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On the Maillet determinant associated with a Dirichlet character

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To the memory of Professor Kentaro Murata

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Abstract. The Maillet determinant associated with a Dirichlet character is considered.

1 Introduction

In this note we consider the Maillet determinant associated with a Dirichlet character. Let $m \geq 3$ be a natural number which is fixed, and a, b stand for rational integers which both are relatively prime to m. Assume that m is odd or divisible by 4. Denote by $R_m(a)$ the least positive residue of a modulo m: $a \equiv R_m(a) \pmod{m}, 1 \leq R_m(a) \leq m-1$. The Maillet determinant D_m is defined by

$$D_m = \det\Big(R_m(ab^{-1})\Big)_{a,b\in S},$$

where $S = \{1 \leq a < m/2 \mid (a,m)=1\}$ and b^{-1} is the multiplicative inverse of b modulo m, and provides the formula for the relative class number h_m^- of the m-th cyclotomic number field K [2], [14], [16]:

$$D_m = \frac{(-m)^{\frac{\varphi(m)}{2}-1}}{Q_K} \prod_{\chi\in X^{(1)}} \prod_{p|m} \Big(1-\chi(p)\Big) h_m^-,$$

where Q_K is the unit index of K (cf. [13] Chap. 3), $\varphi(m)$ is the Euler function, $X^{(1)}$ is the set of primitive Dirichlet characters with conductors dividing m such that $\chi(-1)=-1$ and $\prod\limits_{p|m}$ indicates the product taken over primes dividing m. Carlitz [1] and Fujisaki [6] generalized the Maillet determinant by using the Bernoulli polynomials of higher degree.

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On the other hand the determinant of the Demjanenko matrix, which is one having 0 and 1 as elements, also provides the formula for the relative class number of the cyclotomic number fields [8]. Tsumura [15] and Hirabayashi [10] unified the Maillet determinant and the Demjanenko matrix.

Now, denote by $R'_m(a)$ the absolutely least residue of a modulo m: $a \equiv R'_m(a) \pmod{m}, -m/2 < R'_m(a) < m/2$. Let

$$S'_m(a) = (-1)^{R'_m(a)},$$

$$T'_m(a) = \begin{cases} (-1)^{\frac{R'_m(a)}{2}} & \text{if } R'_m(a) \equiv 0 \pmod{2}, \\ 0 & \text{if } R'_m(a) \equiv 1 \pmod{2}, \end{cases}$$

$$U'_m(a) = \begin{cases} 0 & \text{if } R'_m(a) \equiv 0 \pmod{2}, \\ (-1)^{\frac{R'_m(a)-1}{2}} & \text{if } R'_m(a) \equiv 1 \pmod{2}. \end{cases}$$

Further for odd $m > 3$ denote by $R_m^{(3,k)}(a), k = 0, 1$, the residues of a modulo m which satisfy

$$-\frac{m}{3} < R_m^{(3.k)}(a) < \frac{m}{3}, \text{ or}$$

$$-\frac{2m}{3} < R_m^{(3,k)}(a) < \frac{2m}{3} \text{ and } R_m^{(3,k)}(a) \equiv k \pmod{2}.$$

Let

$$S_m^{(3,0)}(a) = \frac{1}{2}(1 - (-1)^{R_m^{(3,0)}(a)}3)$$

$$= \begin{cases} -1 & \text{if } R_m^{(3,0)}(a) \equiv 0 \pmod{2}, \\ 2 & \text{if } R_m^{(3,0)}(a) \equiv 1 \pmod{2}, \end{cases}$$

$$U_m^{(3,1)}(a) = \begin{cases} 0 & \text{if } R_m^{(3,1)}(a) \equiv 0 \pmod{2}, \\ (-1)^{\frac{R_m^{(3,1)}(a)-1}{2}} & \text{if } R_m^{(3,1)}(a) \equiv 1 \pmod{2}. \end{cases}$$

2 Theorems

Let H be a subgroup of the multiplicative group $G_m = (Z/mZ)^\times$, Z being the ring of rational integers, with index n not containing -1; then H corresponds to an imaginary subfield of degree n in K. Further, let S be a subset of $\{1 \leq a \leq m-1 \mid (a, m) = 1\}$ which forms a complete system of representatives of $G_m/(H, -1)$.

Now, let $B_k(x)$ be the Bernoulli polynomial of degree k: $B_1(x) = x - 1/2, B_2(x) = x^2 - x + 1/6, \ldots$. Let ψ be a real Dirichlet character modulo f, $(f, m) = 1$ and define a matrix $\Delta_{m,k,H}(\psi)$ of degree $n/2$ by

$$\Delta_{m,k,H}(\psi) = \Bigl(\sum_{c\in H}\sum_{d=0}^{f-1}\psi\Bigl(R_m(ab^{-1}c)+dm\Bigr)B_k\Bigl(\frac{R_m(ab^{-1}c)+dm}{fm}\Bigr)\Bigr)_{a,b\in S}.$$

Denote by X the set of primitive Dirichlet characters χ with conductors dividing m which satisfy $\chi(H)=1$, and by $X^{(k)}$ the subset of X consisting of $\chi\in X$such that $\chi(-1)=(-1)^k$. For a Dirichlet character χ with conductor f_χ, let $B_{k,\chi}$ be the generalized Bernoulli number belonging to $\chi: B_{k,\chi}=f_\chi^{k-1}\sum_{a=0}^{f_\chi-1}\chi(a)B_k(a/f_\chi)$.

The following theorem can be proved in a way similar to the proof of Theorem 4 in [5] by using abelian group determinant relation:

Theorem 1. *If $\psi(-1)=(-1)^{k'}$, then*

$$\det\Delta_{m,k,H}(\psi) = \prod_{\chi\in X^{(k+k')}}\frac{1}{2(fm)^{k-1}}\prod_{p\mid m}\Bigl(1-\chi\psi(p)p^{k-1}\Bigr)B_{k,\chi\psi}.$$

Let ψ_0 be the trivial character modulo f, and $\psi_D(a)=\Bigl(\dfrac{D}{a}\Bigr)$ the Kronecker character for the discriminant D of a quadratic number field.

By considering $\det\Delta_{m,1,\{1\}}(\psi)$ where $\psi=\psi_0$ with $f=4, \psi=\psi_{-4}$, $\psi=\psi_{-8}$ and $\psi=\psi_8$ Theorem 1 together with Lemma in the next section allows us to calculate the determinants whose (a,b) entries are $R'_m(ab^{-1})$, $S'_m(ab^{-1}), T'_m(ab^{-1})$ and $U'_m(ab^{-1})$, respectively, because we see

$$R'_m(ab^{-1}) = \pm m\sum_{d=0}^{3}\psi_0\Bigl(R_m(a'b^{-1})+dm\Bigr)B_1\Bigl(\frac{R_m(a'b^{-1})+dm}{4m}\Bigr),$$

$$S'_m(ab^{-1}) = -\psi_{-4}(m)2\sum_{d=0}^{3}\psi_{-4}\Bigl(R_m(a'b^{-1})+dm\Bigr)B_1\Bigl(\frac{R_m(a'b^{-1})+dm}{4m}\Bigr),$$

$$T'_m(ab^{-1}) = -\psi_{-8}(m)\sum_{d=0}^{7}\psi_{-8}\Bigl(R_m(a'b^{-1})+dm\Bigr)B_1\Bigl(\frac{R_m(a'b^{-1})+dm}{8m}\Bigr),$$

$$U'_m(ab^{-1}) = \pm(-1)^{\frac{m-1}{2}}\psi_8(m)\sum_{d=0}^{7}\psi_8\Bigl(R_m(a'b^{-1})+dm\Bigr)B_1\Bigl(\frac{R_m(a'b^{-1})+dm}{8m}\Bigr),$$

where $2a\equiv\pm a' \pmod m), a'\in S$ (cf. [7], [11], [12]). When m is a power of an odd prime, these determinants have already been calculated up to sign in [3], [4].

From now on we assume that $H=\{1\}$. We obtain the following:

Theorem 2. *Assume that $(m,6)=1$ and $H=\{1\}$.*

(1) $\det\Bigl(R_m^{(3,0)}(ab^{-1})\Bigr)_{a,b\in S}$

$$= \Big(-\frac{m}{2}\Big)^{\frac{\varphi(m)}{2}} \prod_{\chi\in X^{(1)}} \chi(2)\Big(1-\chi(3)\Big)\prod_{p\mid m}\Big(1-\chi(p)\Big)B_{1,\chi}.$$

(2) $\det\Big(S_m^{(3,0)}(ab^{-1})\Big)_{a,b\in S}$

$$= \Big(\psi_{-3}(m)\frac{3}{2}\Big)^{\frac{\varphi(m)}{2}} \prod_{\chi\in X^{(0)}} \chi(6)\prod_{p\mid m}\Big(1-\chi\psi_{-3}(p)\Big)B_{1,\chi\psi_{-3}}.$$

(3) $\det\Big(R_m^{(3,1)}(ab^{-1})\Big)_{a,b\in S}$

$$= \Big(-\frac{m}{2}\Big)^{\frac{\varphi(m)}{2}} \prod_{\chi\in X^{(1)}} \Big(1-\chi(2)\Big)\Big(1-\chi(3)\Big)\prod_{p\mid m}\Big(1-\chi(p)\Big)B_{1,\chi}.$$

(4) $\det\Big(U_m^{(3,1)}(ab^{-1})\Big)_{a,b\in S}$

$$= \Big((-1)^{\frac{m-1}{2}}\psi_{12}(m)\frac{1}{2}\Big)^{\frac{\varphi(m)}{2}} \prod_{\chi\in X^{(1)}} \chi(3)\prod_{p\mid m}\Big(1-\chi\psi_{12}(p)\Big)B_{1,\chi\psi_{12}}.$$

In the particular case where $m = p > 3$ is a prime, we have the following:

Corollary. *For an integer c prime to p let $f_p(c)$ indicate the multiplicative order of c modulo p.*

(1) $\det\Big(R_p^{(3,0)}(ab^{-1})\Big) \neq 0$ *if and only if* $f_p(3) \equiv 0 \pmod 2$.

(2) $\det\Big(S_p^{(3,0)}(ab^{-1})\Big) \neq 0$ *if and only if* $p \equiv 2 \pmod 3$.

(3) $\det\Big(R_p^{(3,1)}(ab^{-1})\Big) \neq 0$ *if and only if* $f_p(2) \equiv f_p(3) \equiv 0 \pmod 2$.

(4) $\det\Big(U_p^{(3,1)}(ab^{-1})\Big) \neq 0$ *for all primes* $p > 3$.

3 Proof of Theorem 2

Let u be an integer prime to m. Then, for any $a \in S$ there exists $a' \in S$ such that $ua \equiv \pm a' \pmod{\text{m}}$; thus u induces a permutation σ_u of elements in S as follows: $ua \equiv \pm a' \pmod m, \sigma_u(a) = a' \in S$. Let μ_u be the number of $a \in S$ such that $ua \equiv -\sigma_u(a) \pmod m$, and $\mathrm{sgn}\sigma_u$ the signature of σ_u, that is, $\mathrm{sgn}\sigma_u = 1$ or -1 according as σ_u is even or odd.

Lemma. *For an integer u prime to m, we have*

$$\prod_{\chi\in X}\chi(u) = (-1)^{\mu_u} \ \textit{and} \prod_{\chi\in X^{(0)}}\chi(u) = \mathrm{sgn}\sigma_u.$$

Proof. The first is immediate from that $\prod_{\chi\in X}\chi(u)\equiv u^{\frac{\varphi(m)}{2}}\equiv(-1)^{\mu_u}\pmod{m}$, and for the second see Hirabayashi [10].

Proof of Theorem 2. (1) We consider the case $\psi=\psi_0$ with $f=3$. We have that if $3a\equiv\pm2a'\pmod{m}, a'=\sigma_{2^{-1}3}(a)\in S$, then

$$\sum_{d=0}^{2}\psi_0\Big(R_m(a'b^{-1})+dm\Big)B_1\Big(\frac{R_m(a'b^{-1})+dm}{3m}\Big)$$

$$=\sum_{d=0}^{2}\psi_0\Big(R_m(a'b^{-1})+dm\Big)\Big(\frac{R_m(a'b^{-1})+dm}{3m}-\frac{1}{2}\Big)$$

$$=\begin{cases}\dfrac{2R_m(a'b^{-1})}{3m} & \text{if } R_m(a'b^{-1})\equiv 0 \pmod{3},\\[2ex] \dfrac{2R_m(a'b^{-1})-2m}{3m} & \text{if } R_m(a'b^{-1})\equiv m \pmod{3},\\[2ex] \dfrac{2R_m(a'b^{-1})-m}{3m} & \text{if } R_m(a'b^{-1})\equiv 2m \pmod{3}\end{cases}$$

$$=\pm\frac{R_m^{(3,0)}(ab^{-1})}{m},$$

which implies

$$\Big(R_m^{(3,0)}(ab^{-1})\Big)_{a,b\in S}$$

$$=m\Big(\pm\sum_{d=0}^{2}\psi_0\Big(R_m(a'b^{-1})+dm\Big)B_1\Big(\frac{R_m(a'b^{-1})+dm}{3m}\Big)\Big)_{a,b\in S}.$$

Since $(-1)^{\mu_{2^{-1}3}}\mathrm{sgn}\sigma_{2^{-1}3}=\prod_{\chi\in X^{(1)}}\chi(6)$ by Lemma, it follows from Theorem 1 that

$$\det\Big(R_m^{(3,0)}(ab^{-1})\Big)_{a,b\in S}$$

$$=m^{\frac{\varphi(m)}{2}}\prod_{\chi\in X^{(1)}}\chi(6)\det\Delta_{m,1,\{1\}}(\psi_0)$$

$$=\Big(-\frac{m}{2}\Big)^{\frac{\varphi(m)}{2}}\prod_{\chi\in X^{(1)}}\chi(2)\Big(1-\chi(3)\Big)\prod_{p|m}\Big(1-\chi(p)\Big)B_{1,\chi},$$

because $B_{1,\chi\psi_0} = \Bigl(1-\chi(3)\Bigr)B_{1,\chi}$.

(2) In this case, $f=3$ and $\psi_{-3}(a)=\pm 1$ according as $a\equiv \pm 1 \pmod 3$ unless $a\equiv 0 \pmod 3$. We have that if $3a \equiv \pm 2a' \pmod m, a'=\sigma_{2^{-1}3}(a)\in S$, then

$$\sum_{d=0}^{2}\psi_{-3}\Bigl(R_m(a'b^{-1})+dm\Bigr)B_1\Bigl(\frac{R_m(a'b^{-1})+dm}{3m}\Bigr)$$

$$=\frac{1}{3}\sum_{d=0}^{2}\psi_{-3}\Bigl(R_m(a'b^{-1})+dm)\Bigr)d$$

$$=\begin{cases} -\dfrac{1}{3}\psi_{-3}(m) & \text{if } R_m(a'b^{-1})\equiv 0, m \pmod 3,\\[2ex] \dfrac{2}{3}\psi_{-3}(m) & \text{if } R_m(a'b^{-1})\equiv 2m \pmod 3\end{cases}$$

$$=\frac{\psi_{-3}(m)}{3}S_m^{(3,0)}(ab^{-1}),$$

which yields

$$\Bigl(S_m^{(3,0)}(ab^{-1})\Bigr)_{a,b\in S}$$

$$=\psi_{-3}(m)3\Bigl(\sum_{d=0}^{2}\psi_{-3}\Bigl(R_m(a'b^{-1})+dm\Bigr)B_1\Bigl(\frac{R_m(a'b^{-1})+dm}{3m}\Bigr)\Bigr)_{a,b\in S}.$$

Since $\mathrm{sgn}\sigma_{2^{-1}3}=\prod_{\chi\in X^{(0)}}\chi(6)$ by Lemma it follows from Theorem 1 that

$$\det\Bigl(S_m^{(3,0)}(ab^{-1})\Bigr)_{a,b\in S}$$

$$=\Bigl(\psi_{-3}(m)3\Bigr)^{\frac{\varphi(m)}{2}}\prod_{\chi\in X^{(0)}}\chi(6)\det\Delta_{m,1,\{1\}}(\psi_{-3})$$

$$=\Bigl(\psi_{-3}(m)\frac{3}{2}\Bigr)^{\frac{\varphi(m)}{2}}\prod_{\chi\in X^{(0)}}\chi(6)\prod_{p\mid m}\Bigl(1-\chi\psi_{-3}(p)\Bigr)B_{1,\chi\psi_{-3}}.$$

(3) We consider the case $\psi=\psi_0$ with $f=12$. We have that if $3a\equiv \pm a' \pmod{12}, a'=\sigma_3(a)\in S$, then

$$\sum_{d=0}^{11}\psi_0\Bigl(R_m(a'b^{-1})+dm\Bigr)B_1\Bigl(\frac{R_m(a'b^{-1})+dm}{12m}\Bigr)$$

$$=\sum_{d=0}^{11}\psi_0\Bigl(R_m(a'b^{-1})+dm\Bigr)\Bigl(\frac{R_m(a'b^{-1})+dm}{12m}-\frac{1}{2}\Bigr)$$

$$= \begin{cases} \dfrac{R_m(a'b^{-1})}{3m} & \text{if } R_m(a'b^{-1}) \equiv 0 \text{ mod } 3), \\[2ex] \dfrac{R_m(a'b^{-1}) - m}{3m} & \text{if } R_m(a'b^{-1}) \equiv m \pmod 3, \\[2ex] \dfrac{R_m(a'b^{-1}) + m}{3m} & \text{if } R_m(a'b^{-1}) \equiv 2m \pmod 6, \\[2ex] \dfrac{R_m(a'b^{-1}) - 2m}{3m} & \text{if } R_m(a'b^{-1}) \equiv 5m \pmod 6 \end{cases}$$

$$= \pm \frac{R_m^{(3,1)}(ab^{-1})}{m},$$

which implies

$$\Big(R_m^{(3,1)}(ab^{-1})\Big)_{a,b \in S}$$

$$= m\Big(\pm \sum_{d=0}^{11} \psi_0\Big(R_m(a'b^{-1}) + dm\Big) B_1\Big(\frac{R_m(a''b^{-1}) + dm}{12m}\Big)\Big)_{a,b \in S}.$$

Hence it follows from Theorem 1 and Lemma that

$$\det\Big(R_m^{(3,1)}(ab^{-1})\Big)_{a,b \in S}$$

$$= m^{\frac{\varphi(m)}{2}} \prod_{\chi \in X^{(1)}} \chi(3) \det \Delta_{m,1,\{1\}}(\psi_0)$$

$$= \Big(-\frac{m}{2}\Big)^{\frac{\varphi(m)}{2}} \prod_{\chi \in X^{(1)}} \Big(1 - \chi(2)\Big)\Big(1 - \chi(3)\Big) \prod_{p|m} \Big(1 - \chi(p)\Big) B_{1.\chi},$$

because $B_{1,\chi\psi_0} = \Big(1 - \chi(2)\Big)\Big(1 - \chi(3)\Big) B_{1,\chi}$.

(4) In this case, $f = 12$ and $\psi_{12}(a) = 1$ or -1 according as $a \equiv 1, 11 \pmod{12}$ or $5, 7 \pmod{12}$ unless $(a, 12) \neq 1$. We have that if $3a \equiv \pm a'$, $a' = \sigma_3(a) \in S$, then

$$\sum_{d=0}^{11} \psi_{12}\Big(R_m(a'b^{-1}) + dm\Big) B_1\Big(\frac{R_m(a'b^{-1}) + dm}{12m}\Big)$$

$$= \frac{1}{12} \sum_{d=0}^{11} \psi_{12}\Big(R_m(a'b^{-1}) + dm\Big) d$$

$$= \begin{cases} \psi_{12}(m) & \text{if } R_m(a'b^{-1}) \equiv 2m, 3m, 4m, 5m \pmod{12}, \\[1ex] -\psi_{12}(m) & \text{if } R_m(a'b^{-1}) \equiv 8m, 9m, 10m, 11m \pmod{12}, \\[1ex] 0 & \text{if } R_m(a'b^{-1}) \equiv 0, m \pmod 6) \end{cases}$$

$$= \pm(-1)^{\frac{m-1}{2}}\psi_{12}(m)U_m^{(3,1)}(ab^{-1}),$$

which yields

$$\Big(U_m^{(3,1)}(ab^{-1})\Big)_{a,b\in S}$$
$$= (-1)^{\frac{m-1}{2}}\psi_{12}(m)\Big(\pm\sum_{d=0}^{11}\psi_{12}\Big(R_m(a'b^{-1})+dm\Big)B_1\Big(\frac{R_m(a'b^{-1})+dm}{12m}\Big)\Big)_{a,b\in S}.$$

Therefore it follows from Theorem 1 and Lemma that

$$\det\Big(U_m^{(3,1)}(ab^{-1})\Big)_{a,b\in S}$$
$$=\Big((-1)^{\frac{m-1}{2}}\psi_{12}(m)\Big)^{\frac{\varphi(m)}{2}}\prod_{\chi\in X^{(1)}}\chi(3)\det\Delta_{m,1,\{1\}}(\psi_{12})$$
$$=\Big((-1)^{\frac{m-1}{2}}\psi_{12}(m)\frac{1}{2}\Big)^{\frac{\varphi(m)}{2}}\prod_{\chi\in X^{(1)}}\chi(3)\prod_{p\mid m}\Big(1-\chi\psi_{12}(p)\Big)B_{1,\chi\psi_{12}}.$$

We conclude this note by noting that the products $\det\Big(R_m^{(3,0)}(ab^{-1})\Big)\det\Big(S_m^{(3,0)}(ab^{-1})\Big)$ and $\det\Big(R_m^{(3,1)}(ab^{-1})\Big)\det\Big(U_m^{(3,1)}(ab^{-1})\Big)$, unless they vanish, give the formulae for the relative class numbers of the quadratic extensions $K(\sqrt{-3})$ and $K(\sqrt{3})$ of K, respectively.

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