$ are complex vectors.

Proof. If we set $\mathbf{Z}_i = \mathbf{O}$ $(1 \le i \le n)$ in (3.1), we obtain

$$f(\mathbf{O}) = \sum_{i=1}^{n} \mathbf{A}_i,$$

where $\mathbf{A}_i = f_i(\mathbf{O})$ $(1 \le i \le n)$.

Now, if we put successively $n-1$ variables in (3.1) to be equal to the zero vector, then we get

$$\begin{aligned} f(\mathbf{Z}_1) &= f_1(\mathbf{Z}_1) + \mathbf{A}_2 + \cdots + \mathbf{A}_n, \\ f(\mathbf{Z}_2) &= \mathbf{A}_1 + f_2(\mathbf{Z}_2) + \cdots + \mathbf{A}_n, \\ &\vdots \\ f(\mathbf{Z}_n) &= \mathbf{A}_1 + \mathbf{A}_2 + \cdots + f_n(\mathbf{Z}_n). \end{aligned} \tag{3.4}$$

By summation of the above equations we obtain

$$\sum_{i=1}^{n} f(\mathbf{Z}_i) = \sum_{i=1}^{n} f_i(\mathbf{Z}_i) + (n-1)\sum_{i=1}^{n} \mathbf{A}_i,$$

or

$$\sum_{i=1}^{n} f_i(\mathbf{Z}_i) = \sum_{i=1}^{n} f(\mathbf{Z}_i) - (n-1)\sum_{i=1}^{n} \mathbf{A}_i. \tag{3.5}$$

From (3.5) and (3.1) it follows that

$$f\Big(\sum_{i=1}^{n} \mathbf{Z}_i\Big) = \sum_{i=1}^{n} f(\mathbf{Z}_i) - (n-1)\sum_{i=1}^{n} \mathbf{A}_i,$$

or

$$f\Big(\sum_{i=1}^{n} \mathbf{Z}_i\Big) = \sum_{i=1}^{n} f(\mathbf{Z}_i) + \mathbf{B}, \tag{3.6}$$

where

$$\mathbf{B} = -(n-1)\sum_{i=1}^{n} \mathbf{A}_i. \tag{3.7}$$

The general continuous solution of the equation (3.6) is

$$f(\mathbf{U}) = C\mathbf{U} - \frac{\mathbf{B}}{n-1}, \tag{3.8}$$

where C is a constant complex matrix and $\mathbf{B}$ is an arbitrary constant complex vector.

From (3.8), (3.7) and (3.4) the formulae (3.2) and (3.3) immediately follow. □

Theorem 3.2 *The general continuous solution of the complex vector functional equation*

$$f\Big(\sum_{i=1}^{n} \mathbf{Z}_i\Big) = \sum_{i=1}^{n} f_i(\mathbf{Z}_i) + \mathbf{B} \tag{3.9}$$

is given by

$$f(\mathbf{U}) = C\mathbf{U} + \sum_{i=1}^{n} \mathbf{A}_i - \frac{\mathbf{B}}{n-1}, \tag{3.10}$$

and

$$f_i(\mathbf{U}) = C\mathbf{U} + \mathbf{A}_i - \frac{\mathbf{B}}{n-1} \qquad (1 \le i \le n), \tag{3.11}$$

where C is a constant complex matrix, $\mathbf{B}$ and $\mathbf{A}_i$ $(1 \le i \le n)$ are constant complex vectors.

Proof. The proof of this theorem is analogous to that of the previous Theorem 3.1. □

Theorem 3.3 *The general continuous solution of the functional equation*

$$f\Big(\sum_{i=1}^{n} \mathbf{Z}_i\Big) = \sum_{i=1}^{n} f_i(k\mathbf{Z}_i) \quad (k \ne 0) \tag{3.12}$$

is given by the following formulae

$$f(\mathbf{U}) = C\mathbf{U} + \sum_{i=1}^{n} \mathbf{A}_i \tag{3.13}$$

and

$$f_i(\mathbf{U}) = \frac{1}{k} C\mathbf{U} + \mathbf{A}_i \qquad (1 \le i \le n), \tag{3.14}$$

where C and $\mathbf{A}_i$ are as in the previous theorems.

Proof. The proof of this theorem is completely analogous to the proof of Theorem 3.1. □

Theorem 3.4 *The functional equation*

$$f\Big(\frac{1}{k}\sum_{i=1}^{n} \mathbf{Z}_i\Big) = \sum_{i=1}^{r} f_i\Big(\sum_{p=1}^{m} \mathbf{Z}_{i_p}\Big) \quad (k \ne 0), \tag{3.15}$$

where $n > m$, $r = \binom{n}{m} = C_n^m$ and the summation is over all combinations of m vectors chosen among $\mathbf{Z}_i$ $(1 \le i \le n)$, has a general continuous solution given by

$$f(\mathbf{U}) = kr\frac{m}{n} C\mathbf{U} + \sum_{i=1}^{r} \mathbf{A}_i \tag{3.16}$$

and

$$f_i(\mathbf{U}) = C\mathbf{U} + \mathbf{A}_i \qquad (1 \le i \le r), \tag{3.17}$$

where C is a constant complex matrix and $\mathbf{A}_i$ $(1 \le i \le r)$ are constant complex vectors.

Proof. By introduction of the substitution $\mathbf{T}_i = \sum_{p=1}^{m} \mathbf{Z}_{i_p}$, we obtain

$$\sum_{i=1}^{r} \mathbf{T}_i = \binom{n-1}{m-1} \sum_{i=1}^{n} \mathbf{Z}_i = \frac{m}{n} \binom{n}{m} \sum_{i=1}^{n} \mathbf{Z}_i.$$

Using the above expression, the functional equation (3.15) transforms into the following equation

$$f\Big(\frac{1}{\frac{m}{n}kr} \sum_{i=1}^{r} \mathbf{T}_i\Big) = \sum_{i=1}^{r} f_i(\mathbf{T}_i),$$

whose solution is given by Theorem 3.3. □

Theorem 3.5 *The general continuous solution of the general Jensen's complex vector functional equation*

$$f\Big(\frac{1}{n} \sum_{i=1}^{n} \mathbf{Z}_i\Big) = \frac{1}{n} \sum_{i=1}^{n} f_i(\mathbf{Z}_i) \tag{3.18}$$

is given by

$$f(\mathbf{U}) = C\mathbf{U} + \frac{1}{n} \sum_{i=1}^{n} \mathbf{A}_i \tag{3.19}$$

and

$$f_i(\mathbf{U}) = C\mathbf{U} + \mathbf{A}_i \qquad (1 \le i \le n), \tag{3.20}$$

where C is a constant complex matrix and $\mathbf{A}_i$ $(1 \le i \le n)$ are constant complex vectors.

Proof. If we substitute $f(\mathbf{U}/n) = F(\mathbf{U})$ and $F_i(\mathbf{U}) = f_i(\mathbf{U})/n$ $(1 \le i \le n)$ in equation (3.18), we get the equation form (3.1), and after that the formulae (3.19) and (3.20) immediately follow. □

Theorem 3.6 *The general continuous solution of the functional equation*

$$f\Big(\sum_{i=1}^{p} \mathbf{Z}_{1i}, \ldots, \sum_{i=1}^{p} \mathbf{Z}_{ni}\Big) = \sum_{i=1}^{p} f_i(\mathbf{Z}_{1i}, \ldots, \mathbf{Z}_{ni}) \tag{3.21}$$

is given by the formulae

$$f(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = \sum_{i=1}^{n} C_i\mathbf{Z}_i + \sum_{i=1}^{p} \mathbf{A}_i \tag{3.22}$$

and

$$f_i(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = \sum_{j=1}^{n} C_j\mathbf{Z}_j + \mathbf{A}_i \qquad (1 \le i \le p), \tag{3.23}$$

where C_i $(1 \le i \le n)$ are constant complex matrices and $\mathbf{A}_i$ $(1 \le i \le p)$ are constant complex vectors.

Proof. If we put in (3.21)

$$\mathbf{Z}_{12} = \mathbf{Z}_{13} = \cdots = \mathbf{Z}_{1p} = \mathbf{O}, \quad \ldots \quad , \mathbf{Z}_{n2} = \mathbf{Z}_{n3} = \cdots = \mathbf{Z}_{np} = \mathbf{O},$$

and introduce the notations $\mathbf{A}_i = f_i(\mathbf{O}, \mathbf{O}, \ldots, \mathbf{O})$ $(1 \le i \le p)$, we obtain

$$f(\mathbf{Z}_{11}, \mathbf{Z}_{21}, \ldots, \mathbf{Z}_{n1}) = f_1(\mathbf{Z}_{11}, \mathbf{Z}_{21}, \ldots, \mathbf{Z}_{n1}) + \mathbf{A}_2 + \cdots + \mathbf{A}_p.$$

More generally, if for $2 \le i \le p$ we put

$$\mathbf{Z}_{1j} = \mathbf{O}, \quad \ldots \quad \mathbf{Z}_{nj} = \mathbf{O} \quad \text{for} \quad 1 \le j \le p,\ j \ne i,$$

we obtain

$$f(\mathbf{Z}_{1i}, \mathbf{Z}_{2i}, \ldots, \mathbf{Z}_{ni}) = \mathbf{A}_1 + \cdots + \mathbf{A}_{i-1} + f_i(\mathbf{Z}_{1i}, \mathbf{Z}_{2i}, \ldots, \mathbf{Z}_{ni}) + \mathbf{A}_{i+1} + \cdots + \mathbf{A}_p.$$

We can write the above equalities in the form

$$f_i(\mathbf{Z}_{1i}, \mathbf{Z}_{2i}, \ldots, \mathbf{Z}_{ni}) = f(\mathbf{Z}_{1i}, \mathbf{Z}_{2i}, \ldots, \mathbf{Z}_{ni}) + \mathbf{A}_i - \sum_{j=1}^{p} \mathbf{A}_j \quad (1 \le i \le p). \tag{3.24}$$

After a substitution of (3.24) into (3.21), we get

$$f\Big(\sum_{i=1}^{p} \mathbf{Z}_{1i}, \ldots, \sum_{i=1}^{p} \mathbf{Z}_{ni}\Big) = \sum_{i=1}^{p} f(\mathbf{Z}_{1i}, \mathbf{Z}_{2i}, \ldots, \mathbf{Z}_{ni}) - (p-1) \sum_{i=1}^{p} \mathbf{A}_i,$$

or

$$f\Big(\sum_{i=1}^{p} \mathbf{Z}_{1i}, \ldots, \sum_{i=1}^{p} \mathbf{Z}_{ni}\Big) = \sum_{i=1}^{p} f(\mathbf{Z}_{1i}, \mathbf{Z}_{2i}, \ldots, \mathbf{Z}_{ni}) + \mathbf{B}, \tag{3.25}$$

where

$$\mathbf{B} = -(p-1) \sum_{i=1}^{p} \mathbf{A}_i. \tag{3.26}$$

The general continuous solution of the equation (3.25) is

$$f(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) = \sum_{i=1}^{n} C_i \mathbf{Z}_i - \frac{\mathbf{B}}{p-1}. \tag{3.27}$$

According to (3.27), (3.26) and (3.24) there follow the formulae (3.22) and (3.23). □

Theorem 3.7 *The general continuous solution of the functional equation*

$$f\Big(\sum_{i=1}^{p} \mathbf{Z}_{1i}, \ldots, \sum_{i=1}^{p} \mathbf{Z}_{ni}\Big) = \sum_{i=1}^{p} f_i(k\mathbf{Z}_{1i}, \ldots, k\mathbf{Z}_{ni}) \quad (k \ne 0) \tag{3.28}$$

is given by

$$f(\mathbf{Z}_1, \ldots, \mathbf{Z}_n) = \sum_{i=1}^{n} C_i \mathbf{Z}_i + \sum_{i=1}^{p} \mathbf{A}_i \tag{3.29}$$

and

$$f_i(\mathbf{Z}_1,\ldots,\mathbf{Z}_n)=\frac{1}{k}\sum_{j=1}^{n}C_j\mathbf{Z}_j+\mathbf{A}_i \qquad (1\le i\le p), \tag{3.30}$$

where C_i $(1\le i\le n)$ *are constant complex matrices and* $\mathbf{A}_i$ $(1\le i\le p)$ *are constant complex vectors.*

Proof. Analogous to the previous theorem we obtain the following equalities

$$f_i(k\mathbf{Z}_{1i},\ldots,k\mathbf{Z}_{ni})=f(\mathbf{Z}_{1i},\ldots,\mathbf{Z}_{ni})+\mathbf{A}_i-\sum_{j=1}^{p}\mathbf{A}_j \quad (1\le i\le p)$$

which substituted in (3.28) give an equation of the form (3.25) from where immediately follow the formulae (3.29) and (3.30). □

Now, we will give one more general result.

Theorem 3.8 *The complex vector functional equation*

$$f\Big(\sum_{i=1}^{p}\mathbf{Z}_{1i},\ldots,\sum_{i=1}^{p}\mathbf{Z}_{ni}\Big)=\sum_{i=1}^{p}f_i(k_{1i}\mathbf{Z}_{1i},\ldots,k_{ni}\mathbf{Z}_{ni})+\mathbf{B}, \tag{3.31}$$

where $\mathbf{B}$ *is a constant complex vector and* $k_{si}\neq 0$ $(1\le s\le n;\ 1\le i\le p)$, *has a general continuous solution given by the following formulae*

$$f(\mathbf{Z}_1,\ldots,\mathbf{Z}_n)=\sum_{i=1}^{n}C_i\mathbf{Z}_i+\sum_{i=1}^{p}\mathbf{A}_i-\frac{\mathbf{B}}{p-1} \tag{3.32}$$

and

$$f_i(\mathbf{Z}_1,\ldots,\mathbf{Z}_n)=\sum_{s=1}^{n}\frac{1}{k_{si}}C_s\mathbf{Z}_s+\mathbf{A}_i-\frac{\mathbf{B}}{p-1} \qquad (1\le i\le p), \tag{3.33}$$

where C_i $(1\le i\le n)$ *are constant complex matrices and* $\mathbf{A}_i$ $(1\le i\le p)$ *are constant complex vectors.*

Proof. The proof of this theorem is completely the same as the proof of Theorem 3.6. □

Theorem 3.9 *The general continuous solution of the functional equation*

$$\begin{aligned}&f_1(a_1\mathbf{Z}_1+a_2\mathbf{Z}_2,a_3\mathbf{Z}_3)+f_2(a_1\mathbf{Z}_2+a_2\mathbf{Z}_3,a_3\mathbf{Z}_1)+f_3(a_1\mathbf{Z}_3+a_2\mathbf{Z}_1,a_3\mathbf{Z}_2)\\&=a_4f_1(a_1\mathbf{Z}_1+a_2\mathbf{Z}_2,a_3\mathbf{Z}_3)f_2(a_1\mathbf{Z}_2+a_2\mathbf{Z}_3,a_3\mathbf{Z}_1)f_3(a_1\mathbf{Z}_3+a_2\mathbf{Z}_1,a_3\mathbf{Z}_2),\end{aligned} \tag{3.34}$$

where a_i $(i=1,2,3,4)$ *are complex constants such that* $a_1^3=1$, $a_2^3=1$, *and* $a_3a_4\neq 0$, *is given by the following relations*

$$f_1(\mathbf{U}_1,\mathbf{U}_2)=\frac{1}{\sqrt{-a_4}}\frac{C_1C_2\exp\Big(C_3\mathrm{Re}\frac{a_1^2a_2a_3\mathbf{U}_1-2a_1\mathbf{U}_2}{3a_2a_3}+C_4\mathrm{Im}\frac{a_1^2a_2a_3\mathbf{U}_1-2a_1\mathbf{U}_2}{3a_2a_3}\Big)-\mathbf{I}}{C_1C_2\exp\Big(C_3\mathrm{Re}\frac{a_1^2a_2a_3\mathbf{U}_1-2a_1\mathbf{U}_2}{3a_2a_3}+C_4\mathrm{Im}\frac{a_1^2a_2a_3\mathbf{U}_1-2a_1\mathbf{U}_2}{3a_2a_3}\Big)+\mathbf{I}} \tag{3.35}$$

$$\left(C_i \equiv C_i\left(\frac{a_1^2a_2a_3\mathbf{U}_1 + a_1\mathbf{U}_2}{a_2a_3}\right),\ i = 1,2,3,4\right),$$

$$f_2(\mathbf{U}_1,\mathbf{U}_2) = -\frac{1}{\sqrt{-a_4}}\frac{C_1'\exp\left(C_3'\mathrm{Re}\frac{2\mathbf{U}_2 - a_1a_2a_3\mathbf{U}_1}{3a_3} + C_4'\mathrm{Im}\frac{2\mathbf{U}_2 - a_1a_2a_3\mathbf{U}_1}{3a_3}\right) - \mathbf{I}}{C_1'\exp\left(C_3'\mathrm{Re}\frac{2\mathbf{U}_2 - a_1a_2a_3\mathbf{U}_1}{3a_3} + C_4'\mathrm{Im}\frac{2\mathbf{U}_2 - a_1a_2a_3\mathbf{U}_1}{3a_3}\right) + \mathbf{I}} \tag{3.36}$$

$$\left(C_i' \equiv C_i\left(\frac{a_1a_2a_3\mathbf{U}_1 + \mathbf{U}_2}{a_3}\right),\ i = 1,3,4\right),$$

$$f_3(\mathbf{U}_1,\mathbf{U}_2) = -\frac{1}{\sqrt{-a_4}}\frac{\tilde{C}_2\exp\left(\tilde{C}_3\mathrm{Re}\frac{2a_1^2a_2\mathbf{U}_2 - a_2^2a_3\mathbf{U}_1}{3a_3} + \tilde{C}_4\mathrm{Im}\frac{2a_1^2a_2\mathbf{U}_2 - a_2^2a_3\mathbf{U}_1}{3a_3}\right) - \mathbf{I}}{\tilde{C}_2\exp\left(\tilde{C}_3\mathrm{Re}\frac{2a_1^2a_2\mathbf{U}_2 - a_2^2a_3\mathbf{U}_1}{3a_3} + \tilde{C}_4\mathrm{Im}\frac{2a_1^2a_2\mathbf{U}_2 - a_2^2a_3\mathbf{U}_1}{3a_3}\right) + \mathbf{I}} \tag{3.37}$$

$$\left(\tilde{C}_i \equiv C_i\left(\frac{a_2^2a_3\mathbf{U}_1 + a_1^2a_2\mathbf{U}_2}{a_3}\right),\ i = 2,3,4\right),$$

where C_i $(i = 1,2,3,4)$ are arbitrary functions $\mathcal{V} \mapsto \mathbb{C}$. However, some components of f_i $(i = 1,2,3)$ may be given by one of the following relations:

$$f_1(\mathbf{U}_1,\mathbf{U}_2) = \mp\frac{1}{\sqrt{-a_4}},\quad f_2(\mathbf{U}_1,\mathbf{U}_2) = \pm\frac{1}{\sqrt{-a_4}},\quad f_3(\mathbf{U}_1,\mathbf{U}_2)\ \textit{arbitrary}, \tag{3.38}$$

or

$$f_1(\mathbf{U}_1,\mathbf{U}_2) = \mp\frac{1}{\sqrt{-a_4}},\quad f_2(\mathbf{U}_1,\mathbf{U}_2)\ \textit{arbitrary},\quad f_3(\mathbf{U}_1,\mathbf{U}_2) = \pm\frac{1}{\sqrt{-a_4}}, \tag{3.39}$$

or

$$\begin{aligned} f_1(\mathbf{U}_1,\mathbf{U}_2) &= \frac{1}{\sqrt{-a_4}}, \\ f_2(\mathbf{U}_1,\mathbf{U}_2) &= -\frac{1}{\sqrt{-a_4}}\left(1 + P_1\left(\frac{2\mathbf{U}_2 - a_1a_2a_3\mathbf{U}_1}{3a_3}, \frac{a_1a_2a_3\mathbf{U}_1 + \mathbf{U}_2}{a_3}\right)\right) \\ f_3(\mathbf{U}_1,\mathbf{U}_2) &= -\frac{1}{\sqrt{-a_4}}\left(1 + P_2\left(\frac{2a_1^2a_2\mathbf{U}_2 - a_2^2a_3\mathbf{U}_1}{3a_3}, \frac{a_2^2a_3\mathbf{U}_1 + a_1^2a_2\mathbf{U}_2}{a_3}\right)\right), \end{aligned} \tag{3.40}$$

where $P_1, P_2 :\ \mathcal{V}^2 \mapsto \mathbb{C}$ are such that

$$\mathrm{supp}_2 P_1 \cap \mathrm{supp}_2 P_2 = \emptyset \tag{3.41}$$

with

$$\mathrm{supp}_2 P_i = \{\mathbf{V} \in \mathcal{V} |\ \exists \mathbf{U} \in \mathcal{V} :\ P_i(\mathbf{U},\mathbf{V}) \neq 0\},\quad i = 1,2.$$

Proof. If we introduce the substitutions

$$\mathbf{Z}_1 = \mathbf{U}_1 + \mathbf{U}_3/3,\qquad \mathbf{Z}_2 = a_1a_2^2(\mathbf{U}_2 + \mathbf{U}_3/3),\qquad \mathbf{Z}_3 = a_1^2a_2(-\mathbf{U}_1 - \mathbf{U}_2 + \mathbf{U}_3/3),$$

then from the equation (3.34) we obtain

$$\begin{aligned}
& f_1\left(a_1\Big(\mathbf{U}_1+\mathbf{U}_2+\frac{2}{3}\mathbf{U}_3\Big), a_1^2a_2a_3\Big(-\mathbf{U}_1-\mathbf{U}_2+\frac{\mathbf{U}_3}{3}\Big)\right)\\
+\ & f_2\left(a_1^2a_2^2\Big(\frac{2}{3}\mathbf{U}_3-\mathbf{U}_1\Big), a_3\Big(\frac{\mathbf{U}_3}{3}+\mathbf{U}_1\Big)\right)+f_3\left(a_2\Big(\frac{2}{3}\mathbf{U}_3-\mathbf{U}_2\Big), a_1a_2^2a_3\Big(\mathbf{U}_2+\frac{\mathbf{U}_3}{3}\Big)\right)\\
=\ & a_4 f_1\left(a_1\Big(\mathbf{U}_1+\mathbf{U}_2+\frac{2}{3}\mathbf{U}_3\Big), a_1^2a_2a_3\Big(-\mathbf{U}_1-\mathbf{U}_2+\frac{\mathbf{U}_3}{3}\Big)\right)\\
\times\ & f_2\left(a_1^2a_2^2\Big(\frac{2}{3}\mathbf{U}_3-\mathbf{U}_1\Big), a_3\Big(\frac{\mathbf{U}_3}{3}+\mathbf{U}_1\Big)\right)\cdot f_3\left(a_2\Big(\frac{2}{3}\mathbf{U}_3-\mathbf{U}_2\Big), a_1a_2^2a_3\Big(\mathbf{U}_2+\frac{\mathbf{U}_3}{3}\Big)\right)
\end{aligned} \tag{3.42}$$

By introducing new functions g_1, g_2, g_3 by the formulae

$$\begin{aligned}
& f_1(a_1\mathbf{U}_1, a_1^2a_2a_3\mathbf{U}_2)=\frac{1}{\sqrt{-a_4}}g_1\Big(\frac{\mathbf{U}_1-2\mathbf{U}_2}{3}, \mathbf{U}_1+\mathbf{U}_2\Big)\iff\\
& g_1(\mathbf{U}_1,\mathbf{U}_2)=\sqrt{-a_4}f_1\left(a_1\Big(\frac{2}{3}\mathbf{U}_2+\mathbf{U}_1\Big), a_1^2a_2a_3\Big(\frac{\mathbf{U}_2}{3}-\mathbf{U}_1\Big)\right),\\
& f_2(a_1^2a_2^2\mathbf{U}_1, a_3\mathbf{U}_2)=-\frac{1}{\sqrt{-a_4}}g_2\Big(\frac{2\mathbf{U}_2-\mathbf{U}_1}{3}, \mathbf{U}_1+\mathbf{U}_2\Big)\iff\\
& g_2(\mathbf{U}_1,\mathbf{U}_2)=-\sqrt{-a_4}f_2\left(a_1^2a_2^2\Big(\frac{2}{3}\mathbf{U}_2-\mathbf{U}_1\Big), a_3\Big(\frac{\mathbf{U}_2}{3}+\mathbf{U}_1\Big)\right),
\end{aligned} \tag{3.43}$$

$$\begin{aligned}
& f_3(a_2\mathbf{U}_1, a_1a_2^2a_3\mathbf{U}_2)=-\frac{1}{\sqrt{-a_4}}f_3\Big(\frac{2\mathbf{U}_2-\mathbf{U}_1}{3}, \mathbf{U}_1+\mathbf{U}_2\Big)\iff\\
& g_3(\mathbf{U}_1,\mathbf{U}_2)=-\sqrt{-a_4}f_3\left(a_2\Big(\frac{2}{3}\mathbf{U}_2-\mathbf{U}_1\Big), a_1a_2^2a_3\Big(\frac{\mathbf{U}_2}{3}+\mathbf{U}_1\Big)\right),
\end{aligned}$$

the functional equation (3.42) takes the following form

$$g_1(\mathbf{U}_1+\mathbf{U}_2,\mathbf{U}_3)=\frac{g_2(\mathbf{U}_1,\mathbf{U}_3)+g_3(\mathbf{U}_2,\mathbf{U}_3)}{\mathbf{I}+g_2(\mathbf{U}_1,\mathbf{U}_3)g_3(\mathbf{U}_2,\mathbf{U}_3)}. \tag{3.44}$$

Now we will distinguish the following cases:

1^0 Let $g_i(\mathbf{U}_1,\mathbf{U}_2)\not\equiv\mathbf{I}$ $(i=1,2,3)$. For simplicity of notation we assume that all components of g_1, g_2, g_3 are not 1. If this is not the case, we obtain the formulae for the respective components in 2^0-4^0.

Introducing new functions h_i $(i=1,2,3)$ given by

$$g_i(\mathbf{U}_1,\mathbf{U}_3)=\frac{h_i(\mathbf{U}_1,\mathbf{U}_3)-\mathbf{I}}{h_i(\mathbf{U}_1,\mathbf{U}_3)+\mathbf{I}}\iff h_i(\mathbf{U}_1,\mathbf{U}_3)=\frac{\mathbf{I}+g_i(\mathbf{U}_1,\mathbf{U}_3)}{\mathbf{I}-g_i(\mathbf{U}_1,\mathbf{U}_3)}\quad(i=1,2,3), \tag{3.45}$$

the equation (3.44) reduces to the following form

$$h_1(\mathbf{U}_1+\mathbf{U}_2,\mathbf{U}_3)=h_2(\mathbf{U}_1,\mathbf{U}_3)h_3(\mathbf{U}_2,\mathbf{U}_3),$$

whose general continuous solution is

$$\begin{aligned} h_1(\mathbf{U}_1,\mathbf{U}_3) &= C_1(\mathbf{U}_3)C_2(\mathbf{U}_3)\exp\big[C_3(\mathbf{U}_3)\mathrm{Re}\mathbf{U}_1 + C_4(\mathbf{U}_3)\mathrm{Im}\mathbf{U}_1\big], \\ h_i(\mathbf{U}_1,\mathbf{U}_3) &= C_{i-1}(\mathbf{U}_3)\exp\big[C_3(\mathbf{U}_3)\mathrm{Re}\mathbf{U}_1 + C_4(\mathbf{U}_3)\mathrm{Im}\mathbf{U}_1\big] \quad (i=2,3), \end{aligned} \tag{3.46}$$

where C_i $(\mathcal{V} \mapsto \mathbb{C})$ are arbitrary functions. However, some components may satisfy

$$h_1 \equiv \mathbf{O}, \qquad h_2 \equiv \mathbf{O}, \qquad h_3 \ \text{arbitrary}, \tag{3.47}$$

or

$$h_1 \equiv \mathbf{O}, \qquad h_2 \ \text{arbitrary}, \qquad h_3 \equiv \mathbf{O}. \tag{3.48}$$

Therefore, according to (3.46), (3.47), (3.48), (3.45) and (3.43) we obtain (3.35)–(3.37), or (3.38), or (3.39) (with the upper signs).

2^0 Let $g_3 \equiv 1$. Then $g_1 \equiv 1$, g_2 can be arbitrary and we obtain (3.39) with the lower sign.

3^0 Let $g_2 \equiv 1$. Then $g_1 \equiv 1$, g_3 can be arbitrary and we obtain (3.38) with the lower sign.

4^0 Let $g_1 \equiv 1$. Then we obtain the equation

$$\big(g_2(\mathbf{U}_1,\mathbf{U}_3) - 1\big)\big(g_3(\mathbf{U}_2,\mathbf{U}_3) - 1\big) = 0.$$

Its general solution is

$$g_2(\mathbf{U}_1,\mathbf{U}_3) = 1 + P_1(\mathbf{U}_1,\mathbf{U}_3), \qquad g_3(\mathbf{U}_2,\mathbf{U}_3) = 1 + P_2(\mathbf{U}_2,\mathbf{U}_3),$$

where the functions $P_1, P_2 : \mathcal{V}^2 \mapsto \mathbb{C}$ satisfy (3.41). This leads to (3.40) and the proof is complete. □

Next we will consider some other complex vector functional equations, which will be interesting for the enrichment of this relatively new theory.

Theorem 3.10 *The general solution of the complex vector functional equation*

$$f(\mathbf{Z}_1\mathbf{Z}_2) = \mathbf{Z}_1^{\alpha_1} f(\mathbf{Z}_2) + \mathbf{Z}_2^{\alpha_2} f(\mathbf{Z}_1) + \mathbf{A}\mathbf{Z}_1^{\alpha_1}\mathbf{Z}_2^{\alpha_2}, \tag{3.49}$$

where $\mathbf{A}$ *is a constant complex vector and* $\alpha_1,\ \alpha_2 \in \mathbb{N}$, $\alpha_1 \neq \alpha_2$, *is given by the formula*

$$f(\mathbf{U}) = \mathbf{B}\left(\mathbf{U}^{\alpha_2} - \mathbf{U}^{\alpha_1}\right) - \mathbf{A}\mathbf{U}^{\alpha_2}, \tag{3.50}$$

where $\mathbf{B}$ *is an arbitrary constant complex vector.*

Proof. Since $\alpha_1 \neq \alpha_2$, we can choose a vector $\mathbf{C}$ so that all components of $\mathbf{C}^{\alpha_2} - \mathbf{C}^{\alpha_1}$ are nonzero. Setting $\mathbf{Z}_1 = \mathbf{U}$, $\mathbf{Z}_2 = \mathbf{C}$ and using $f(\mathbf{UC}) = f(\mathbf{CU})$, we deduce the relation

$$\mathbf{U}^{\alpha_1} f(\mathbf{C}) + \mathbf{C}^{\alpha_2} f(\mathbf{U}) + \mathbf{A}\mathbf{U}^{\alpha_1}\mathbf{C}^{\alpha_2} = \mathbf{C}^{\alpha_1} f(\mathbf{U}) + \mathbf{U}^{\alpha_2} f(\mathbf{C}) + \mathbf{A}\mathbf{C}^{\alpha_1}\mathbf{U}^{\alpha_2}$$

or

$$f(\mathbf{U}) = \frac{f(\mathbf{C})}{\mathbf{C}^{\alpha_2} - \mathbf{C}^{\alpha_1}}\left(\mathbf{U}^{\alpha_2} - \mathbf{U}^{\alpha_1}\right) + \frac{\mathbf{A}\mathbf{C}^{\alpha_2}}{\mathbf{C}^{\alpha_2} - \mathbf{C}^{\alpha_1}}\left(\mathbf{U}^{\alpha_2} - \mathbf{U}^{\alpha_1}\right) - \mathbf{A}\mathbf{U}^{\alpha_2}.$$

Therefore the general solution of the functional equation (3.49) is (3.50) with

$$\mathbf{B} = \frac{f(\mathbf{C}) + \mathbf{A}\mathbf{C}^{\alpha_2}}{\mathbf{C}^{\alpha_2} - \mathbf{C}^{\alpha_1}}. \qquad \square$$

Now we shall find the general solution of equation (3.49) in the case $\alpha_1 = \alpha_2$.

Theorem 3.11 *The general solution of the complex vector functional equation*

$$f(\mathbf{Z}_1\mathbf{Z}_2) = \mathbf{Z}_1^\alpha f(\mathbf{Z}_2) + \mathbf{Z}_2^\alpha f(\mathbf{Z}_1) + \mathbf{A}(\mathbf{Z}_1\mathbf{Z}_2)^\alpha, \tag{3.51}$$

where $\mathbf{A}$ *is a constant complex vector and* $\alpha \in \mathbb{N}$*, is given by the formula*

$$f(\mathbf{U}) = \mathbf{U}^\alpha (\mathbf{B} \ln |\mathbf{U}| - \mathbf{A}), \tag{3.52}$$

where $\mathbf{B}$ *is an arbitrary constant complex vector and* $\ln |\mathbf{U}| := (\ln |u_1|, \dots, \ln |u_n|)^T$ *with the convention that if* $u_k = 0$ *for some* $k \in \{1, \dots, n\}$*, then the corresponding component of* $\ln |\mathbf{U}|$ *is assumed* 0.

Proof. We can write equation (3.51) in the form

$$f(\mathbf{Z}_1\mathbf{Z}_2) + \mathbf{A}(\mathbf{Z}_1\mathbf{Z}_2)^\alpha = \mathbf{Z}_1^\alpha [f(\mathbf{Z}_2) + \mathbf{A}\mathbf{Z}_2^\alpha] + \mathbf{Z}_2^\alpha [f(\mathbf{Z}_1) + \mathbf{A}\mathbf{Z}_1^\alpha].$$

Clearly, the function $g(\mathbf{U}) := f(\mathbf{U}) + \mathbf{A}\mathbf{U}^\alpha$ satisfies

$$g(\mathbf{Z}_1\mathbf{Z}_2) = \mathbf{Z}_1^\alpha g(\mathbf{Z}_2) + \mathbf{Z}_2^\alpha g(\mathbf{Z}_1).$$

In particular, we have $g(\mathbf{O}) = g(\mathbf{I}) = \mathbf{O}$. Further on, for $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{U}$ we obtain

$$g(\mathbf{U}^2) = 2\mathbf{U}^\alpha g(\mathbf{U}).$$

For $\mathbf{U}$ with nonzero components we have

$$\frac{g(\mathbf{U}^2)}{\mathbf{U}^{2\alpha}} = 2\frac{g(\mathbf{U})}{\mathbf{U}^\alpha}. \tag{3.53}$$

In general, for all nonzero components of $\mathbf{U}$ the corresponding components of g satisfy an equation of the form (3.53).

Thus, $h(\mathbf{U}) := g(\mathbf{U})/\mathbf{U}^\alpha$ satisfies

$$h(\mathbf{U}^2) = 2h(\mathbf{U}).$$

The general solution of this equation is $h(\mathbf{U}) = \mathbf{B} \ln |\mathbf{U}|$, where $\mathbf{B}$ is an arbitrary vector and $\ln |\mathbf{U}|$ was defined in the statement of the theorem. Now we derive successively $g(\mathbf{U}) = \mathbf{B}\mathbf{U}^\alpha \ln |\mathbf{U}|$ and (3.52). A straightforward verification shows that (3.52) satisfies equation (3.51). $\square$

Theorem 3.12 *The functional equation*

$$f(\mathbf{Z}_1\mathbf{Z}_2) = \mathbf{Z}_1^{\alpha_1} g(\mathbf{Z}_2) + \mathbf{Z}_2^{\alpha_2} f(\mathbf{Z}_1) \qquad (\alpha_1,\ \alpha_2 \in \mathbb{N}) \tag{3.54}$$

has a general solution given by

$$\begin{aligned} f(\mathbf{U}) &= \mathbf{B}\,(\mathbf{U}^{\alpha_2} - \mathbf{U}^{\alpha_1}) + \mathbf{C}\mathbf{U}^{\alpha_2}, \\ g(\mathbf{U}) &= \mathbf{B}\,(\mathbf{U}^{\alpha_2} - \mathbf{U}^{\alpha_1}) \quad \text{if} \quad \alpha_1 \neq \alpha_2 \end{aligned} \tag{3.55}$$

or

$$\begin{aligned} f(\mathbf{U}) &= \mathbf{U}^{\alpha}\,(\mathbf{B}\ln|\mathbf{U}| + \mathbf{C}), \\ g(\mathbf{U}) &= \mathbf{B}\ln|\mathbf{U}| \quad \text{if} \quad \alpha_1 = \alpha_2 = \alpha, \end{aligned} \tag{3.56}$$

where $\mathbf{B}$ *and* $\mathbf{C} = f(\mathbf{I})$ *are arbitrary constant complex vectors and* $\ln|\mathbf{U}|$ *was defined in Theorem 3.11.*

Proof. By the substitution of $\mathbf{Z}_1 = \mathbf{I}$ into (3.54) we obtain

$$g(\mathbf{Z}_2) = f(\mathbf{Z}_2) - f(\mathbf{I})\mathbf{Z}_2^{\alpha_2}. \tag{3.57}$$

Using (3.54) and (3.57), we get

$$f(\mathbf{Z}_1\mathbf{Z}_2) = \mathbf{Z}_1^{\alpha_1} f(\mathbf{Z}_2) + \mathbf{Z}_2^{\alpha_2} f(\mathbf{Z}_1) - f(\mathbf{I})\mathbf{Z}_1^{\alpha_1}\mathbf{Z}_2^{\alpha_2}.$$

This is an equation of the form (3.49). In the case $\alpha_1 \neq \alpha_2$ we apply Theorem 3.10 and obtain (3.55). In the case $\alpha_1 = \alpha_2$ we apply Theorem 3.11 and obtain (3.56). □

Theorem 3.13 *The general solution of the functional equation*

$$f(\mathbf{Z}_1\mathbf{Z}_2) = \mathbf{Z}_1^{\alpha_1} h(\mathbf{Z}_2) + \mathbf{Z}_2^{\alpha_2} g(\mathbf{Z}_1) \qquad (\alpha_1,\ \alpha_2 \in \mathbb{N}) \tag{3.58}$$

is given by the following formulae

$$\begin{aligned} f(\mathbf{U}) &= \mathbf{B}\,(\mathbf{U}^{\alpha_2} - \mathbf{U}^{\alpha_1}) + \mathbf{C}\mathbf{U}^{\alpha_2}, \\ g(\mathbf{U}) &= \mathbf{B}\,(\mathbf{U}^{\alpha_2} - \mathbf{U}^{\alpha_1}) + \mathbf{C}\mathbf{U}^{\alpha_2} - \mathbf{D}\mathbf{U}^{\alpha_1}, \\ h(\mathbf{U}) &= \mathbf{B}\,(\mathbf{U}^{\alpha_2} - \mathbf{U}^{\alpha_1}) + \mathbf{D}\mathbf{U}^{\alpha_2} \quad \text{if} \ \alpha_1 \neq \alpha_2 \end{aligned} \tag{3.59}$$

or

$$\begin{aligned} f(\mathbf{U}) &= \mathbf{U}^{\alpha}\,(\mathbf{B}\ln|\mathbf{U}| + \mathbf{C}), \\ g(\mathbf{U}) &= \mathbf{U}^{\alpha}\,(\mathbf{B}\ln|\mathbf{U}| + \mathbf{C} - \mathbf{D}), \\ h(\mathbf{U}) &= \mathbf{U}^{\alpha}\,(\mathbf{B}\ln|\mathbf{U}| + \mathbf{D}) \quad \text{if} \ \alpha_1 = \alpha_2 = \alpha, \end{aligned} \tag{3.60}$$

where $\mathbf{B}$, $\mathbf{C} = f(\mathbf{I})$ *and* $\mathbf{D} = h(\mathbf{I})$ *are constant complex vectors and* $\ln|\mathbf{U}|$ *is as in Theorem 3.11.*

Proof. If we put $\mathbf{Z}_1 = \mathbf{I}$ in (3.58), there follows the relation

$$f(\mathbf{Z}_2) = h(\mathbf{Z}_2) + g(\mathbf{I})\mathbf{Z}_2^{\alpha_2}. \tag{3.61}$$

Analogously, for $\mathbf{Z}_2 = \mathbf{I}$ we obtain

$$f(\mathbf{Z}_1) = \mathbf{Z}_1^{\alpha_1} h(\mathbf{I}) + g(\mathbf{Z}_1). \tag{3.62}$$

Finally, for $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{I}$, we have $f(\mathbf{I}) = g(\mathbf{I}) + h(\mathbf{I})$. By substituting $h(\mathbf{Z}_2)$ from (3.61) and $g(\mathbf{Z}_1)$ from (3.62) into (3.58), we get

$$f(\mathbf{Z}_1\mathbf{Z}_2) = \mathbf{Z}_1^{\alpha_1} f(\mathbf{Z}_2) + \mathbf{Z}_2^{\alpha_2} f(\mathbf{Z}_1) - f(\mathbf{I})\mathbf{Z}_1^{\alpha_1}\mathbf{Z}_2^{\alpha_2}.$$

To this equation we can apply Theorem 3.10 for $\alpha_1 \neq \alpha_2$ or Theorem 3.11 for $\alpha_1 = \alpha_2$ to obtain respectively the formulae (3.59) or (3.60). □

Theorem 3.14 *The general solution of the functional equation*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(\mathbf{U}_i\mathbf{V}_j) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\mathbf{U}_i^{\alpha_1} f(\mathbf{V}_j) + \mathbf{V}_j^{\alpha_2} f(\mathbf{U}_i) + \mathbf{A}\mathbf{U}_i^{\alpha_1}\mathbf{V}_j^{\alpha_2}\right], \tag{3.63}$$

where $\mathbf{A}$ *is a constant complex vector and* $\alpha_1,\ \alpha_2 \in \mathbb{N}$, *is given by the formula* (3.50) *for* $\alpha_1 \neq \alpha_2$ *and by the formula* (3.52) *for* $\alpha_1 = \alpha_2$.

Proof. Equation (3.63) can be obtained by summation of equations of the form (3.49) for each pair of variables $(\mathbf{U}_i, \mathbf{V}_j)$, so the formula (3.50) for $\alpha_1 \neq \alpha_2$ and (3.52) for $\alpha_1 = \alpha_2$ give a solution of equation (3.63). It remains to show that each solution of equation (3.63) has the form given by (3.50) for $\alpha_1 \neq \alpha_2$ and (3.52) for $\alpha_1 = \alpha_2$.

First we see that $f(\mathbf{O}) = \mathbf{O}$. Next we put into (3.63) $\mathbf{U}_1 = \mathbf{Z}_1$, $\mathbf{V}_1 = \mathbf{Z}_2$, $\mathbf{U}_i = \mathbf{V}_j = \mathbf{O}$ for $2 \leq i \leq m$, $2 \leq j \leq n$ and we obtain (3.49). Thus any solution of (3.63) must be given by (3.50) or (3.52). □

Theorem 3.15 *The general solution of the functional equation*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(\mathbf{U}_i\mathbf{V}_j) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\mathbf{U}_i^{\alpha_1} g(\mathbf{V}_j) + \mathbf{V}_j^{\alpha_2} f(\mathbf{U}_i)\right] \qquad (\alpha_1,\ \alpha_2 \in \mathbb{N})$$

is given by the formula (3.55) *for* $\alpha_1 \neq \alpha_2$ *and by the formula* (3.56) *for* $\alpha_1 = \alpha_2$.

Proof. The proof of this theorem is analogous to that of the previous Theorem 3.14. □

Theorem 3.16 *The general solution of the functional equation*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(\mathbf{U}_i\mathbf{V}_j) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\mathbf{U}_i^{\alpha_1} h(\mathbf{V}_j) + \mathbf{V}_j^{\alpha_2} g(\mathbf{U}_i)\right] \qquad (\alpha_1,\ \alpha_2 \in \mathbb{N})$$

is given by the formula (3.59) *for* $\alpha_1 \neq \alpha_2$ *and by the formula* (3.60) *for* $\alpha_1 = \alpha_2$.

Proof. For the proof of this theorem holds the same as for the previous Theorem 3.15. □

Theorem 3.17 *The general continuous solution of the complex vector functional equation*

$$f\left(\prod_{i=1}^{n}\mathbf{Z}_i\right)=\sum_{i=1}^{n}f_i(\mathbf{Z}_i), \tag{3.64}$$

where f and f_i $(1 \le i \le n)$ map the subset of $\mathcal{V}$ consisting of all vectors with nonzero components into $\mathcal{V}$, is given by

$$f(\mathbf{Z})=C\ln|\mathbf{Z}|+\sum_{i=1}^{n}\mathbf{A}_i, \qquad f_i(\mathbf{Z})=C\ln|\mathbf{Z}|+\mathbf{A}_i \quad (1\le i\le n), \tag{3.65}$$

where C is a constant complex matrix, $\mathbf{A}_i$ $(1 \le i \le n)$ are arbitrary constant complex vectors, and $\ln|\mathbf{Z}| = (\ln|z_1|, \ln|z_1|, \dots, \ln|z_n|)^T$ *for* $\mathbf{Z} = (z_1, z_2, \dots, z_n)^T$.

Proof. First we shall prove the theorem under the assumption that $\mathbf{Z}_i$ $(1 \le i \le n)$ are real vectors with positive components.

By the identity $\mathbf{Z} = \exp(\ln \mathbf{Z})$, from (3.64) it follows that

$$f\left(\exp\sum_{i=1}^{n}\ln\mathbf{Z}_i\right)=\sum_{i=1}^{n}f_i\big(\exp(\ln\mathbf{Z}_i)\big).$$

If we introduce the notations

$$\ln\mathbf{Z}_i=\mathbf{U}_i, \qquad f(\exp(\mathbf{U}))=g(\mathbf{U}), \quad f_i(\exp(\mathbf{U}))=g_i(\mathbf{U}) \quad (1\le i\le n),$$

we obtain the equation

$$g\left(\sum_{i=1}^{n}\mathbf{U}_i\right)=\sum_{i=1}^{n}g_i(\mathbf{U}_i).$$

According to Theorem 3.1, the general continuous solution of this equation is

$$g(\mathbf{U})=C\mathbf{U}+\sum_{i=1}^{n}\mathbf{A}_i, \qquad g_i(\mathbf{U})=C\mathbf{U}+\mathbf{A}_i \quad (1\le i\le n),$$

where C is a constant complex matrix, and $\mathbf{A}_i$ $(1 \le i \le n)$ are arbitrary constant complex vectors. Thus

$$f(\mathbf{Z})=C\ln\mathbf{Z}+\sum_{i=1}^{n}\mathbf{A}_i, \qquad f_i(\mathbf{Z})=C\ln\mathbf{Z}+\mathbf{A}_i \quad (1\le i\le n), \tag{3.66}$$

where $\mathbf{Z}$ is a real vector with positive components.

From (3.66) it follows that

$$f_i(\mathbf{I})=\mathbf{A}_i \quad (1\le i\le n), \qquad f(\mathbf{I})=\sum_{i=1}^{n}\mathbf{A}_i.$$

Now we shall prove (3.65) for all complex vectors with nonzero components.

If in (3.64) we put $\mathbf{Z}_i = \mathbf{Z}$, $\mathbf{Z}_j = \mathbf{I}$ for $j \neq i$, we obtain

$$f_i(\mathbf{Z}) = f(\mathbf{Z}) - \sum_{j\neq i} \mathbf{A}_j \qquad (1 \leq i \leq n). \tag{3.67}$$

Thus (3.64) can be transformed into a functional equation with one unknown function f:

$$f\left(\prod_{i=1}^{n} \mathbf{Z}_i\right) = \sum_{i=1}^{n} f(\mathbf{Z}_i) - (n-1)\sum_{j=1}^{n} \mathbf{A}_j. \tag{3.68}$$

Let us denote $\omega_n = \exp\left(\frac{2\pi i}{n}\right)$. Then $\omega_n^n = 1$, $\omega_n \neq 1$. If we put in (3.68) $\mathbf{Z}_1 = \cdots = \mathbf{Z}_n = \omega_n \mathbf{I}$, we obtain

$$f(\mathbf{I}) = nf(\omega_n \mathbf{I}) - (n-1)\sum_{j=1}^{n} \mathbf{A}_j,$$

i.e., $nf(\omega_n \mathbf{I}) = f(\mathbf{I}) + (n-1)f(\mathbf{I}) = nf(\mathbf{I})$ and $f(\omega_n \mathbf{I}) = f(\mathbf{I})$.

Next, for any k $(1 \leq k < n)$ we put in (3.68) $\mathbf{Z}_1 = \cdots = \mathbf{Z}_k = \omega_n \mathbf{I}$, $\mathbf{Z}_{k+1} = \cdots = \mathbf{Z}_n = \mathbf{I}$ to obtain $f(\omega_n^k \mathbf{I}) = f(\mathbf{I})$.

Now let us consider a complex vector $\boldsymbol{\omega}$ with $|\boldsymbol{\omega}| = \mathbf{I}$. Any component of $\boldsymbol{\omega}$ has the form $\exp(2\pi i r)$, $0 \leq r < 1$. We can represent the real number r in the form

$$r = \sum_{k=1}^{\infty} r_k n^{-k}, \qquad 0 \leq r_k < n. \tag{3.69}$$

Then $f(\boldsymbol{\omega}) = f(\mathbf{I})$. This equality is easy to see for those components for which the sum (3.69) is finite. For an infinite sum we apply the continuity of f.

Finally, for a complex vector $\mathbf{Z}$ with nonzero components we have $\mathbf{Z} = |\mathbf{Z}|\boldsymbol{\omega}$ with $|\boldsymbol{\omega}| = \mathbf{I}$ and

$$f(\mathbf{Z}) = f(|\mathbf{Z}|) + f(\boldsymbol{\omega}) + (n-2)f(\mathbf{I}) - (n-1)f(\mathbf{I}),$$

i.e., $f(\mathbf{Z}) = f(|\mathbf{Z}|)$. Now from (3.67) we find $f_i(\mathbf{Z}) = f_i(|\mathbf{Z}|)$ $(1 \leq i \leq n)$ and this implies (3.65). □

Theorem 3.18 *The general continuous solution of the functional equation*

$$f\left(\prod_{i=1}^{n} \mathbf{Z}_{i1}, \ldots, \prod_{i=1}^{n} \mathbf{Z}_{im}\right) = \sum_{j=1}^{n} f_j(\mathbf{Z}_{j1}, \ldots, \mathbf{Z}_{jm}) + \mathbf{A},$$

where $m \geq 1$, $n \geq 2$, $\mathbf{Z}_{rj}$ $(1 \leq r \leq n;\ 1 \leq j \leq m)$ *are vectors with nonzero components and* $\mathbf{A}$ *is a constant complex vector, is given by*

$$\begin{aligned} f(\mathbf{Z}_{r1}, \ldots, \mathbf{Z}_{rm}) &= \sum_{j=1}^{m} C_j \ln|\mathbf{Z}_{rj}| + \sum_{j=1}^{n} \mathbf{D}_j - \frac{\mathbf{A}}{n-1}, \\ f_r(\mathbf{Z}_{r1}, \ldots, \mathbf{Z}_{rm}) &= \sum_{j=1}^{m} C_j \ln|\mathbf{Z}_{rj}| + \mathbf{D}_r - \frac{\mathbf{A}}{n-1} \quad (1 \leq r \leq n), \end{aligned}$$

where C_j *are arbitrary constant complex matrices and* $\mathbf{D}_j$ *are arbitrary constant complex vectors.*

Proof. The proof is similar to the proof of the previous theorem. □

Theorem 3.19 *The general continuous solution of the functional equation*

$$f\left(\prod_{i=1}^{n}\mathbf{Z}_{i1},\ldots,\prod_{i=1}^{n}\mathbf{Z}_{im}\right)=\prod_{j=1}^{n}f_j(\mathbf{Z}_{j1},\ldots,\mathbf{Z}_{jm}),$$

where $m \geq 1$, $n \geq 2$, $\mathbf{Z}_{rj}$ $(1 \leq r \leq n;\ 1 \leq j \leq m)$ *are vectors with nonzero components, is given by*

$$\begin{aligned} f(\mathbf{Z}_{r1},\ldots,\mathbf{Z}_{rm}) &= \mathbf{C}\prod_{j=1}^{m}\mathbf{Z}_{rj}^{\alpha}, \\ f_r(\mathbf{Z}_{r1},\ldots,\mathbf{Z}_{rm}) &= \mathbf{C}^{1/n}\prod_{j=1}^{m}\mathbf{Z}_{rj}^{\alpha} \quad (1 \leq r \leq n), \end{aligned}$$

where $\mathbf{C}$ *is an arbitrary constant complex vector and* α *is an arbitary integer number, or*

$$\begin{aligned} f(\mathbf{Z}_{r1},\ldots,\mathbf{Z}_{rm}) &= \mathbf{C}\prod_{j=1}^{m}|\mathbf{Z}_{rj}|^{\alpha}, \\ f_r(\mathbf{Z}_{r1},\ldots,\mathbf{Z}_{rm}) &= \mathbf{C}^{1/n}\prod_{j=1}^{m}|\mathbf{Z}_{rj}|^{\alpha} \quad (1 \leq r \leq n), \end{aligned}$$

where $\mathbf{C}$ *is as above and* α *is an arbitrary complex number.*

Proof. It is similar to that of the previous theorem. □

Theorem 3.20 *The general continuous solution of the equation*

$$f\left(\sum_{i=1}^{p}\mathbf{Z}_i+n\sum_{1\leq i<j\leq p}\mathbf{Z}_i\mathbf{Z}_j+n^2\sum_{1\leq i<j<k\leq p}\mathbf{Z}_i\mathbf{Z}_j\mathbf{Z}_k+\cdots+n^{p-1}\prod_{i=1}^{p}\mathbf{Z}_i\right)=\sum_{i=1}^{p}f_i(\mathbf{Z}_i),$$

where the complex vectors $\mathbf{Z}_i$ $(1 \leq i \leq p)$ *and the complex constant* n *are such that* $\mathbf{Z}_i + n\mathbf{I}$ *are vectors with nonzero components, is given by*

$$f(\mathbf{Z}) = K\ln|\mathbf{I}+n\mathbf{Z}| + \sum_{i=1}^{p}\mathbf{C}_i, \qquad f_i(\mathbf{Z}) = K\ln|\mathbf{I}+n\mathbf{Z}| + \mathbf{C}_i \quad (1 \leq i \leq p),$$

where K *is an arbitrary constant complex matrix and* $\mathbf{C}_i$ $(1 \leq i \leq p)$ *are arbitrary constant complex vectors.*

Proof. The proof of this theorem is similar to that of Theorem 3.1. Namely, we put $p-2$ of the variables to be equal to the zero vector, etc. □

4 Some quadratic complex vector functional equations

Here we will first consider a homogeneous quadratic complex vector functional equation of the form

$$\begin{aligned} &\mathbf{a}_1 f(\mathbf{Z}_1)g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+\mathbf{a}_2 f(\mathbf{Z}_2)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1)\\ +\;&\mathbf{a}_3 f(\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2)+\mathbf{a}_4 f(\mathbf{Z}_4)g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=\mathbf{O}, \end{aligned} \tag{4.1}$$

where $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ are constant complex vectors, $f:\ \mathcal{V}\mapsto\mathcal{V}$ and $g:\ \mathcal{V}^3\mapsto\mathcal{V}$. It is easy to see that if a component of the function f is identically 0, then the respective component of the function g may be arbitrary. Thus it suffices to consider the scalar equation

$$\begin{aligned} &a_1 f(\mathbf{Z}_1)g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+a_2 f(\mathbf{Z}_2)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1)\\ +\;&a_3 f(\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2)+a_4 f(\mathbf{Z}_4)g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=0, \end{aligned} \tag{4.2}$$

where a_1,a_2,a_3,a_4 are complex constants, $f:\ \mathcal{V}\mapsto\mathbb{C}$ and $g:\ \mathcal{V}^3\mapsto\mathbb{C}$. Equation (4.2) has trivial solutions for which $f\equiv 0$ and g is arbitrary. We shall call a *general solution* of (4.2) the set of all other solutions of this equation. Obviously, the general solution includes other trivial solutions of the form $g\equiv 0$, f arbitrary.

First we will find the general solution of some particular cases of equation (4.2), and after that we will prove a general result for equation (4.2) by reducing it to one of these particular cases using a matrix method. Finally, we will consider some equations which generalize equation (4.2).

Theorem 4.1 *The general solution of the functional equation*

$$\begin{aligned} &f(\mathbf{Z}_1)g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+f(\mathbf{Z}_2)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1)\\ +\;&f(\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2)+f(\mathbf{Z}_4)g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=0 \end{aligned} \tag{4.3}$$

is given by

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}),\\ g(\mathbf{U},\mathbf{V},\mathbf{W}) &= F(\mathbf{U})G_1(\mathbf{V},\mathbf{W})-F(\mathbf{W})G_1(\mathbf{U},\mathbf{V})\\ &+ F(\mathbf{V})\big[G_2(\mathbf{W},\mathbf{U})-G_2(\mathbf{U},\mathbf{W})\big], \end{aligned} \tag{4.4}$$

where $F:\ \mathcal{V}\mapsto\mathbb{C}$ and G_1, $G_2:\ \mathcal{V}^2\mapsto\mathbb{C}$ are arbitrary functions.

Proof. If we exclude the trivial case $f(\mathbf{Z}_1)\equiv 0$, g arbitrary, then there is a constant complex vector $\mathbf{A}_1$ such that $f(\mathbf{A}_1)=C\neq 0$.

Putting $\mathbf{Z}_1=\mathbf{Z}_2=\mathbf{Z}_3=\mathbf{Z}_4=\mathbf{A}_1$, from (4.3) it follows that

$$g(\mathbf{A}_1,\mathbf{A}_1,\mathbf{A}_1)=0. \tag{4.5}$$

For $\mathbf{Z}_1=\mathbf{Z}_2=\mathbf{A}_1$, from (4.3) we obtain

$$Cg(\mathbf{A}_1,\mathbf{Z}_3,\mathbf{Z}_4)+Cg(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{A}_1)+f(\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{A}_1,\mathbf{A}_1)+f(\mathbf{Z}_4)g(\mathbf{A}_1,\mathbf{A}_1,\mathbf{Z}_3)=0. \tag{4.6}$$

From (4.6), for $\mathbf{Z}_3 = \mathbf{A}_1$, having in view (4.5), we get

$$g(\mathbf{A}_1, \mathbf{A}_1, \mathbf{Z}_4) + g(\mathbf{A}_1, \mathbf{Z}_4, \mathbf{A}_1) + g(\mathbf{Z}_4, \mathbf{A}_1, \mathbf{A}_1) = 0. \tag{4.7}$$

If we set $\mathbf{Z}_1 = \mathbf{Z}_3 = \mathbf{A}_1$ and if we introduce the notations

$$g(\mathbf{U}, \mathbf{A}_1, \mathbf{V}) = K(\mathbf{U}, \mathbf{V}) \quad \text{and} \quad g(\mathbf{A}_1, \mathbf{U}, \mathbf{A}_1) = -2CH(\mathbf{U}), \tag{4.8}$$

(4.3) yields

$$\left[K(\mathbf{Z}_2, \mathbf{Z}_4) - f(\mathbf{Z}_2)H(\mathbf{Z}_4) - f(\mathbf{Z}_4)H(\mathbf{Z}_2)\right] + \left[K(\mathbf{Z}_4, \mathbf{Z}_2) - f(\mathbf{Z}_4)H(\mathbf{Z}_2) - f(\mathbf{Z}_2)H(\mathbf{Z}_4)\right] = 0, \tag{4.9}$$

which is a cyclic equation, so that its general solution is

$$K(\mathbf{U}, \mathbf{V}) \equiv g(\mathbf{U}, \mathbf{A}_1, \mathbf{V}) = P(\mathbf{U}, \mathbf{V}) - P(\mathbf{V}, \mathbf{U}) + f(\mathbf{U})H(\mathbf{V}) + f(\mathbf{V})H(\mathbf{U}), \tag{4.10}$$

where $P : \mathcal{V}^2 \mapsto \mathbb{C}$ is an arbitrary function.

With the notation $g(\mathbf{U}, \mathbf{A}_1, \mathbf{A}_1) = -CL(\mathbf{U})$, the relation (4.6), because of (4.7) and (4.8), implies

$$g(\mathbf{A}_1, \mathbf{Z}_3, \mathbf{Z}_4) = -g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{A}_1) + f(\mathbf{Z}_3)L(\mathbf{Z}_4) - f(\mathbf{Z}_4)L(\mathbf{Z}_3) - 2f(\mathbf{Z}_4)H(\mathbf{Z}_3). \tag{4.11}$$

With $\mathbf{Z}_1 = \mathbf{A}_1$, taking into account (4.10) and (4.11), from (4.3) we obtain

$$\begin{aligned} Cg(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) = {} & -f(\mathbf{Z}_2)S(\mathbf{Z}_3, \mathbf{Z}_4) + f(\mathbf{Z}_4)S(\mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_3)\left[P(\mathbf{Z}_2, \mathbf{Z}_4) - P(\mathbf{Z}_4, \mathbf{Z}_2)\right] \\ & -f(\mathbf{Z}_2)f(\mathbf{Z}_3)H(\mathbf{Z}_4) + f(\mathbf{Z}_3)f(\mathbf{Z}_4)H(\mathbf{Z}_2) - f(\mathbf{Z}_2)f(\mathbf{Z}_4)L(\mathbf{Z}_3) + f(\mathbf{Z}_3)f(\mathbf{Z}_4)L(\mathbf{Z}_2) \end{aligned}$$

with $S(\mathbf{U}, \mathbf{V}) = g(\mathbf{U}, \mathbf{V}, \mathbf{A}_1)$.

Now if we introduce the notations

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}), \\ G_1(\mathbf{U}, \mathbf{V}) &= -\frac{1}{C}\left[S(\mathbf{U}, \mathbf{V}) + F(\mathbf{V})L(\mathbf{U})\right], \\ G_2(\mathbf{U}, \mathbf{V}) &= -\frac{1}{C}\left[P(\mathbf{U}, \mathbf{V}) - F(\mathbf{V})H(\mathbf{U})\right], \end{aligned}$$

we obtain (4.4).

On the other hand, all functions of the form (4.4) satisfy the equation (4.3). □

Theorem 4.2 *The general solution of the complex vector functional equation*

$$\begin{aligned} & f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) - f(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) \\ + {} & f(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) - f(\mathbf{Z}_4)g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = 0 \end{aligned} \tag{4.12}$$

is given by

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}), \\ g(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F(\mathbf{U})G_1(\mathbf{V}, \mathbf{W}) + F(\mathbf{W})G_1(\mathbf{U}, \mathbf{V}) \\ &- F(\mathbf{V})\left[G_2(\mathbf{W}, \mathbf{U}) - G_2(\mathbf{U}, \mathbf{W})\right] + AF(\mathbf{U})F(\mathbf{V})F(\mathbf{W}), \end{aligned} \tag{4.13}$$

where $F : \mathcal{V} \mapsto \mathbb{C}$ and $G_1, G_2 : \mathcal{V}^2 \mapsto \mathbb{C}$ are arbitrary functions and A is an arbitrary constant.

Proof. In this case we also suppose that there is a constant complex vector $\mathbf{A}_1$ such that $f(\mathbf{A}_1) = C \neq 0$.

If we put $\mathbf{Z}_1 = \mathbf{Z}_3 = \mathbf{A}_1$ and if we introduce the notations $g(\mathbf{U}, \mathbf{A}_1, \mathbf{V}) = K(\mathbf{U}, \mathbf{V})$ and $g(\mathbf{A}_1, \mathbf{U}, \mathbf{A}_1) = 2CH(\mathbf{U})$, equation (4.12) is reduced to the equation (4.9) whose general solution is determined by (4.10).

If we substitute $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{A}_1$ into (4.12), we get

$$g(\mathbf{A}_1, \mathbf{Z}_3, \mathbf{Z}_4) = g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{A}_1) - f(\mathbf{Z}_3)L(\mathbf{Z}_4) - f(\mathbf{Z}_4)L(\mathbf{Z}_3) + 2H(\mathbf{Z}_3)f(\mathbf{Z}_4) + Bf(\mathbf{Z}_3)f(\mathbf{Z}_4), \tag{4.14}$$

where we have introduced the notations $g(\mathbf{U}, \mathbf{A}_1, \mathbf{A}_1) = CL(\mathbf{U})$ and $g(\mathbf{A}_1, \mathbf{A}_1, \mathbf{A}_1) = BC^2$ and have used the equality

$$g(\mathbf{A}_1, \mathbf{A}_1, \mathbf{U}) = g(\mathbf{A}_1, \mathbf{U}, \mathbf{A}_1) - g(\mathbf{U}, \mathbf{A}_1, \mathbf{A}_1) + BCf(\mathbf{U}),$$

which is obtained from (4.12) for $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{A}_1$ and $\mathbf{Z}_4 = \mathbf{U}$.

Now, if we again put $\mathbf{Z}_1 = \mathbf{A}_1$ into (4.12), then because of (4.10) and (4.14), we obtain

$$\begin{aligned} Cg(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) &= f(\mathbf{Z}_2)S(\mathbf{Z}_3, \mathbf{Z}_4) + f(\mathbf{Z}_4)S(\mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_3)\left[P(\mathbf{Z}_2, \mathbf{Z}_4) - P(\mathbf{Z}_4, \mathbf{Z}_2)\right] \\ &\quad - f(\mathbf{Z}_2)f(\mathbf{Z}_3)H(\mathbf{Z}_4) + f(\mathbf{Z}_3)f(\mathbf{Z}_4)H(\mathbf{Z}_2) - f(\mathbf{Z}_3)f(\mathbf{Z}_4)L(\mathbf{Z}_2) \\ &\quad - f(\mathbf{Z}_2)f(\mathbf{Z}_4)L(\mathbf{Z}_3) + Bf(\mathbf{Z}_2)f(\mathbf{Z}_3)f(\mathbf{Z}_4), \end{aligned}$$

with $S(\mathbf{U}, \mathbf{V}) = g(\mathbf{U}, \mathbf{V}, \mathbf{A}_1)$.

By introducing the notations

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}), \\ G_1(\mathbf{U}, \mathbf{V}) &= \frac{1}{C}\left[S(\mathbf{U}, \mathbf{V}) - F(\mathbf{U})L(\mathbf{V})\right], \\ G_2(\mathbf{U}, \mathbf{V}) &= \frac{1}{C}\left[P(\mathbf{U}, \mathbf{V}) - F(\mathbf{U})H(\mathbf{V})\right], \end{aligned}$$

we arrive at the formulae (4.13). □

Theorem 4.3 *The general solution of the functional equation*

$$\begin{aligned} & f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) \pm if(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) \\ - \; & f(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) \mp if(\mathbf{Z}_4)g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = 0 \end{aligned} \tag{4.15}$$

is given by

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}), \\ g(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= \mp iF(\mathbf{U})G(\mathbf{V}, \mathbf{W}) + F(\mathbf{W})G(\mathbf{U}, \mathbf{V}) \qquad (4.16) \\ &+ F(\mathbf{V})\left[P(\mathbf{U}, \mathbf{W}) + P(\mathbf{W}, \mathbf{U})\right] + BF(\mathbf{U})F(\mathbf{V})F(\mathbf{W}), \end{aligned}$$

where $F : \mathcal{V} \mapsto \mathbb{C}$ and G, $P : \mathcal{V}^2 \mapsto \mathbb{C}$ are arbitrary functions and B is an arbitrary constant.

Proof. Again we suppose that there is a constant complex vector $\mathbf{A}_1$ such that $f(\mathbf{A}_1) = C \neq 0$.

If we set $\mathbf{Z}_1 = \mathbf{Z}_3 = \mathbf{A}_1$ and if we introduce the notations $g(\mathbf{U}, \mathbf{A}_1, \mathbf{V}) = K(\mathbf{U}, \mathbf{V})$ and $g(\mathbf{A}_1, \mathbf{U}, \mathbf{A}_1) = \pm iCH(\mathbf{U})$, (4.15) yields

$$K(\mathbf{Z}_2, \mathbf{Z}_4) - f(\mathbf{Z}_2)H(\mathbf{Z}_4) = K(\mathbf{Z}_4, \mathbf{Z}_2) - f(\mathbf{Z}_4)H(\mathbf{Z}_2),$$

which is a cyclic equation, so that its general solution can be written in the form

$$K(\mathbf{U}, \mathbf{V}) = f(\mathbf{U})H(\mathbf{V}) + CP(\mathbf{U}, \mathbf{V}) + CP(\mathbf{V}, \mathbf{U}), \tag{4.17}$$

where $P : \mathcal{V}^2 \mapsto \mathbb{C}$ is an arbitrary function.

Setting $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{A}_1$, the functional equation (4.15) yields

$$Cg(\mathbf{A}_1, \mathbf{Z}_3, \mathbf{Z}_4) \pm iCg(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{A}_1) - f(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{A}_1, \mathbf{A}_1) \mp if(\mathbf{Z}_4)g(\mathbf{A}_1, \mathbf{A}_1, \mathbf{Z}_3) = 0. \tag{4.18}$$

We put into equation (4.18) $\mathbf{Z}_3 = \mathbf{A}_1$, $\mathbf{Z}_4 = \mathbf{U}$ and obtain

$$g(\mathbf{A}_1, \mathbf{A}_1, \mathbf{U}) = CH(\mathbf{U}) + CL(\mathbf{U}) - BCf(\mathbf{U}), \tag{4.19}$$

where we have introduced the notations $g(\mathbf{U}, \mathbf{A}_1, \mathbf{A}_1) = CL(\mathbf{U})$ and $\mp ig(\mathbf{A}_1, \mathbf{A}_1, \mathbf{A}_1) = BC^2$. Now we substitute (4.19) into (4.18) and obtain

$$g(\mathbf{A}_1, \mathbf{Z}_3, \mathbf{Z}_4) = \mp ig(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{A}_1) + f(\mathbf{Z}_3)L(\mathbf{Z}_4) \pm if(\mathbf{Z}_4)H(\mathbf{Z}_3) \pm if(\mathbf{Z}_4)L(\mathbf{Z}_3) \mp iBf(\mathbf{Z}_3)f(\mathbf{Z}_4). \tag{4.20}$$

With $\mathbf{Z}_1 = \mathbf{A}_1$, taking into account (4.17) and (4.20), from (4.15) we obtain

$$\begin{aligned} Cg(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) = {} & \mp if(\mathbf{Z}_2)S(\mathbf{Z}_3, \mathbf{Z}_4) + f(\mathbf{Z}_4)S(\mathbf{Z}_2, \mathbf{Z}_3) + f(\mathbf{Z}_3)\left[P(\mathbf{Z}_2, \mathbf{Z}_4) + P(\mathbf{Z}_4, \mathbf{Z}_2)\right] \\ & \pm if(\mathbf{Z}_2)f(\mathbf{Z}_4)L(\mathbf{Z}_3) - f(\mathbf{Z}_3)f(\mathbf{Z}_4)L(\mathbf{Z}_2) + Bf(\mathbf{Z}_2)f(\mathbf{Z}_3)f(\mathbf{Z}_4) \end{aligned}$$

with $S(\mathbf{U}, \mathbf{V}) = g(\mathbf{U}, \mathbf{V}, \mathbf{A}_1)$.

Now if we introduce the notations

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}), \\ G(\mathbf{U}, \mathbf{V}) &= \mp \frac{i}{C}\left[S(\mathbf{U}, \mathbf{V}) - F(\mathbf{V})L(\mathbf{U})\right], \end{aligned}$$

we obtain (4.16). □

Theorem 4.4 *The equation*

$$f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + (1 \pm i)f(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) \pm if(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) = 0. \tag{4.21}$$

has a general solution

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}), \qquad (4.22) \\ g(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= F(\mathbf{U})F(\mathbf{V})H(\mathbf{W}) - (1 \mp i)F(\mathbf{U})F(\mathbf{W})H(\mathbf{V}) \mp iF(\mathbf{V})F(\mathbf{W})H(\mathbf{U}), \end{aligned}$$

where F, $H : \mathcal{V} \mapsto \mathbb{C}$ are arbitrary functions.

Proof. Equation (4.21) can be written in the form

$$f(\mathbf{Z}_1)g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+f(\mathbf{Z}_2)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1)\pm i\big[f(\mathbf{Z}_2)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1)+f(\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2)\big]=0.$$

By a cyclic permutation of the variables this equation implies

$$f(\mathbf{Z}_2)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1)+f(\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2)\pm i\big[f(\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2)+f(\mathbf{Z}_4)g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)\big]=0.$$

From the last two equations we deduce

$$f(\mathbf{Z}_1)g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)+f(\mathbf{Z}_2)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1)+f(\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2)+f(\mathbf{Z}_4)g(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3)=0,$$

whose general solution according to Theorem 4.1 is given by the formulae (4.4).

We substitute these into (4.21) and obtain

$$\begin{aligned}&\mp iF(\mathbf{Z}_1)F(\mathbf{Z}_2)G_1(\mathbf{Z}_3,\mathbf{Z}_4)+F(\mathbf{Z}_2)F(\mathbf{Z}_3)G_1(\mathbf{Z}_4,\mathbf{Z}_1)\\&\pm iF(\mathbf{Z}_3)F(\mathbf{Z}_4)G_1(\mathbf{Z}_1,\mathbf{Z}_2)-F(\mathbf{Z}_1)F(\mathbf{Z}_4)G_1(\mathbf{Z}_2,\mathbf{Z}_3)\\&+(1\pm i)F(\mathbf{Z}_2)F(\mathbf{Z}_4)\big[G_2(\mathbf{Z}_1,\mathbf{Z}_3)-G_2(\mathbf{Z}_3,\mathbf{Z}_1)\big]\\&+(1\mp i)F(\mathbf{Z}_1)F(\mathbf{Z}_3)\big[G_2(\mathbf{Z}_4,\mathbf{Z}_2)-G_2(\mathbf{Z}_2,\mathbf{Z}_4)\big]=0.\end{aligned}\tag{4.23}$$

We put into this equation $\mathbf{Z}_1=\mathbf{Z}_2=\mathbf{A}_1,\ \mathbf{Z}_3=\mathbf{U},\ \mathbf{Z}_4=\mathbf{V}$ and obtain

$$\begin{aligned}&\mp iC^2G_1(\mathbf{U},\mathbf{V})+CF(\mathbf{U})G_1(\mathbf{V},\mathbf{A}_1)\pm iF(\mathbf{U})F(\mathbf{V})G_1(\mathbf{A}_1,\mathbf{A}_1)-CF(\mathbf{V})G_1(\mathbf{A}_1,\mathbf{U})\\&+(1\pm i)CF(\mathbf{V})\big[G_2(\mathbf{A}_1,\mathbf{U})-G_2(\mathbf{U},\mathbf{A}_1)\big]\\&+(1\mp i)CF(\mathbf{U})\big[G_2(\mathbf{V},\mathbf{A}_1)-G_2(\mathbf{A}_1,\mathbf{V})\big]\doteq 0.\end{aligned}\tag{4.24}$$

If we introduce the notation

$$B=G_1(\mathbf{A}_1,\mathbf{A}_1)/C^2,$$

$$\begin{aligned}M(\mathbf{U})&=\pm\frac{i}{C}G_1\mathbf{A}_1,\mathbf{U})+\frac{1\pm i}{C}\big[G_2(\mathbf{A}_1,\mathbf{U})-G_2(\mathbf{U},\mathbf{A}_1)\big],\\N(\mathbf{V})&=\mp\frac{i}{C}G_1(\mathbf{V},\mathbf{A}_1)-\frac{1\pm i}{C}\big[G_2(\mathbf{V},\mathbf{A}_1)-G_2(\mathbf{A}_1,\mathbf{V})\big],\end{aligned}$$

from equation (4.24) we can express

$$G_1(\mathbf{U},\mathbf{V})=BF(\mathbf{U})F(\mathbf{V})+F(\mathbf{U})N(\mathbf{V})+F(\mathbf{V})M(\mathbf{U}).\tag{4.25}$$

We substitute this expression into (4.23) and obtain

$$\begin{aligned}&F(\mathbf{Z}_1)F(\mathbf{Z}_2)F(\mathbf{Z}_3)\big[M(\mathbf{Z}_4)\mp iN(\mathbf{Z}_4)\big]\mp iF(\mathbf{Z}_1)F(\mathbf{Z}_2)F(\mathbf{Z}_4)\big[M(\mathbf{Z}_3)\mp iN(\mathbf{Z}_3)\big]\\-\;&F(\mathbf{Z}_1)F(\mathbf{Z}_3)F(\mathbf{Z}_4)\big[M(\mathbf{Z}_2)\mp iN(\mathbf{Z}_2)\big]\pm iF(\mathbf{Z}_2)F(\mathbf{Z}_3)F(\mathbf{Z}_4)\big[M(\mathbf{Z}_1)\mp iN(\mathbf{Z}_1)\big]\\&+(1\pm i)F(\mathbf{Z}_2)F(\mathbf{Z}_4)\big[G_2(\mathbf{Z}_1,\mathbf{Z}_3)-G_2(\mathbf{Z}_3,\mathbf{Z}_1)\big]\\&+(1\mp i)F(\mathbf{Z}_1)F(\mathbf{Z}_3)\big[G_2(\mathbf{Z}_4,\mathbf{Z}_2)-G_2(\mathbf{Z}_2,\mathbf{Z}_4)\big]=0.\end{aligned}$$

From this equality for $\mathbf{Z}_1=\mathbf{Z}_2=\mathbf{A}_1$ we find

$$G_2(\mathbf{Z}_1,\mathbf{Z}_3)-G_2(\mathbf{Z}_3,\mathbf{Z}_1)=\frac{1\pm i}{2}\big\{F(\mathbf{Z}_1)\big[M(\mathbf{Z}_3)\mp iN(\mathbf{Z}_3)\big]-F(\mathbf{Z}_3)\big[M(\mathbf{Z}_1)\mp iN(\mathbf{Z}_1)\big]\big\}.$$

We substitute this expression and (4.25) into the formula for g in (4.4) and deduce (4.22) with

$$H(\mathbf{U}) = -\frac{1 \pm i}{2}\big[M(\mathbf{U}) - N(\mathbf{U})\big]. \qquad \square$$

Theorem 4.5 *The equation*

$$f(\mathbf{Z}_1)g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) - (1 \mp i)f(\mathbf{Z}_2)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1) \mp i f(\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2) = 0.$$

has a general solution

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}), \\ g(\mathbf{U},\mathbf{V},\mathbf{W}) &= F(\mathbf{U})F(\mathbf{V})H(\mathbf{W}) + (1 \pm i)F(\mathbf{U})F(\mathbf{W})H(\mathbf{V}) \pm iF(\mathbf{V})F(\mathbf{W})H(\mathbf{U}), \end{aligned} \tag{4.26}$$

where $F,\ H : \mathcal{V} \mapsto \mathbb{C}$ *are arbitrary functions.*

Proof. The proof is similar to that of the previous theorem, making use of the result of Theorem 4.2. □

Theorem 4.6 *The general solution of the functional equation*

$$f(\mathbf{Z}_1)g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + f(\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2) = 0 \tag{4.27}$$

is given by the formulae

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}) \\ g(\mathbf{U},\mathbf{V},\mathbf{W}) &= F(\mathbf{V})\big[G(\mathbf{W},\mathbf{U}) - G(\mathbf{U},\mathbf{W})\big], \end{aligned} \tag{4.28}$$

where $F : \mathcal{V} \mapsto \mathbb{C}$ *and* $G : \mathcal{V}^2 \mapsto \mathbb{C}$ *are arbitrary functions.*

Proof. Let us assume that $f(\mathbf{A}_1) = C \neq 0$. Then, for $\mathbf{Z}_1 = \mathbf{A}_1$, from (4.27) we get

$$g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) = f(\mathbf{Z}_3)M(\mathbf{Z}_4,\mathbf{Z}_2), \tag{4.29}$$

where $M(\mathbf{U},\mathbf{V}) = -\frac{1}{C}g(\mathbf{U},\mathbf{A}_1,\mathbf{V})$.

Substituting (4.29) into (4.27), we obtain

$$f(\mathbf{Z}_1)f(\mathbf{Z}_3)M(\mathbf{Z}_4,\mathbf{Z}_2) + f(\mathbf{Z}_3)f(\mathbf{Z}_1)M(\mathbf{Z}_2,\mathbf{Z}_4) = 0. \tag{4.30}$$

For $\mathbf{Z}_1 = \mathbf{Z}_3 = \mathbf{A}_1$ equation (4.30) yields

$$M(\mathbf{Z}_2,\mathbf{Z}_4) + M(\mathbf{Z}_4,\mathbf{Z}_2) = 0,$$

whose general solution is

$$M(\mathbf{Z}_2,\mathbf{Z}_4) = G(\mathbf{Z}_2,\mathbf{Z}_4) - G(\mathbf{Z}_4,\mathbf{Z}_2),$$

where $G : \mathcal{V}^2 \mapsto \mathbb{C}$ is an arbitrary function.

Therefore, with the notation $f(\mathbf{U}) = F(\mathbf{U})$, we get the formulae (4.28). □

Theorem 4.7 *The general solution of the functional equation*

$$f(\mathbf{Z}_1)g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) - f(\mathbf{Z}_3)g(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2) = 0 \tag{4.31}$$

is given by the formulae

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}) \\ g(\mathbf{U},\mathbf{V},\mathbf{W}) &= F(\mathbf{V})\big[G(\mathbf{U},\mathbf{W}) + G(\mathbf{W},\mathbf{U})\big], \end{aligned} \tag{4.32}$$

where $F: \mathcal{V} \mapsto \mathbb{C}$ and $G: \mathcal{V}^2 \mapsto \mathbb{C}$ are arbitrary functions.

Proof. If $f(\mathbf{A}_1) = C \neq 0$, then the equation (4.31) for $\mathbf{Z}_1 = \mathbf{A}_1$ yields

$$g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) = f(\mathbf{Z}_3)M(\mathbf{Z}_4,\mathbf{Z}_2), \tag{4.33}$$

where $M(\mathbf{U},\mathbf{V}) = \frac{1}{C}g(\mathbf{U},\mathbf{A}_1,\mathbf{V})$.

Substituting (4.33) into (4.31), we get

$$f(\mathbf{Z}_1)f(\mathbf{Z}_3)M(\mathbf{Z}_4,\mathbf{Z}_2) - f(\mathbf{Z}_3)f(\mathbf{Z}_1)M(\mathbf{Z}_2,\mathbf{Z}_4) = 0,$$

from which for $\mathbf{Z}_1 = \mathbf{Z}_3 = \mathbf{A}_1$ we have

$$M(\mathbf{Z}_2,\mathbf{Z}_4) - M(\mathbf{Z}_4,\mathbf{Z}_2) = 0.$$

The general solution of this equation is

$$M(\mathbf{U},\mathbf{V}) = G(\mathbf{U},\mathbf{V}) + G(\mathbf{V},\mathbf{U}), \tag{4.34}$$

where $G: \mathcal{V}^2 \mapsto \mathbb{C}$ is an arbitrary function.

If we put $f(\mathbf{U}) = F(\mathbf{U})$, by virtue of (4.33) and (4.34) we have (4.32). □

Theorem 4.8 *The general solution of the functional equation*

$$f(\mathbf{Z}_1)g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + f(\mathbf{Z}_2)g(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1) = 0 \tag{4.35}$$

is

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}), \\ g(\mathbf{U},\mathbf{V},\mathbf{W}) &\equiv 0, \end{aligned} \tag{4.36}$$

where $F: \mathcal{V} \mapsto \mathbb{C}$ is an arbitrary function.

Proof. If $f(\mathbf{A}_1) = C \neq 0$, then for $\mathbf{Z}_j = \mathbf{A}_1$ $(1 \leq j \leq 4)$, from (4.35) we obtain $g(\mathbf{A}_1,\mathbf{A}_1,\mathbf{A}_1) = 0$. For $\mathbf{Z}_1 = \mathbf{A}_1$, the equation (4.35) is reduced to

$$g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) = f(\mathbf{Z}_2)M(\mathbf{Z}_3,\mathbf{Z}_4), \tag{4.37}$$

where $M(\mathbf{U},\mathbf{V}) = -\frac{1}{C}g(\mathbf{U},\mathbf{V},\mathbf{A}_1)$.

Inserting (4.37) into (4.35), we have

$$f(\mathbf{Z}_1)f(\mathbf{Z}_2)M(\mathbf{Z}_3,\mathbf{Z}_4) + f(\mathbf{Z}_2)f(\mathbf{Z}_3)M(\mathbf{Z}_4,\mathbf{Z}_1) = 0, \tag{4.38}$$

from which, for $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{A}_1$, we get

$$M(\mathbf{Z}_3, \mathbf{Z}_4) = f(\mathbf{Z}_3)N(\mathbf{Z}_4), \tag{4.39}$$

where

$$N(\mathbf{U}) = -\frac{1}{C}M(\mathbf{U}, \mathbf{A}_1) = \frac{1}{C^2}g(\mathbf{U}, \mathbf{A}_1, \mathbf{A}_1).$$

Therefore, by virtue of (4.39), from (4.38) we get

$$f(\mathbf{Z}_1)f(\mathbf{Z}_2)f(\mathbf{Z}_3)N(\mathbf{Z}_4) + f(\mathbf{Z}_2)f(\mathbf{Z}_3)f(\mathbf{Z}_4)N(\mathbf{Z}_1) = 0. \tag{4.40}$$

Equation (4.40), for $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{A}_1$ and $\mathbf{Z}_4 = \mathbf{U}$, yields

$$CN(\mathbf{U}) + f(\mathbf{U})N(\mathbf{A}_1) = 0.$$

Since $N(\mathbf{A}_1) = \frac{1}{C^2}g(\mathbf{A}_1, \mathbf{A}_1, \mathbf{A}_1) = 0$, we have

$$N(\mathbf{U}) \equiv 0. \tag{4.41}$$

By virtue of (4.41), (4.39) and (4.37) we may conclude that

$$g(\mathbf{U}, \mathbf{V}, \mathbf{W}) \equiv 0. \qquad \square$$

Theorem 4.9 *The general solution of the functional equation*

$$f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) - f(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) = 0 \tag{4.42}$$

is given by

$$\begin{aligned} f(\mathbf{U}) &= F(\mathbf{U}), \\ g(\mathbf{U}, \mathbf{V}, \mathbf{W}) &= BF(\mathbf{U})F(\mathbf{V})F(\mathbf{W}), \end{aligned} \tag{4.43}$$

where $F : \mathcal{V} \mapsto \mathbb{C}$ is an arbitrary function and B is an arbitrary constant.

Proof. Let $f(\mathbf{A}_1) = C \neq 0$. Then, for $\mathbf{Z}_1 = \mathbf{A}_1$, equation (4.42) becomes

$$g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) = f(\mathbf{Z}_2)M(\mathbf{Z}_3, \mathbf{Z}_4), \tag{4.44}$$

where the notation

$$M(\mathbf{U}, \mathbf{V}) = \frac{1}{C}g(\mathbf{U}, \mathbf{V}, \mathbf{A}_1)$$

has been introduced.

If (4.44) is inserted into (4.42), then the following equation is obtained

$$f(\mathbf{Z}_1)f(\mathbf{Z}_2)M(\mathbf{Z}_3, \mathbf{Z}_4) - f(\mathbf{Z}_2)f(\mathbf{Z}_3)M(\mathbf{Z}_4, \mathbf{Z}_1) = 0, \tag{4.45}$$

which, for $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{A}_1$, $\mathbf{Z}_3 = \mathbf{U}$ and $\mathbf{Z}_4 = \mathbf{V}$, yields

$$M(\mathbf{U}, \mathbf{V}) = f(\mathbf{U})N(\mathbf{V}), \tag{4.46}$$

where

$$N(\mathbf{U}) = \frac{1}{C}M(\mathbf{U}, \mathbf{A}_1) = \frac{1}{C^2}g(\mathbf{U}, \mathbf{A}_1, \mathbf{A}_1).$$

By virtue of (4.46), equation (4.45) becomes

$$f(\mathbf{Z}_1)f(\mathbf{Z}_2)f(\mathbf{Z}_3)N(\mathbf{Z}_4) - f(\mathbf{Z}_2)f(\mathbf{Z}_3)f(\mathbf{Z}_4)N(\mathbf{Z}_1) = 0$$

which, for $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{A}_1$ and $\mathbf{Z}_4 = \mathbf{U}$, gives

$$N(\mathbf{U}) = Bf(\mathbf{U}), \tag{4.47}$$

where

$$B = \frac{1}{C}N(\mathbf{A}_1) = \frac{1}{C^2}g(\mathbf{A}_1, \mathbf{A}_1, \mathbf{A}_1).$$

Using (4.44), (4.46) and (4.47), we find

$$g(\mathbf{U}, \mathbf{V}, \mathbf{W}) = Bf(\mathbf{U})f(\mathbf{V})f(\mathbf{W}).$$ □

Theorem 4.10 *The general solution of the functional equation*

$$f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) \pm if(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) = 0 \tag{4.48}$$

is (4.36).

Proof. The theorem can be proved in the same way as Theorem 4.8. Here we give an alternative proof.

The equation (4.48) implies

$$f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + f(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) = 0.$$

According to Theorem 4.6 its general solution is given by (4.28). We substitute this into (4.48) to obtain

$$F(\mathbf{Z}_1)F(\mathbf{Z}_3)\left[G(\mathbf{Z}_4, \mathbf{Z}_2) - G(\mathbf{Z}_2, \mathbf{Z}_4)\right] \pm iF(\mathbf{Z}_2)F(\mathbf{Z}_4)\left[G(\mathbf{Z}_1, \mathbf{Z}_3) - G(\mathbf{Z}_3, \mathbf{Z}_1)\right] = 0. \tag{4.49}$$

Let $\mathbf{A}_1$ be such that $F(\mathbf{A}_1) = C \neq 0$. We put into (4.49) $\mathbf{Z}_1 = \mathbf{Z}_3 = \mathbf{A}_1$, $\mathbf{Z}_2 = \mathbf{U}$ and $\mathbf{Z}_4 = \mathbf{W}$. Thus

$$C^2\left[G(\mathbf{W}, \mathbf{U}) - G(\mathbf{U}, \mathbf{W})\right] = 0.$$

This shows that $G(\mathbf{W}, \mathbf{U}) \equiv G(\mathbf{U}, \mathbf{W})$ and a substitution into (4.28) yields

$$g(\mathbf{U}, \mathbf{V}, \mathbf{W}) \equiv 0.$$ □

Now we shall prove the following general result.

Theorem 4.11 *The general solution of the functional equation*

$$\begin{aligned} &a_1 f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + a_2 f(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) \\ +\ &a_3 f(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) + a_4 f(\mathbf{Z}_4)g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = 0, \end{aligned} \tag{4.50}$$

where a_i $(i = 1,2,3,4)$ *are complex constants, is determined by the formulae*

1° (4.36) *if* $(a_1 + a_3)^2 - (a_2 + a_4)^2 \neq 0$ *and* $(a_1 - a_3)^2 + (a_2 - a_4)^2 \neq 0$;
2° (4.36) *if* $(a_1 + a_3)^2 - (a_2 + a_4)^2 \neq 0$ *and* $a_2 - a_4 = \pm i(a_1 - a_3) \neq 0$;
3° (4.28) *if* $a_1^2 - a_2^2 \neq 0$, $a_3 = a_1$, $a_4 = a_2$;
4° (4.36) *if* $a_1 + a_3 = a_2 + a_4 \neq 0$, $(a_1 - a_3)^2 + (a_2 - a_4)^2 \neq 0$;
5° (4.22) *if* $a_1 + a_3 = a_2 + a_4 \neq 0$, $a_2 - a_4 = \pm i(a_1 - a_3)$;
6° (4.4) *if* $a_1 = a_2 = a_3 = a_4 \neq 0$;
7° (4.43) *if* $a_2 + a_4 = -(a_1 + a_3) \neq 0$, $(a_1 - a_3)^2 + (a_2 - a_4)^2 \neq 0$;
8° (4.26) *if* $a_2 + a_4 = -(a_1 + a_3) \neq 0$, $a_2 - a_4 = \pm i(a_1 - a_3) \neq 0$;
9° (4.13) *if* $a_1 = -a_2 = a_3 = -a_4 \neq 0$;
10° (4.32) *if* $a_1 + a_3 = a_2 + a_4 = 0$, $a_1^2 + a_2^2 \neq 0$;
11° (4.16) *if* $a_1 + a_3 = a_2 + a_4 = 0$, $a_2 = \pm i a_1 \neq 0$;
12° $f(\mathbf{U}) = F(\mathbf{U})$ *and* $g(\mathbf{U},\mathbf{V},\mathbf{W}) = G(\mathbf{U},\mathbf{V},\mathbf{W})$ *where* $F : \mathcal{V} \mapsto \mathbb{C}$ *and* $G : \mathcal{V}^3 \mapsto \mathbb{C}$ *are arbitrary functions if* $a_1 = a_2 = a_3 = a_4 = 0$.

Proof. If we introduce the substitution $f(\mathbf{Z}_1)g(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) = h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4)$, from (4.50) there follows the equation

$$\begin{aligned} & a_1 h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + a_2 h(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1) \\ + \; & a_3 h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2) + a_4 h(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = 0. \end{aligned} \tag{4.51}$$

Now we can use Theorem 2.1 from [1]. By a cyclic permutation of the vectors in (4.51) we obtain the following system of equations

$$\begin{aligned} & a_1 h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + a_2 h(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1) \\ + \; & a_3 h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2) + a_4 h(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = 0, \\ & a_4 h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + a_1 h(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1) \\ + \; & a_2 h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2) + a_3 h(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = 0, \\ & a_3 h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + a_4 h(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1) \\ + \; & a_1 h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2) + a_2 h(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = 0, \\ & a_2 h(\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4) + a_3 h(\mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1) \\ + \; & a_4 h(\mathbf{Z}_3,\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2) + a_1 h(\mathbf{Z}_4,\mathbf{Z}_1,\mathbf{Z}_2,\mathbf{Z}_3) = 0. \end{aligned} \tag{4.52}$$

If we put $h(\mathbf{Z}_i,\mathbf{Z}_{i+1},\mathbf{Z}_{i+2},\mathbf{Z}_{i+3}) = h^i$ $(\mathbf{Z}_{i+4} \equiv \mathbf{Z}_i,\ i = 1,2,3,4)$, the system (4.52) becomes

$$\begin{aligned} a_1 h^1 + a_2 h^2 + a_3 h^3 + a_4 h^4 &= 0, \\ a_4 h^1 + a_1 h^2 + a_2 h^3 + a_3 h^4 &= 0, \\ a_3 h^1 + a_4 h^2 + a_1 h^3 + a_2 h^4 &= 0, \\ a_2 h^1 + a_3 h^2 + a_4 h^3 + a_1 h^4 &= 0. \end{aligned} \tag{4.53}$$

By the notations $a_1 + a_3 = 2M$, $a_2 + a_4 = 2P$, $a_1 - a_3 = 2N$, $a_2 - a_4 = 2Q$, *i.e.* $a_1 = M + N$, $a_2 = P + Q$, $a_3 = M - N$, $a_4 = P - Q$, the above system (4.53) takes the form

$$\begin{aligned}
M(h^1+h^3)+P(h^2+h^4)+N(h^1-h^3)+Q(h^2-h^4)&=0,\\
P(h^1+h^3)+M(h^2+h^4)-Q(h^1-h^3)+N(h^2-h^4)&=0,\\
M(h^1+h^3)+P(h^2+h^4)-N(h^1-h^3)-Q(h^2-h^4)&=0,\\
P(h^1+h^3)+M(h^2+h^4)+Q(h^1-h^3)-N(h^2-h^4)&=0,
\end{aligned}$$

which is equivalent to the system

$$\begin{aligned}
M(h^1+h^3)+P(h^2+h^4)&=0,\\
P(h^1+h^3)+M(h^2+h^4)&=0,\\
N(h^1-h^3)+Q(h^2-h^4)&=0,\\
-\ Q(h^1-h^3)+N(h^2-h^4)&=0.
\end{aligned} \tag{4.54}$$

The determinant of the linear system (4.54) is

$$\Delta=\left[(a_1+a_3)^2-(a_2+a_4)^2\right]\left[(a_1-a_3)^2+(a_2-a_4)^2\right]=16(M^2-P^2)(N^2+Q^2).$$

We can decompose the system (4.54) into two subsystems:

$$M(h^1+h^3)+P(h^2+h^4)=0, \qquad P(h^1+h^3)+M(h^2+h^4)=0, \tag{4.55}$$

$$N(h^1-h^3)+Q(h^2-h^4)=0, \qquad -Q(h^1-h^3)+N(h^2-h^4)=0, \tag{4.56}$$

whose determinants are respectively M^2-P^2 and N^2+Q^2.

The following statements hold:

1° If $M^2-P^2\neq 0$, the system (4.55) is equivalent to the equation $h^1+h^3=0$;

2° If $M=P=0$, the system (4.55) is equivalent to the equation $0=0$;

3° If $P=M\neq 0$, the system (4.55) is equivalent to the equation $h^1+h^2+h^3+h^4=0$;

4° If $P=-M\neq 0$, the system (4.55) is equivalent to the equation $h^1-h^2+h^3-h^4=0$;

5° If $N^2+Q^2\neq 0$, the system (4.56) is equivalent to the equation $h^1-h^3=0$;

6° If $N=Q=0$, the system (4.56) is equivalent to the equation $0=0$;

7° If $Q=\pm iN\neq 0$, the system (4.56) is equivalent to the equation $h^1\pm ih^2-h^3\mp ih^4=0$.

By virtue of the above seven statements we conclude that:

1′ If $M^2-P^2\neq 0$, $N^2+Q^2\neq 0$, the system (4.54) is equivalent to the system $h^1+h^3=0$, $h^1-h^3=0$, *i.e.* to the equation $h=0$;

2′ If $M^2-P^2\neq 0$, $Q=\pm iN\neq 0$, the system (4.54) is equivalent to the system $h^1+h^3=0$, $h^1\pm ih^2-h^3\mp ih^4=0$, *i.e.* to the equation $h^1\pm ih^2=0$;

3′ If $M^2-P^2\neq 0$, $N=Q=0$, the system (4.54) is equivalent to the system $h^1+h^3=0$, $0=0$, *i.e.* to the equation $h^1+h^3=0$;

4′ If $P=M\neq 0$, $N^2+Q^2\neq 0$, the system (4.54) is equivalent to the system $h^1-h^3=0$, $h^1+h^2+h^3+h^4=0$, *i.e.* to the equation $h^1+h^2=0$;

5′ If $P = M \neq 0$, $Q = \pm iN \neq 0$, the system (4.54) is equivalent to the system $h^1 + h^2 + h^3 + h^4 = 0$, $h^1 \pm ih^2 - h^3 \mp ih^4 = 0$, *i.e.* to the equation $h^1 + (1 \pm i)h^2 \pm ih^3 = 0$;

6′ If $P = M \neq 0$, $N = Q = 0$, the system (4.54) is equivalent to the equation $h^1 + h^2 + h^3 + h^4 = 0$;

7′ If $P = -M \neq 0$, $N^2 + Q^2 \neq 0$, the system (4.54) is equivalent to the system $h^1 - h^3 = 0$, $h^1 - h^2 + h^3 - h^4 = 0$, *i.e.* to the equation $h^1 - h^2 = 0$;

8′ If $P = -M \neq 0$, $Q = \pm iN \neq 0$, the system (4.54) is equivalent to the system $h^1 - h^2 + h^3 - h^4 = 0$, $h^1 \pm ih^2 - h^3 \mp ih^4 = 0$, *i.e.* to the equation $h^1 - (1 \mp i)h^2 \mp ih^3 = 0$;

9′ If $P = -M \neq 0$, $N = Q = 0$, the system (4.54) is equivalent to the equation $h^1 - h^2 + h^3 - h^4 = 0$;

10′ If $M = P = 0$, $N^2 + Q^2 \neq 0$, the system (4.54) is equivalent to the equation $h^1 - h^3 = 0$;

11′ If $M = P = 0$, $Q = \pm iN \neq 0$, the system (4.54) is equivalent to the equation $h^1 \pm ih^2 - h^3 \mp ih^4 = 0$;

12′ If $M = P = 0$, $N = Q = 0$, the system (4.54) is equivalent to the equation $0 = 0$.

Therefore, the following result holds:

Lemma 4.12 *The functional equation* (4.50) *is equivalent to the equation:*

1° $f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) = 0$ *if* $(a_1 + a_3)^2 - (a_2 + a_4)^2 \neq 0$ *and* $(a_1 - a_3)^2 + (a_2 - a_4)^2 \neq 0$;

2° $f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) \pm if(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) = 0$ *if* $(a_1 + a_3)^2 - (a_2 + a_4)^2 \neq 0$ *and* $a_2 - a_4 = \pm i(a_1 - a_3) \neq 0$;

3° $f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + f(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) = 0$ *if* $a_3 = a_1$, $a_4 = a_2$ *and* $a_1^2 - a_2^2 \neq 0$;

4° $f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + f(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) = 0$ *if* $a_1 + a_3 = a_2 + a_4 \neq 0$ *and* $(a_1 - a_3)^2 + (a_2 - a_4)^2 \neq 0$;

5° $f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + (1 \pm i)f(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) \pm if(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) = 0$ *if* $a_1 + a_3 = a_2 + a_4 \neq 0$ *and* $a_2 - a_4 = \pm i(a_1 - a_3) \neq 0$;

6° $f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) + f(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) + f(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) + f(\mathbf{Z}_4)g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = 0$ *if* $a_1 = a_2 = a_3 = a_4 \neq 0$;

7° $f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) - f(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) = 0$ *if* $a_2 + a_4 = -(a_1 + a_3) \neq 0$ *and* $(a_1 - a_3)^2 + (a_2 - a_4)^2 \neq 0$;

8° $f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) - (1 \mp i)f(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) \mp if(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) = 0$ *if* $a_2 + a_4 = -(a_1 + a_3) \neq 0$ *and* $a_2 - a_4 = \pm i(a_1 - a_3) \neq 0$;

9° $f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) - f(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) + f(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) - f(\mathbf{Z}_4)g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = 0$ *if* $a_1 = -a_2 = a_3 = -a_4 \neq 0$;

10° $f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) - f(\mathbf{Z}_3)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) = 0$ *if* $a_1 + a_3 = a_2 + a_4 = 0$ *and* $a_1^2 + a_2^2 \neq 0$;

11° $f(\mathbf{Z}_1)g(\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4) \pm if(\mathbf{Z}_2)g(\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_1) - f(\mathbf{Z}_3)g(\mathbf{Z}_4, \mathbf{Z}_1, \mathbf{Z}_2) \mp if(\mathbf{Z}_4)g(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3) = 0$ *if* $a_1 + a_3 = a_2 + a_4 = 0$ *and* $a_2 = \pm ia_1 \neq 0$;

12° $0 = 0$ *if* $a_1 = a_2 = a_3 = a_4 = 0$.

Combining Lemma 4.12 and Theorems 4.1-4.10 we get the proof of Theorem 4.11. □

Now we will give a general result.

Theorem 4.13 *The general solution of the functional equation*

$$\sum_{i=1}^{n} a_i f(\mathbf{Z}_i) g(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \ldots, \mathbf{Z}_{i+n-1}) = 0 \qquad (n > 1, \quad \mathbf{Z}_{n+i} \equiv \mathbf{Z}_i), \tag{4.57}$$

where a_i $(1 \leq i \leq n)$ *are complex constants,* $f : \mathcal{V} \mapsto \mathbb{C}$, $g : \mathcal{V}^{n-1} \mapsto \mathbb{C}$, *is determined by the following formulae*

$$f(\mathbf{Z}) = F(\mathbf{Z}),$$

$$\begin{bmatrix} g(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_n) \\ g(\mathbf{Z}_3, \ldots, \mathbf{Z}_n, \mathbf{Z}_1) \\ \vdots \\ g(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix} = \operatorname{diag}\left\{ \frac{1}{F(\mathbf{Z}_1)}, \frac{1}{F(\mathbf{Z}_2)}, \ldots, \frac{1}{F(\mathbf{Z}_n)} \right\} B \begin{bmatrix} v(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ v(\mathbf{Z}_2, \ldots, \mathbf{Z}_n, \mathbf{Z}_1) \\ \vdots \\ v(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix}, \tag{4.58}$$

where

$$A \equiv \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & & & \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \\ b_n & b_1 & \cdots & b_{n-1} \\ \vdots & & & \\ b_2 & b_3 & \cdots & b_1 \end{bmatrix}$$

is a nonzero $n \times n$ *constant cyclic matrix with complex entries such that* $AB = O$, $F : \mathcal{V} \mapsto \mathbb{C}$ *is an arbitrary nonzero function, and the function* $v : \mathcal{V}^n \mapsto \mathbb{C}$ *is chosen in such a way that the* j*−th component of the vector in the right-hand side of* (4.58) *does not depend on* $\mathbf{Z}_j$.

Proof. By the substitutions

$$f(\mathbf{Z}_i) g(\mathbf{Z}_{i+1}, \mathbf{Z}_{i+2}, \ldots, \mathbf{Z}_{i+n-1}) = h(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+n-1}) \qquad (1 \leq i \leq n) \tag{4.59}$$

the functional equation (4.57) reduces to the linear equation

$$\sum_{i=1}^{n} a_i h(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+n-1}) = 0 \qquad (n > 1, \quad \mathbf{Z}_{n+i} \equiv \mathbf{Z}_i). \tag{4.60}$$

By a cyclic permutation of the vectors $\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n$ in (4.60) we obtain the following linear system of equations

$$\begin{aligned}
a_1h(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_n) &+ a_2h(\mathbf{Z}_2,\dots,\mathbf{Z}_n,\mathbf{Z}_1)\\
&+ \cdots + a_nh(\mathbf{Z}_n,\mathbf{Z}_1,\dots,\mathbf{Z}_{n-1}) = 0,\\
a_nh(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_n) &+ a_1h(\mathbf{Z}_2,\dots,\mathbf{Z}_n,\mathbf{Z}_1)\\
&+ \cdots + a_{n-1}h(\mathbf{Z}_n,\mathbf{Z}_1,\dots,\mathbf{Z}_{n-1}) = 0,\\
\vdots&\\
a_2h(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_n) &+ a_3h(\mathbf{Z}_2,\dots,\mathbf{Z}_n,\mathbf{Z}_1)\\
&+ \cdots + a_1h(\mathbf{Z}_n,\mathbf{Z}_1,\dots,\mathbf{Z}_{n-1}) = 0.
\end{aligned} \tag{4.61}$$

If we introduce the notation

$$H = \begin{bmatrix} h(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_n) \\ h(\mathbf{Z}_2,\dots,\mathbf{Z}_n,\mathbf{Z}_1) \\ \vdots \\ h(\mathbf{Z}_n,\mathbf{Z}_1,\dots,\mathbf{Z}_{n-1}) \end{bmatrix},$$

then the system (4.61) can be written in a matrix form as

$$AH = \mathbf{O}. \tag{4.62}$$

According to Theorem 2.1 from [1] (see also Theorem 18.1 in the monograph [5]) the general solution of (4.62) is given by

$$H = BV, \tag{4.63}$$

where

$$V = \begin{bmatrix} v(\mathbf{Z}_1,\mathbf{Z}_2,\dots,\mathbf{Z}_n) \\ v(\mathbf{Z}_2,\dots,\mathbf{Z}_n,\mathbf{Z}_1) \\ \vdots \\ v(\mathbf{Z}_n,\mathbf{Z}_1,\dots,\mathbf{Z}_{n-1}) \end{bmatrix},$$

$v : \mathcal{V}^n \mapsto \mathbb{C}$ is an arbitrary function, the matrix B is as in the statement of the theorem. If $f \equiv 0$, then in (4.63) $H \equiv V \equiv \mathbf{O}$ and g may be arbitrary. If this is not the case, we put $f(\mathbf{Z}) = F(\mathbf{Z})$, where $F : \mathcal{V} \mapsto \mathbb{C}$ is an arbitrary nonzero function, and from (4.63) and (4.59) we deduce (4.58). □

5 Some nonhomogeneous hypercomplex vector functional equations

In this section we shall consider some linear vector functional equations such that their coefficients and the components of the vector functions do not necessarily commute. So we may assume that these coefficients belong to some noncommutative associative division algebra, say, the algebra of quaternions $\mathbb{H}$, and $\mathcal{V}$ is an n-dimensional vector space over $\mathbb{H}$.

Theorem 5.1 *The nonhomogeneous vector functional equation*

$$\sum_{i=1}^{n} a_i f(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+n-1}) = \sum_{i=1}^{n} g(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+n-1}) b_i + h(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \qquad (n > 1,\ \mathbf{Z}_{n+i} \equiv \mathbf{Z}_i), \tag{5.1}$$

where a_i, $b_i \in \mathbb{H}$ $(1 \le i \le n)$ are constants, $f, g : \mathcal{V}^n \mapsto \mathcal{V}$ are unknown functions and $h : \mathcal{V}^n \mapsto \mathcal{V}$ is a given function, has a general solution given by the following formulae

$$\begin{bmatrix} f(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ f(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ f(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix} = R \begin{bmatrix} h(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ h(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ h(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix} \tag{5.2}$$

$$+R \begin{bmatrix} u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) & u(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) & \cdots & u(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \\ u(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) & u(\mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_2) & \cdots & u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ \vdots & & & \\ u(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) & u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) & \cdots & u(\mathbf{Z}_{n-1}, \mathbf{Z}_n, \ldots, \mathbf{Z}_{n-2}) \end{bmatrix} \mathbf{b}$$

$$+(I_n - RA) \begin{bmatrix} v(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ v(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ v(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix}$$

and

$$\begin{bmatrix} g(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) & g(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) & \cdots & g(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \\ g(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) & g(\mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_2) & \cdots & g(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ \vdots & & & \\ g(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) & g(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) & \cdots & g(\mathbf{Z}_{n-1}, \mathbf{Z}_n, \ldots, \mathbf{Z}_{n-2}) \end{bmatrix} \tag{5.3}$$

$$= -(I_n - AR) \begin{bmatrix} h(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ h(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ h(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix} \mathbf{s}$$

$$+ \begin{bmatrix} u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) & u(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) & \cdots & u(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \\ u(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) & u(\mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_2) & \cdots & u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ \vdots & & & \\ u(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) & u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) & \cdots & u(\mathbf{Z}_{n-1}, \mathbf{Z}_n, \ldots, \mathbf{Z}_{n-2}) \end{bmatrix}$$

$$-(I_n - AR) \begin{bmatrix} u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) & u(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) & \cdots & u(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \\ u(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) & u(\mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_2) & \cdots & u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ \vdots & & & \\ u(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) & u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) & \cdots & u(\mathbf{Z}_{n-1}, \mathbf{Z}_n, \ldots, \mathbf{Z}_{n-2}) \end{bmatrix} \mathbf{bs},$$

if and only if

$$(I_n - AR)\begin{bmatrix} h(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) \\ h(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_1) \\ \vdots \\ h(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}) \end{bmatrix}(1-\mathbf{sb}) = \begin{bmatrix} \mathbf{O} \\ \mathbf{O} \\ \vdots \\ \mathbf{O} \end{bmatrix}, \tag{5.4}$$

where

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & & & \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

I_n is the unit $n \times n$ matrix, the $n \times n$ matrix R and the n-dimensional row vector $\mathbf{s}$ *satisfy the relations $ARA = A$ and* $\mathbf{bsb} = \mathbf{b}$ *respectively, and u and v are arbitrary functions $\mathcal{V}^n \mapsto \mathcal{V}$.*

Proof. By a cyclic permutation of the vectors in (5.1) we obtain

$$\begin{aligned}
&a_1 f(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) + a_2 f(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_1) + \cdots + a_n f(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}) \\
&\quad = g(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n)b_1 + g(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_1)b_2 + \cdots \\
&\quad\quad + g(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1})b_n + h(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n), \\
&a_n f(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) + a_1 f(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_1) + \cdots + a_{n-1} f(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}) \\
&\quad = g(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_1)b_1 + g(\mathbf{Z}_3,\mathbf{Z}_4,\ldots,\mathbf{Z}_2)b_2 + \cdots \\
&\quad\quad + g(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n)b_n + h(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_1), \\
&\vdots \\
&a_2 f(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) + a_3 f(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_1) + \cdots + a_1 f(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}) \\
&\quad = g(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1})b_1 + g(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n)b_2 + \cdots \\
&\quad\quad + g(\mathbf{Z}_{n-1},\mathbf{Z}_n,\ldots,\mathbf{Z}_{n-2})b_n + h(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}),
\end{aligned}$$

i.e., in a matrix form

$$AF = G\mathbf{b} + H, \tag{5.5}$$

where

$$F = \begin{bmatrix} f(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) \\ f(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_1) \\ \vdots \\ f(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}) \end{bmatrix}, \qquad H = \begin{bmatrix} h(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) \\ h(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_1) \\ \vdots \\ h(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}) \end{bmatrix}$$

and

$$G = \begin{bmatrix} g(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) & g(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_1) & \cdots & g(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}) \\ g(\mathbf{Z}_2,\mathbf{Z}_3,\ldots,\mathbf{Z}_1) & g(\mathbf{Z}_3,\mathbf{Z}_4,\ldots,\mathbf{Z}_2) & \cdots & g(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) \\ \vdots & & & \\ g(\mathbf{Z}_n,\mathbf{Z}_1,\ldots,\mathbf{Z}_{n-1}) & g(\mathbf{Z}_1,\mathbf{Z}_2,\ldots,\mathbf{Z}_n) & \cdots & g(\mathbf{Z}_{n-1},\mathbf{Z}_n,\ldots,\mathbf{Z}_{n-2}) \end{bmatrix}.$$

The necessity of (5.4) follows from the premultiplication and postmultiplication of (5.5) with $I_n - AR$ and $1 - \mathbf{sb}$, respectively. Here we should note that if $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{sb} = 1$. In this case (5.4) reduces to $\mathbf{O} = \mathbf{O}$. On the other hand, if $\mathbf{b} = \mathbf{0}$, then $\mathbf{s}$ can be arbitrary.

To establish the sufficiency of condition (5.4), note that it can be written in the form

$$ARH + (I_n - AR)H\mathbf{sb} = H, \tag{5.6}$$

thus showing that (5.5) admits $F = RH$ and $G = -(I_n - AR)H\mathbf{s}$ as a solution.

Now let us suppose that (5.4), or equivalently (5.6) is satisfied. If we introduce the matrices

$$U = \begin{bmatrix} u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) & u(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) & \cdots & u(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \\ u(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) & u(\mathbf{Z}_3, \mathbf{Z}_4, \ldots, \mathbf{Z}_2) & \cdots & u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ \vdots & & & \\ u(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) & u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) & \cdots & u(\mathbf{Z}_{n-1}, \mathbf{Z}_n, \ldots, \mathbf{Z}_{n-2}) \end{bmatrix}$$

and

$$V = \begin{bmatrix} v(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ v(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ v(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix},$$

then formulae (5.2) and (5.3) take the form

$$F = RH + RU\mathbf{b} + (I_n - RA)V, \qquad G = -(I_n - AR)H\mathbf{s} + U - (I_n - AR)U\mathbf{bs}. \tag{5.7}$$

Substituting the above expressions for F and G into (5.5), it is easily seen that they really satisfy this equation.

Now we shall show that an arbitrary solution of (5.5) can be obtained from the formulae (5.7) for a suitable choice of U and V. Indeed, let F_0 and G_0 satisfy (5.5), *i.e.*,

$$AF_0 = G_0\mathbf{b} + H. \tag{5.8}$$

Taking $V = F_0$ and $U = G_0$, the equations (5.7) become

$$F = F_0 - R(AF_0 - G_0\mathbf{b} - H)$$

and

$$G = G_0 - (I_n - AR)(G_0\mathbf{b} + H)\mathbf{s},$$

respectively. But in view of (5.8), they provide in fact $F = F_0$ and $G = G_0$, thus completing the proof. □

It is easy to show that in the case $b_i = 0$ $(1 \leq i \leq n)$ the above theorem reduces to Theorem 3.1 proved in [1], here restated as

Corollary 5.2 *The equation $AF = H$ is consistent if and only if $ARH = H$, in which case the general solution is $F = RH + (I - RA)V$, where V is an arbitrary vector.*

If the given vector function $h(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) = \mathbf{O}$, then the above corollary reduces to Theorem 2.1 proved in [1], here given as

Corollary 5.3 *The equation*

$$AF = [\mathbf{O}, \mathbf{O}, \ldots, \mathbf{O}]^T$$

has a general solution $F = (I - RA)V$, where V is any vector.

Corollary 5.4 *The vector functional equation*

$$\sum_{i=1}^{n} a_i f(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \ldots, \mathbf{Z}_{i+n-1}) b = h(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n)$$
$$(n > 1, \quad \mathbf{Z}_{n+i} \equiv \mathbf{Z}_i,\ 1 \le i \le n),$$

has a general solution

$$\begin{bmatrix} f(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ f(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ f(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix} = R \begin{bmatrix} h(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ h(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ h(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix} b^{-1}$$
$$+(I_n - RA) \begin{bmatrix} u(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ u(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ u(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix} \tag{5.9}$$

if and only if

$$AR \begin{bmatrix} h(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ h(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ h(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix} = \begin{bmatrix} h(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ h(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ h(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix}, \tag{5.10}$$

where

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & & & \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}$$

and the matrix R is such that $ARA = A$ holds, and $u: \mathcal{V}^n \mapsto \mathcal{V}$ is an arbitrary function.

Proof. If we put

$$F = \begin{bmatrix} f(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ f(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ f(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix} b,$$

H as in Theorem 5.1, then we can apply Corollary 5.2. The consistency condition $ARH = H$ is, in fact, (5.10). From the general solution $F = RH + (I - RA)V$ we can derive (5.9) by postmultiplication with b^{-1} and replacing the arbitrary function v by u. □

Theorem 5.5 *The vector functional equations*

$$\sum_{i=1}^{n} a_{ji} f(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \dots, \mathbf{Z}_{i+n-1}) b_j = h^j(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) \quad (j = 1, 2) \tag{5.11}$$

$$(n > 1, \qquad \mathbf{Z}_{n+i} \equiv \mathbf{Z}_i,\ 1 \le i \le n),$$

where $a_{ji} \in \mathbb{H}$ $(j = 1, 2;\ 1 \le i \le n)$ *and* $b_j \in \mathbb{H}$ $(j = 1, 2)$ *are constants and* $h^j : \mathcal{V}^n \mapsto \mathcal{V}$ $(j = 1, 2)$ *are given functions, have a common solution with a projector* P

$$\begin{bmatrix} f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) \\ f(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_1) \\ \vdots \\ f(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) \end{bmatrix} = R_2 \begin{bmatrix} h^2(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) \\ h^2(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_1) \\ \vdots \\ h^2(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) \end{bmatrix} b_2^{-1}$$
$$+P \left\{ R_1 \begin{bmatrix} h^1(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) \\ h^1(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_1) \\ \vdots \\ h^1(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) \end{bmatrix} b_1^{-1} - R_2 \begin{bmatrix} h^2(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) \\ h^2(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_1) \\ \vdots \\ h^2(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) \end{bmatrix} b_2^{-1} \right\} \tag{5.12}$$

if and only if

$$AR_1 \begin{bmatrix} h^1(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) \\ h^1(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_1) \\ \vdots \\ h^1(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) \end{bmatrix} b_1^{-1} = AR_2 \begin{bmatrix} h^2(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) \\ h^2(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_1) \\ \vdots \\ h^2(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) \end{bmatrix} b_2^{-1} \tag{5.13}$$

for some matrices R_1, R_2 *and* A *such that the row space of* A *is*

$$\text{Row}\, A = \text{Row}\, A_1 \cap \text{Row}\, A_2,$$

where

$$A_j = \begin{bmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{jn} & a_{j1} & \cdots & a_{j,n-1} \\ \vdots & & & \\ a_{j2} & a_{j3} & \cdots & a_{j1} \end{bmatrix} \qquad (j = 1, 2).$$

Proof. We transform the functional equation (5.11) into a matrix form. By a cyclic permutation of the vectors in (5.11) we obtain

$$a_{j1} f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) b_j + a_{j2} f(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_1) b_j + \dots + a_{jn} f(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) = h_1^j,$$
$$a_{jn} f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) b_j + a_{j1} f(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_1) b_j + \dots + a_{j,n-1} f(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) = h_2^j,$$
$$\vdots$$
$$a_{j2} f(\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n) b_j + a_{j3} f(\mathbf{Z}_2, \mathbf{Z}_3, \dots, \mathbf{Z}_1) b_j + \dots + a_{j1} f(\mathbf{Z}_n, \mathbf{Z}_1, \dots, \mathbf{Z}_{n-1}) = h_n^j$$

$$(j = 1, 2),$$

i.e., in a matrix form

$$A_j F b_j = H_j \qquad (j = 1, 2), \tag{5.14}$$

where

$$F = \begin{bmatrix} f(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ f(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ f(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} h_1^j \\ h_2^j \\ \vdots \\ h_n^j \end{bmatrix} \equiv \begin{bmatrix} h^j(\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n) \\ h^j(\mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_1) \\ \vdots \\ h^j(\mathbf{Z}_n, \mathbf{Z}_1, \ldots, \mathbf{Z}_{n-1}) \end{bmatrix}.$$

If for some F (5.14) holds, then

$$AR_1 H_1 b_1^{-1} = AR_1 A_1 F b_1 b_1^{-1} = AF = AR_2 A_2 F b_2 b_2^{-1} = AR_2 H_2 b_2^{-1},$$

which shows the necessity of (5.13).

To prove the sufficiency it is important to observe that for some subspace $\mathcal{L}$

$$\operatorname{Row} A_2 = \operatorname{Row} A \oplus \mathcal{L},$$

in which case

$$\operatorname{Row} A_1 \cap \mathcal{L} = \{0\},$$

and consequently there exists an idempotent matrix P such that $A_1 P = A_1$, while $\operatorname{Row} A_2 P \subset \operatorname{Row} A$. Assuming now that (5.13) is satisfied, then

$$F = F_2 + P(F_1 - F_2)$$

with $F_j = R_j H_j b_j^{-1}$ $(j = 1, 2)$ satisfies the two equations in (5.14). □

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Ice B. Risteski
2 Milepost Place # 606
Toronto, M4H 1C7, Ontario, Canada
e-mail: ice@scientist.com