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$D_{p+1}(3)(p \geq 3)$ Can Be Characterized by Its Order Components

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Abstract. It is proved that if $M = D_{p+1}(3)(p \geq 3)$, G is a finite group and has the same order components of M, then $G \cong M$.

1 Introduction

If G is a finite group, we define the prime graph $\Gamma(G)$ as follows: its vertices are the primes dividing the order of G, and two vertices p and q are joined by an edge if and only if there is an element in G of order pq. We denote the set of all the connected components of graph $\Gamma(G)$ by $T(G) = \{\pi_i(G)$, for $i = 1, 2, \cdots, t(G)\}$ where $t(G)$ is the number of connected components of $\Gamma(G)$, and if G is of even order we always assume that 2 belongs to $\pi_1(G)$. We denote by $\pi(n)$ the set of all primes dividing n where n is a natural number. Obviously $|G|$ can be expressed as a product of $m_1, m_2, \cdots, m_{t(G)}$, where m_i is a positive integer with $\pi(m_i) = \pi_i(G)$. All m_i are called the order components of G. Let $OC(G) = \{m_1, m_2, \cdots, m_{t(G)}\}$ be the set of order components of G. The order components of non-abelian simple groups having at least two prime graph components have been obtained in [3].

Professor J.G.Thompson proposed the following conjecture. Let M be a non-abelian simple group, if G is a finite group satisfying $Z(G) = 1$ and $N(G) = N(M)$, where $N(G) = \{n \in N | G$ has a conjugacy class C, such that $|C| = n\}$, then $G \cong M$. To formulate another conjecture put forward by Professor W. J.Shi, let us introduce the notation $\pi_e(G)$ that denotes the set of orders of elements of G. After the conjecture, if both M and G are finite groups with properties $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$ then $G \cong M$. In [5], we had proved that if M is a simple group with non-connected prime graph and G is a finite group satisfying the conditions of J.G. Thompson's conjecture then $OC(G) = OC(M)$. Obviously, if G is a finite group satisfying the conditions of W.J. Shi's conjecture, we have $OC(G) = OC(M)$. As a consequence, these two conjectures naturally hold for a simple group M characterized by it's order components that consist at least of two

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elements. Hence, it's an important topic to find out those simple groups satisfying above mentioned properties.

We have established that the following simple groups have a non-connected prime graph and can be characterized by their order components: a finite simple group with at least three prime graph components [5], sporadic simple groups [3], Suzuki-Ree groups [6], $G_2(q)$ where $q \equiv 0 \pmod 3$ [4], $E_8(q)$ [1], $PSL_2(q)$ [7], ${}^3D_4(q)$ [8], ${}^2D_n(3), 9 \leq n = 2^m + 1 \neq p$ [9], ${}^2D_{p+1}(2)$, $5 \leq p \neq 2^m - 1$ [28], A_p where p and $p-2$ are primes [12], $PSL(5,q)$ [13], $PSL(3,q)$ where q is an odd prime power [14], $PSL(3,q)$ for $q = 2^n$ [15], $F_4(q)$ where q is even [16], $C_2(q)$ where $q > 5$ [17], $PSU_5(q)$ [18], $PSU(3,q)$ for $q > 5$ [19], ${}^2D_4(q)$ [20], ${}^2E_6(q)$ [22], $E_6(q)$ [21], $PSL(p,q)$.[23], $PSU(p,q)$[24], $PSL(p+1,q)$ [25], $PSU(p+1,q)$[26], $C_p(2)$[29].

In this paper we continue this work and will prove the following theorem:

Theorem 1. *Let $M = D_{p+1}(3)$ where p is an odd prime. If a finite group G satisfies the condition $OC(G) = OC(M)$, then $G \cong M$.*

2 Preliminary Results

Lemma 2. *[[3] Lemma 6] If $t(G) \geq 2$, H is a π_i subgroup of G, and $H \lhd G$,then $(\Pi_{j=1, j\neq i}^{t(G)} m_i) \mid (|H| - 1)$.*

Lemma 3. *[[2] Theorem 2] Let G be a 2-Frobenius group of even order. Then $t(G) = 2$, G has a normal series $1 \unlhd H \unlhd K \unlhd G$ such that $|K/H| = m_2$, $|H| \cdot |G/K| = m_1$, $|G/K| \mid (|K/H| - 1)$, $|G/K| \mid \varphi(|K/H|)$,and H is nilpotent.*

Lemma 4. *[[30] Lemma 3]If M is a simple group with $t(M) = 2$, G is a finite group and $OC(G) = OC(M)$, then one of the following holds:*

(1) G is a Frobenius group or 2-Frobenius group.

(2) G has a normal series $1 \unlhd H \unlhd K \unlhd G$ such that H is a nilpotent π_1-group, K/H is a non-abelian simple group, the odd order component of M is equal to one of those of K/H, G/K is a cyclic π_1-group, and $|G/K| \mid |Out(K/H)|$.

Lemma 5. *[[11] Remark] The only solution of the equation $p^m - q^n = 1$, where p , q are primes and $m, n > 1$, is $3^2 - 2^3 = 1$.*

Lemma 6. *[31] Let p be a prime and n be a natural number, $n \geq 2$. Then there exists a prime divisor r of $p^n - 1$ which does not divide $p^m - 1$ for any natural number $m \leq n$, except $n = 6$, $p = 2$ or $n = 2$, $p + 1$ is a power of 2. Such r is called a primitive prime divisor of $p^n - 1$.*

Of course a primitive prime divisor of $p^n - 1$ can not divide $p^n + 1$ or $p^m - 1$ for $n \nmid m$.

Lemma 7. *[[27] Lemma 1] If $n \geq 6$ is a natural number then there exists at least $s(n)$ primes p_i such that $\frac{n+1}{2} < p_i < n$:*
$s(n) = 6$ for $n \geq 49$;
$s(n) = 5$ for $42 \leq n \leq 47$;
$s(n) = 4$ for $38 \leq n \leq 41$;
$s(n) = 3$ for $18 \leq n \leq 37$;
$s(n) = 2$ for $14 \leq n \leq 17$;
$s(n) = 1$ for $6 \leq n \leq 13$.

Suppose p is a prime number, a is a natural number, We denote the power of p in the standard factorization of a by a_p, and if $a_p = p^n$, then we write $p^n \parallel a$. Of course $p^n \parallel a$ denotes that $p^n | a$ and $p^{n+1} \nmid a$.

Lemma 8. *[[29] Lemma 7, Lemma 8, Lemma 9] Let p be a prime, $q > 1$ be a natural number $e = min\{d > 0 : p \mid (q^d - 1)\}$, $q^e = 1 + p^r k$, $p \nmid k$, $s = \prod_{i=1}^{n}(q^i - 1)$, t is a natural number and $p^u \parallel t$. If $p > 2$ or $r > 2$ then $p^{r+u} \parallel (q^{et} - 1)$, $s_p < q^{\frac{np}{p-1}}$ and $s_p < q^{1.5n}$ if $p = 2$.*

Definition 9. *Let a and f be expressions of integers with integral coefficients, if $f \mid a$ and $(f,\ a/f) = 1$, then we say that f is a $hall\ factor\ of\ a$.*

Lemma 10. *[[10] Theorem 1]If q is a power of a prime number, $c = \prod_{i=1}^{n}(q^{2i} - 1)$ or $(q^n \pm 1) \cdot \prod_{i=1}^{n-1}(q^{2i} - 1)$, then there exists a hall factor f of c satisfying:*
(1) If $n \geq 23$ then $f > q^{8n}$;
(2) If $n = 22$ then $f > q^{7n}$;
(3) If $18 \leq n \leq 21$ then $f > q^{6n}$;
(4) If $16 \leq n \leq 17$ then $f > q^{5n}$;
(5) If $14 \leq n \leq 15$ then $f > q^{4n}$.
And if the standard factorization of $f = \prod_{k=1}^{t} r_k^{\delta_k}$, then $r_k^{\delta_k} \leq \frac{q^{n-1}-1}{q-1}$.

3 Proof of the theorem

Proof: Because $M = D_{p+1}(3)$, $p \geq 3$, and G has the same order components with M, so the even order component of G is $m_1 = 2 \cdot 3^{p(p+1)}(3^{p+1} - 1)(3^p + 1)\prod_{i=1}^{p-1}(3^{2i} - 1)$, the odd order component of G is $m_2 = (3^p - 1)/2$.

We divide the proof into several cases based on Lemma 4 and Tables 1-4 in [3].

Case 1. G is not be a Frobenius group or a 2-Frobenius group.

Subcase 1.1 If G is a Frobenius group with Frobenius kernel H and complement K, then $|H| = m_1,\ |K| = m_2$ since $|K| \mid (|H|-1)$. There exists a primitive prime divisor r of $3^{2p}-1$ by Lemma 6. Set $S_r \in Syl_r(H)$, of course $|S_r| \mid (3^p+1)/4$ and $S_r \trianglelefteq G$ since H is nilpotent and so $S_r charH$. $|S_r| \equiv 1 \pmod{m_2}$ by Lemma 2, which is impossible.

Subcase 1.2 If G is a 2-Frobenius group , there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H is a nilpotent π_1 group, $|K/H| = m_2$, $|G/K| \mid (|K/H| - 1) = (3 \cdot (3^{p-1} - 1))/2$. Hence $(3^p + 1) \mid |H|$. Similarly to Subcase 1.1, we can show it's impossible.

From Subcase 1.1, Subcase 1.2 and Lemma 4 we have the following properties:

1. There is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a simple group, H and G/K are π_1 group and H is nilpotent.

2. The odd order component of G is one of those of K/H, consequently $t(K/H) \geq 2$. Hence K/H may be one of the simple groups listed in Tables 1-4 in [3].

Case 2. $K/H \ncong E_7(2),\ E_7(3),\ A_2(2),\ A_2(4),\ {}^2A_5(2),\ {}^2E_6(2),\ {}^2F_4(2)'$ or one of the sporadic simple groups.

Any odd order component of $E_7(2)$, $E_7(3)$, $A_2(2)$, $A_2(4)$, ${}^2A_5(2)$, or one of the sporadic simple groups(except Suz and F_{22}) can not be written into the form $(3^p-1)/2$ for $p \geq 3$. Though ${}^2E_6(2)$, Suz, F_{22} or ${}^2F_4(2)'$ has an order component 13 can be written into the form $(3^p-1)/2$ and $p = 3$, but the order of ${}^2E_6(2)$, Suz, F_{22} or ${}^2F_4(2)'$ can not divides $|D_4(3)|$. So $K/H \ncong E_7(2)$, $E_7(3)$, $A_2(2)$, $A_2(4)$, ${}^2A_5(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$ or one of the sporadic simple groups.

Case 3. $K/H \ncong A_n$.

If $K/H \cong A_n$ then A_n has an odd component equal to $(3^p-1)/2$. Thus $|A_{(3^p-1)/2}| \mid |A_n|$ and $|A_n| \mid |D_{p+1}(3)|$. by Lemma 7, there exists at least six primes p_i satisfying $(3^p+1)/4 < p_i < (3^p-1)/2$ for $p \geq 5$,On the other hand there exists at most three prime divisors of $|D_{p+1}(3)|$ between $(3^p+1)/4$ and $(3^p-1)/2$ by Lemma 7, a contradiction.

By trivial calculation, we conclude that p can not be 3.

Case 4 $K/H \ncong A_n(q)$, ${}^2A_n(q)$, $E_6(q)$, $E_8(q)$, $F_4(q)$, ${}^2F_4(q)$ or ${}^2E_6(q)$.

Subcase 4.1 If $K/H \cong A_1(q)$, then $(3^p-1)/2 = q,\ (q\pm1)/(2, q-1)$. Whenever in any case we have that $q \leq 3^{p+1}$, hence $|K/H| < 3^{3(p+1)}$. Assume $q = r^f$, we have that $|G/K| < 3^{p+1}$ since $2^{3^{p+1}/2} > 3^{p+1}$ and $|G/K| \mid |Out(K/H)| = 2f$. If $p \geq 14$ then there exists a hall factor g of $|G| = 3^{p(p+1)}(3^{p+1}-1)\prod_{i=1}^{p}(3^{2i}-1)$ satisfying that $g > 3^{4p}$ and for any prime number $r' \mid g$ we have that $g_{r'} < (3^p-1)/2$ by Lemma 10. Clearly $(g,\ |H|) \neq 1$. Let p' be a prime number satisfy $p' \mid (g,\ |H|)$ and $S_{p'} \in Syl_{p'}(G)$. $S_{p'}$ is a normal π_1-subgroup of G and $|S_{p'}| < (3^p-1)/2$, which contradicts Lemma 2.

By trivial calculation, we can show p can not be 3, 5, 7 or 11.

Subcase 4.2 If $K/H \cong A_{p'}(q),\ q-1 \mid p'-1$ then $(3^p-1)/2 = (q^{p'}-1)/(q-1)$. Thus $q^{p'+1} \geq 3^p$.

If $p' > 7$ then $q^{p'(p'+1)/2} > 3^{3(p+1)}$, which implies q is a power of 3 by Lemma 8 and Lemma 8. It follows that $3^p - 3 = 2(q^{p'-1} + q^{p'-2} + \cdots + q)$, consequently $q = 3$ and $p = p'$, furthermore $3^p + 1 \mid |H|$. Similarly to Subcase 1.1, we can get a contradiction.

By trivial calculation we can show that p' can not be 3 or 5.

Similarly, we can show that $K/H \ncong {}^2A_n(q),\ E_6(q),\ E_8(q),\ F_4(q),\ {}^2F_4(q)$ or ${}^2E_6(q)$.

Subcase 4.3 If $K/H \cong A_{p'-1}(q)$ then $(q^{p'}-1)/(q-1)(p',\ q-1) = (3^p-1)/2$.

Similarly to Subcase 4.2, we can prove that p' can not be greater than 7. And similarly to Subcase 4.1, it is easy to prove that p' can not be 3, 5 or 7.

Case 5 $K/H \ncong B_n(q)$ or $C_n(q)$.

Subcase 5.1 If $K/H \cong C_{p'}(2)$, then $2^{p'} - 1 = (3^p - 1)/2$, $2^{p'+1} - 1 = 3^p$, which contradicts Lemma 5.

Subcase 5.2 If $K/H \cong B_{p'}(3)$ or $C_{p'}(3)$, then $(3^p-1)/2 = (3^{p'}-1)/2$, $p = p'$, $|B_p(3)|$ and $|C_p(3)|$ are divisors of $|D_{p+1}(3)|$, which is impossible.

Subcase 5.3 Similarly to Subcase 4.2, we can show that $K/H \cong B_n(q)$ or $C_n(q)(4 \leq n = 2^m)$.

Case 6 $K/H \ncong {}^2D_n(q)$.

Subcase 6.1 If $K/H \cong {}^2D_{p'}(3)(5 \leq p' \neq 2^k+1)$, then $(3^p-1)/2 = (3^{p'}+1)/4$ $2 \cdot 3^p - 3^{p'} = 3$ a contradiction. Similarly, we can prove that $K/H \ncong {}^2D_n(3)(9 \leq n = 2^k + 1$ is not a prime); $K/H \ncong {}^2D_{p'}(3)(5 \leq p' = 2^k + 1)$.

Subcase 6.2 If $K/H \cong {}^2D_{p'+1}(2)(p' \neq 2^m - 1)$, then $(3^p - 1)/2 = 2^{p'} - 1$, $3^p = 2^{p'+1} - 1$, which contradicts Lemma 5. Similarly we can prove $K/H \ncong {}^2D_{p'+1}(2)(3 \leq p' = 2^k - 1)$.

Subcase 6.3 Similarly to Subcase 4.2, we can prove that $K/H \ncong {}^2D_n(q)(2 \leq n = 2^k)$.

Case 7 $K/H \ncong G_2(q)$; $K/H \ncong {}^3D_4(q)$; $K/H \ncong {}^2G_2(q)(q = 3^{2k+1})$.

Subcase 7.1 If $K/H \cong G_2(q)(3 \mid q)$, then $(3^p-1)/2 = q^2 \pm q + 1$, $3^p - 3 = q^2 \pm q$, a contradiction. Similarly we have that $K/H \ncong {}^2G_2(q)(q = 3^{2k+1})$.

Subcase 7.2 Similarly to Subcase 4.1, we can prove that $K/H \ncong G_2(q)(3 \mid q \pm 1)$ or ${}^3D_4(q)$.

Case 8 $K/H \ncong {}^2B_2(q)(4 \leq q = 2^{2k+1})$.

If $K/H \cong {}^2B_2(q)(4 \leq q = 2^{2k+1})$, then $(3^p - 1)/2 = q \pm \sqrt{2q} + 1$ or $q - 1$. Clearly $(3^p - 1)/2 \neq q - 1$。

If $(3^p - 1)/2 = q + \sqrt{2q} + 1$, then $3(3^{p-1} - 1) = 2^{k+2}(2^k + 1)$, $2^k \mid p - 1$ by Lemma 8, furthermore, $2^{k+2}(2^k + 1) = 3(3^{p-1} - 1) > 3^{p-1} > 3^{2^k} > 2^{2^k} > 2^{2k+3} > 2^{k+2}(2^k + 1)$ for $k \geq 4$, a contradiction. By calculation we can prove that k can not be 1, 2 or 3.

Similarly we have $(3^p - 1)/2 \neq q - \sqrt{2q} + 1$.

Case 9 From Cases 1-8 and Lemma 4 we have K/H is isomorphic to one of $D_n(q)$.

If $K/H \cong D_{p'}(3)(p' \geq 5)$, then $(3^p - 1)/2 = (3^{p'} - 1)/2$, $p = p'$, hence $|G/K| \cdot |H| = 3^{2p}(3^p + 1)(3^{p+1} - 1)$, furthermore, $3^p + 1 \mid |H|$ since $|G/K| \mid |Out(K/H)| = 4$, similarly to Subcase 1.1 we can get a contradiction.

If $K/H \cong D_{P'}(5)$, $p' \geq 5$, then $(5^{p'} - 1)/4 = (3^p - 1)/2$, $5^{p'} > 3^P$. Furthermore, $5^{p'(p'-1)} > 3^{3(p+1)}$, which contradicts Lemma 8.

So $K/H \cong D_{p'+1}(3)$, $(3^{p'} - 1)/2 = (3^p - 1)/2$, $p = p'$, hence $K/H = 1$, $H = 1$, $G \cong M$. $\square$

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