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A characterization of the automorphism groups of sporadic groups by the set of orders of maximal Abelian subgroups

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Abstract. In this paper as the main result we prove that the automorphism groups of some sporadic simple groups are uniquely determined by the set of orders of maximal abelian subgroups. Also we determine finite groups with the same prime graph as these groups.

1 Introduction

In group theory it is usual to get information on the structure of a group G by studying the subgroups of G. It is proved that if G is one of the following groups: $PSL(2, 2^n)$, $Sz(2^{2m+1})$, A_n ($n \leq 10$), A_p where p and $p-2$ are prime numbers or $B_n(q)$, where $n = 2^m \geq 4$, then G is uniquely determined by the set of orders of maximal abelian subgroups of G [1, 3, 30]. Chen and et. al. in [12] proved that every sporadic simple group is uniquely determined by the set of orders of maximal abelian subgroups. In this paper we continue this work and show that the automorphism groups of some sporadic simple groups are uniquely determined by the set of orders of maximal abelian subgroups. We use the notation $M(G)$ for the set of orders of maximal abelian subgroups of G. There is a close relation between this result and the prime graph of G. Now we recall the definition of the prime graph of a finite group.

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then the set $\pi(|G|)$ is denoted by $\pi(G)$. We construct the prime graph of G as follows: *The prime graph* $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$, and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1(G), \pi_2(G), \ldots, \pi_{t(G)}(G)$ be the connected components of $\Gamma(G)$. We use the notation π_i instead of $\pi_i(G)$.

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If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$. Now $|G| = m_1 m_2 \ldots m_{t(G)}$, where $\pi(m_i) = \pi_i$ $(1 \leq i \leq t(G))$. The positive numbers $m_1, \ldots, m_{t(G)}$ are called the order components of G and $m_2, \ldots, m_{t(G)}$ are called the odd order components of G (see [20]). Also the set of order elements of G is denoted by $\pi_e(G)$. In [2, 17, 22, 23] finite groups with the same prime graph as a CIT simple group, ${}^2F_4(q)$, $PGL(2,p)$ and $PSL(2,q)$ are determined. It is proved that if $q = 3^{2n+1}$ $(n > 0)$, then the simple group ${}^2G_2(q)$ is uniquely determined by its prime graph [21, 32]. Also in [24] it is proved that $PSL(2,p)$, where $p > 11$ is a prime number and $p \not\equiv 1 \pmod{12}$ is uniquely determined by its prime graph. Hagie in [11] determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. In this paper, also we determine finite groups G such that their prime graph is $\Gamma(A)$, where A is the automorphism group of a sporadic simple group, except $Aut(J_2)$. The structure of the automorphism groups of sporadic simple groups are described in [6]. Let S be a sporadic simple group. Then the prime graph of $Aut(S)$ is connected if and only if $S = J_2$ or McL.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [6]. If p is a prime number, then $p^k \| n$ means that $p^k \mid n$, but $p^{k+1} \nmid n$.

2 Preliminary Results

The next lemma summarizes the basic structural properties of a Frobenius group [9, 27]:

Lemma 2.1. Let G be a Frobenius group and let H, K be Frobenius complement and Frobenius kernel of G, respectively. Then $t(G) = 2$, and the prime graph components of G are $\pi(H)$, $\pi(K)$. Also the following conditions hold:

(1) $|H|$ divides $|K| - 1$.

(2) K is nilpotent and if $|H|$ is even, then K is abelian.

(3) Sylow p-subgroups of H are cyclic for odd p and are cyclic or generalized quaternion for $p = 2$.

(4) If H is a non-solvable Frobenius complement, then H has a normal subgroup H_0 such that $|H : H_0| \leq 2$, $H_0 = SL(2,5) \times Z$, where the Sylow subgroups of Z are cyclic and $(|Z|, 30) = 1$.

Also the next lemma follows from [10] and the properties of Frobenius groups [13]:

Lemma 2.2. Let G be a 2-Frobenius group, i.e. G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then

(1) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;

(2) G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq Aut(K/H)$;

(3) H is nilpotent and G is a solvable group.

Definition. A group G is called a C_{pp} group if the centralizers in G of its elements of order p are $p-$groups.

Lemma 2.3. ([5]) **(a)** The $C_{13,13}$-simple groups are: A_{13}, A_{14}, A_{15}; Suz, Fi_{22}; $L_2(q)$, $q = 3^3$, 5^2, 13^n or $2 \times 13^n - 1$ which is a prime, $n \geq 1$; $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(5)$, $S_6(3)$, $O_8^+(3)$, $G_2(q)$, $q = 2^2$, 3; $F_4(2)$,$U_3(q)$, $q = 2^2$, 23; $Sz(2^3)$, ${}^3D_4(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$.
(b) The $C_{19,19}$-simple groups are: A_{19}, A_{20}, A_{21}; J_1, J_3, $O'N$, Th, HN; $L_2(q)$, $q = 19^n$, $2 \times 19^n - 1$ which is a prime, $(n \geq 1)$; $L_3(7)$, $U_3(2^3)$, $R(3^3)$, ${}^2E_6(2)$.

By using Theorem A and Lemma 2.3 in [31] we have the following result:
Lemma 2.4. Let G be a finite group and A be the automorphism group of a sporadic simple group S. If $\Gamma(A)$ is not connected and $\Gamma(G) = \Gamma(A)$, then one of the following holds:
(a) G is a Frobenius or a 2-Frobenius group;
(b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group with $t(K/H) \geq 2$ and $|G/K| \mid |Out(K/H)|$. Also $\pi_2(A) = \pi_i(K/H)$ for some $i \geq 2$ and $\pi_2(A) \subseteq \pi(K/H) \subseteq \pi(S)$.

Lemma 2.5. ([31, Corollary]) If G is a solvable group with at least two prime graph components, then G is either a Frobenius group or a 2-Frobenius group and G has exactly two prime graph components one of which consists of the primes dividing the lower Frobenius complement.

As a corollary of Lemma 2.5, it follows that if G is a solvable finite group, then $t(G) \leq 2$.

Lemma 2.6. [3, Lemma 2] For G and M assume $M(G) = M(M)$. Then G and M have the same prime graph.

Lemma 2.7. [3, Lemma 3] For G and M assume $M(G) = M(M)$. If the prime graph of M has isolated points and the Sylow subgroups corresponding to these primes are of prime order, then the set of odd order components of K/H in Lemma 2.4 is a subset of order components of G.

Lemma 2.8. ([4]) Let $t(G) \geq 2$ and $N \trianglelefteq G$. If N is a π_1-group and let $m_2, \ldots, m_r$ are odd order components of G, then each of the order components $m_2, \ldots, m_r$ is a divisor of $|N| - 1$.

Lemma 2.9. [25] Let G be a finite group, N a normal subgroup of G, and G/N a Frobenius group with Frobenius kernel F and cyclic complement C. If $(|F|, |N|) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$ for some prime divisor p of $|N|$.

Lemma 2.10. [28] Let G be a finite group and N a nontrivial normal p-subgroup, for some prime p, and set $K = G/N$. Suppose that K contains an element x of order m coprime to p such that $\langle \varphi|_{\langle x\rangle}, 1|_{\langle x\rangle}\rangle > 0$ for every Brauer character φ of (an absolutely irreducible representation of) K in characteristic p. Then G contains elements of order pm.

The next lemma was introduced by Crescenzo and modified by Bugeaud:
Lemma 2.11. ([7, 20]) With the exceptions of the relations $(239)^2 - 2(13)^4 = -1$ and $(3)^5 - 2(11)^2 = 1$ every solution of the equation

$$p^m - 2q^n = \pm 1; \quad p, q \ \text{ prime}; \quad m, n > 1,$$

has exponents $m = n = 2$; i.e. it comes from a unit $p - q.2^{\frac{1}{2}}$ of the quadratic field $Q(2^{\frac{1}{2}})$ for which the coefficients p, q are prime.

Lemma 2.12. ([20]) The only solution of the equation $p^m - q^n = 1$; p, q prime; and $m, n > 1$ is $3^2 - 2^3 = 1$.

Lemma 2.13. (Zsigmondy's Theorem) ([33])
Let p be a prime and n be a positive integer. Then one of the following holds: (i) $p = 2$, $n = 1$ or 6; (ii) p is a Mersenne prime and $n = 2$; (iii) there is a *primitive prime* p' for $p^n - 1$, that is, $p'|(p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \le m < n$.

3 Main Results

In this section we prove that the automorphism group of each finite sporadic simple group, except $Aut(J_2)$ and $Aut(McL)$, is uniquely determined by the set of orders of maximal abelian subgroups of G. For the proof of these results we determine finite groups with the same prime graph as the automorphism group of a sporadic simple group.

If S is a sporadic simple group and $A = Aut(S)$, then $|A : S| \le 2$ (see [6]), so $A = S$ or $|A : S| = 2$. Let S be M_{11}, M_{23}, M_{24}, J_1, J_4, Ru, Ly, Co_1, Co_2, Co_3, Fi_{23}, M, B or Th. Then $Aut(S) = S$ [6] and for these groups Hagie in [11] determined finite groups with the same prime graph as $\Gamma(S)$. Also Chen and et. al. in [12] proved that in this case S and so $Aut(S)$ are uniquely determined by the set of orders of maximal abelian subgroups. Therefore in the sequel we consider the rest of sporadic simple groups. Hence we consider the case $|A : S| = 2$.

If S is one of the following groups: M_{12}, He, Fi_{22} or HN, then $Aut(S) \neq S$ but $\Gamma(S) = \Gamma(Aut(S))$. Therefore groups with the same prime graph of these groups are determined in [11]. So in the next three theorems we prove that these groups are uniquely determined by $M(Aut(S))$.

Theorem 3.1. Let G be a finite group such that $M(G) = M(Aut(M_{12}))$. Then $G \cong Aut(M_{12})$.
Proof. By using Lemma 2.6, we know that $\Gamma(G) = \Gamma(M_{12})$. Now by using Theorem 3(5) in [11] it follows that $G \cong 11^{2n} : SL_2(5)$ for some $n \in \mathbb{N}$; $G \cong 11^{2n} : SL_2(5).2$ for some $2 \le n \in \mathbb{N}$; $G/O_\pi(G) \cong L_2(11)$, $L_2(11).2$, where $\pi \subseteq \{2, 5\}$; $G/O_2(G) \cong M_{11}$, where $O_2(G) \neq 1$; or $G/O_2(G) \cong M_{12}$ or $Aut(M_{12})$. Now we consider each case.

If $G \cong 11^{2n} : SL_2(5)$ for some $n \in \mathbb{N}$ or $G \cong 11^{2n} : SL_2(5).2$ for some $2 \leq n \in \mathbb{N}$, then $11^2 \mid |G|$ and so there is an abelian subgroup of order 11^2 in G. Therefore 11^2 divides an element of $M(G)$, which is a contradiction, since 11^2 does not divide any element of $M(Aut(M_{12}))$.

If $G/O_\pi(G) \cong L_2(11)$ or $L_2(11).2$, where $\pi \subseteq \{2,5\}$, then $3 \| \, |G|$. But we know that $Aut(M_{12})$ has an abelian subgroup of order 9, which is a contradiction.

If $G/O_2(G) \cong M_{11}$, where $O_2(G) \neq 1$, then let $Z = Z(O_2(G))$. Hence $Z \trianglelefteq G$ and so $11 \mid (|Z| - 1)$ and $|Z| \mid 2^5$. Therefore $|Z| = 1$, which is a contradiction.

If $G/O_2(G) \cong M_{12}$ or $Aut(M_{12})$, then $Z = Z(O_2(G))$ is a normal subgroup of G and similarly to the above discussion it follows that $O_2(G) = 1$. Therefore $G \cong M_{12}$ or $Aut(M_{12})$. But by using Theorem 2.1 in [12] we know that M_{12} is uniquely determined by $M(M_{12})$. Therefore $G \cong Aut(M_{12})$. □

Theorem 3.2. Let G be a finite group such that $M(G) = M(Aut(He))$. Then $G \cong Aut(He)$.

Proof. By using Lemma 2.6 and Theorem 3(4) in [11] it follows that $G/O_\pi(G) \cong L_2(16)$, where $2, 7 \in \pi$; $L_2(16).2$, where $7 \in \pi$; $L_2(16).4$, where $7 \in \pi$; $O_8^-(2)$, where 2 or $7 \in \pi$; $O_8^-(2).2$; $S_8(2)$ where $\pi \subseteq \{2,3,5\}$; He or $Aut(He)$ where $\pi \subseteq \{2,3\}$. We note that always in the above discussion, $\pi \subseteq \{2,3,5\}$ or $\pi \subseteq \{2,3,7\}$.

If $7 \in \pi$, then let $Z = Z(P)$, where P is a 7-Sylow subgroup of $O_\pi(G)$. Then $|Z| = 7^i$, where $1 \leq i \leq 3$, since $7^3 \| \, |Aut(He)|$. Also Z is a normal subgroup of G and by using Lemma 2.8, $17 \mid (7^i - 1)$, where $1 \leq i \leq 3$, which is a contradiction. Therefore $7 \nmid |O_\pi(G)|$. Also $7^2 \mid |Aut(He)|$ and so $Aut(He)$ has an abelian subgroup of order 7^2. But $7 \nmid |O_\pi(G)|$ and 7^2 does not divide $|O_8^-(2)|$ and $|S_8(2)|$. Therefore $G/O_\pi(G) \cong He$ or $Aut(He)$. Similarly to the above we can prove that $3, 5 \notin \pi$. Therefore we conclude that $G/O_2(G) \cong He$ or $Aut(He)$. Now by using GAP we have the Brauer Character table of He (mod 2). Let x be an element of order 17 in He and let $X = \langle x \rangle$. Now for each irreducible character φ of He (mod 2) we can see that $\langle \varphi|_X, 1|_X \rangle > 0$. Now by using Lemma 2.10, it follows that $34 \in \pi_e(G)$, which is a contradiction. Therefore $G \cong He$ or $Aut(He)$. By using Theorem 2.1 in [12] we know that He is uniquely determined by $M(He)$ and so we conclude that $G \cong Aut(He)$. □

Theorem 3.3. The automorphism group of HN and the automorphism group of Fi_{22} are uniquely determined by the set of orders of maximal abelian subgroups.

Proof. First we prove the theorem for $Aut(HN)$. Similarly to the proof of the last theorems, by using Theorem 3(4) in [11] we know that $G/O_\pi(G) \cong HN$ or $Aut(HN)$, where $\pi \subseteq \{2,3,5,7\}$. Also 19 is an odd order component of G and similarly to the last theorems we conclude that $O_\pi(G) = 1$. For convenience we omit the proof. Now since HN is uniquely determined by $M(HN)$, we conclude that $G \cong Aut(HN)$.

Now we prove the theorem for $Aut(Fi_{22})$. Using Theorem 3(4) in [11] we conclude that $G/O_\pi(G) \cong Suz$, $Aut(Suz)$, Fi_{22} or $Aut(Fi_{22})$, where $\pi \subseteq \{2,3,5\}$. Let $G/O_\pi(G) \cong Suz$. By using the Brauer character table of Suz mod 2 or 3, we conclude that if $X = \langle x \rangle$, where $o(x) = 13$, then for every irreducible character φ of Suz mod 2 (or 3) we have $\langle \varphi|_X, 1|_X \rangle > 0$. Therefore we conclude that G has an element of order 26 (or 39), which is a contradiction. Similarly if $G/O_\pi(G) \cong Aut(Suz)$, then we can prove that $O_\pi(G) = 1$. On the other hand $Aut(Fi_{22})$ has an abelian subgroup of order 42, but $Aut(Suz)$ has not an abelian subgroup of this order. Therefore $G/O_\pi(G) \cong Fi_{22}$ or $Aut(Fi_{22})$. Similarly to the above we can prove that $\pi \subseteq \{2, 3\}$. We know that ${}^2F_4'(2)$ is isomorphic to a subgroup of Fi_{22}. Now by using the Brauer character table of ${}^2F_4'(2)$ mod 2 or 3 we can prove that $O_\pi(G) = 1$ and for convenience we omit the proof. Also by Theorem 2.1 in [12] it is proved that Fi_{22} is uniquely determined by $M(Fi_{22})$, and so $G \cong Aut(Fi_{22})$. □

In the sequel we consider the rest of sporadic simple groups. We know that every sporadic simple group is a C_{pp} simple group, for some prime p. In order to determine finite groups with the same prime graph as these groups, we consider the following diophantine equations which have many applications in the theory of finite groups (for example see [15] or [20]):

$$(i)\quad \frac{q^p-1}{q-1} = y^n, \qquad\qquad (ii)\quad \frac{q^p-1}{(q-1)(p,q-1)} = y^n,$$

$$(iii)\quad \frac{q^p+1}{q+1} = y^n, \qquad\qquad (iv)\quad \frac{q^p+1}{(q+1)(p,q+1)} = y^n.$$

We note that the odd order components of some non-abelian simple groups of Lie type are of the form $(q^p \pm 1)/((q \pm 1)(p, q \pm 1))$ [15] and there exists some results about these diophantine equations [19]. Now we prove the following lemma about these diophantine equations to determine some C_{pp}-simple groups.

Lemma 3.4. Let $p \geq 3$ and p_0 be prime numbers and $q = p_0^\alpha$.
(a) If $y = 11$ and $p_0 \in \{2, 3, 5, 7\}$, then $(p, q, n) = (5, 3, 2)$ is the only solution of (i) and (ii). Also $(p, q, n) = (5, 2, 1)$ is the only solution of (iii) and (iv).
(b) If $y = 29$ and $p_0 \in \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$, then the diophantine equations (i)-(iv) have no solution.
(c) If $y = 31$ and $p_0 \in \{2, 3, 5, 7, 11, 19\}$, then $(p, q, n) = (5, 2, 1)$ and $(3, 5, 1)$ are the only solutions of (i) and (ii). Also (iii) and (iv) have no solution.
Proof. Let $q = p_0^\alpha$ and $(q^p - 1)/(q - 1) = 11^n$ or $(q^p - 1)/((q - 1)(p, q - 1)) = 11^n$. Then $11 \mid (p_0^{\alpha p} - 1)$, which implies that $p_0^{\alpha p} \equiv 1 \pmod{11}$ and hence $\beta := ord_{11}(p_0)$ is a divisor of αp. Since $p \geq 3$ and $(p_0^{\alpha p} - 1)/(p_0^\alpha - 1) = 11^n$ or $(p_0^{\alpha p} - 1)/(p_0^\alpha - 1)(p, p_0^\alpha - 1)) = 11^n$, it follows that 11 is a primitive prime for $p_0^{\alpha p} - 1$. Also 11 is a primitive prime for $p_0^\beta - 1$, by the definition of $ord_{11}(p_0)$. Therefore $\beta = \alpha p$, by the definition of the primitive prime (see Lemma 2.3). Also by using the Fermat theorem we know that β is a divisor of 10. Hence the only possibility for p is 5 and so $1 \leq \alpha \leq 2$. Now by checking the possibilities for q it follows that $(p, q, n) = (5, 3, 2)$ is the only solution of the diophantine equations (i) and (ii). Similarly consider the diophantine equations

$$\frac{q^p+1}{q+1} = 11^n, \quad \text{and} \quad \frac{q^p+1}{(q+1)(p,q+1)} = 11^n,$$

Then 11 is a divisor of $p_o^{2\alpha p} - 1$ and in a similar manner it follows that $p = 5$ and

$\alpha = 1$. Therefore the only solution of these diophantine equations is $(p, q, n) = (5, 2, 1)$.

The proofs of (b) and (c) are similar and for convenience we omit the proof of them. $\Box$

The odd order components of finite non-abelian simple groups are listed in Table 1 in [15]. Now by using Lemmas 2.11, 2.12, 2.13 and 3.4 we can prove the following lemma. For convenience we omit the proof.

Lemma 3.5. Let M be a simple group of Lie type over $GF(q)$, where $q = p^\alpha$.

(a) If $p \in \{2, 3, 5, 7, 11\}$ and M is a $C_{11,11}$-group, then M is $L_2(11)$, $L_5(3)$, $L_6(3)$, $U_5(2)$, $U_6(2)$, $O_{11}(3)$, $S_{10}(3)$ or $O^+_{10}(3)$.

(b) If $p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$ and M is a $C_{29,29}$-group, then $M = L_2(29)$.

(c) If $p \in \{2, 3, 5, 7, 11, 19, 31\}$ and M is a $C_{31,31}$-group, then M is $L_5(2)$, $L_3(5)$, $L_6(2)$, $L_4(5)$, $O^+_{10}(2)$, $O^+_{12}(2)$, $L_2(31)$, $L_2(32)$, $G_2(5)$ or $Sz(32)$.

Theorem 3.6. The automorphism group of the sporadic simple group $O'N$ is uniquely determined by its prime graph. Therefore $Aut(O'N)$ is characterizable by the set of orders of maximal abelian subgroups.

Proof. Let G be a finite group such that $\Gamma(G) = \Gamma(Aut(O'N))$. Since there exists no edge between 3, 11, 31 in $\Gamma(G)$, we conclude that G is a non-solvable group, otherwise G has a Hall $\{3, 11, 31\}$-subgroup T, which has three components and this is a contradiction. Therefore G is not a 2-Frobenius group, by Lemma 2.5. If G is a non-solvable Frobenius group and H, K be the Frobenius complement and the Frobenius kernel of G, respectively, then by using Lemma 2.1 it follows that H has a normal subgroup H_0 with $|H : H_0| \leq 2$ such that $H_0 = SL(2, 5) \times Z$ where the Sylow subgroups of Z are cyclic and $(|Z|, 30) = 1$. We know that $3 \nsim 7$, $3 \nsim 11$ and $3 \nsim 19$ in $\Gamma(G)$. Therefore $Z = 1$ and hence $\{7, 11\} \subseteq \pi(K)$, but this is a contradiction, since K is nilpotent and $7 \nsim 11$ in $\Gamma(G)$. Hence G is neither a Frobenius group nor a 2-Frobenius group. Therefore Lemma 2.4 implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is a $C_{31,31}$-simple group and $\pi(K/H) \subseteq \pi(G)$. Hence K/H is $L_3(5)$, $L_5(2)$, $L_6(2)$, $L_2(31)$, $L_2(32)$, $G_2(5)$ or $O'N$. If $K/H \cong L_2(5)$, $L_6(2)$, $L_2(31)$ or $G_2(5)$, then $11, 19 \in \pi(H)$, which is a contradiction, since $11 \nsim 19$ in $\Gamma(G)$ and H is nilpotent. If $K/H \cong L_3(5)$ or $L_2(32)$, then $\{7, 19\} \subseteq \pi(H)$, which is a contradiction, since $7 \nsim 19$ in $\Gamma(G)$. Therefore $K/H \cong O'N$ and since $Out(O'N) = 2$, it follows that $G/H \cong O'N$ or $Aut(O'N)$. We know that $O'N$ has a $11 : 5$ subgroup by [6]. If $7 \in \pi(H)$, then let T be a $\{5, 7, 11\}$-subgroup of G which is solvable and hence $t(T) \leq 2$, which is a contradiction since there exists no edge between 5, 7 and 11 in $\Gamma(G)$. Therefore $7 \notin \pi(H)$. If we consider $\{5, 11, p\}$-subgroup of G, where $p \in \{19, 31\}$, it follows that $\pi(H) \cap \{7, 19, 31\} = \emptyset$. Therefore $\pi(H) \subseteq \{2, 3, 5, 11\}$. Also $O'N$ has a $19 : 3$ subgroup, which implies that $\pi(H) \cap \{11\} = \emptyset$. Let $p \in \{3, 5\}$. If $p \in \pi(H)$, then let P be the p-Sylow subgroup of H. If $Q \in Syl_7(G)$, then Q acts fixed point freely on P, since $7 \nsim 3$ and $7 \nsim 5$ in $\Gamma(G)$. Therefore PQ is a Frobenius group and hence Q is a cyclic group. But this is a contradiction since Sylow 7-subgroups of $O'N$ are elementary abelian by [6]. Therefore $\pi(H) \cap \{3, 5\} = \emptyset$. Hence H is a 2-group. Let $H \neq 1$. Since $K/H \cong O'N$, consider the modular character table of $O'N$ mod 2. Let $X = \langle x \rangle$ and $o(x) = 31$. By using GAP we can see that for every irreducible character of $O'N$ mod 2, say φ, we have $\langle \varphi|_X, 1|_X \rangle > 0$. Therefore by using Lemma 2.10 we conclude that $2 \sim 31$, which is a contradiction. Therefore $H = 1$ and so $G \cong O'N$ or $Aut(O'N)$. We know that $2 \nsim 11$, in $\Gamma(O'N)$ and so $G \cong Aut(O'N)$. Therefore $Aut(O'N)$ is characterizable by its prime graph and consequently by $M(Aut(O'N))$. $\square$

Theorem 3.7. Let G be a finite group.

(a) If $\Gamma(G) = \Gamma(Aut(M_{22}))$, then $G/O_2(G) \cong M_{22}$ and $O_2(G) \neq 1$ or $G/O_2(G) \cong Aut(M_{22})$.

(b) The automorphism group of M_{22} is uniquely determined by the set of orders of maximal abelian subgroups.

Proof. (a) Let $\Gamma(G) = \Gamma(Aut(M_{22}))$. Since there exists no edge between 3, 5 and 7, we conclude that G is a non-solvable group. Since $5 \nsim 7$, $3 \nsim 11$ and $7 \nsim 11$ in $\Gamma(G)$, we conclude that G is not a non-solvable Frobenius group. Therefore G is not a Frobenius group or a 2-Frobenius group. By using Lemma 2.4, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a $C_{11,11}$-simple group. Now by using Lemma 3.5 we know that if M is a simple group of Lie type, then M is isomorphic to $L_2(11)$, $L_5(3)$, $L_6(3)$, $U_5(2)$, $U_6(2)$, $O_{11}(3)$, $S_{10}(3)$ or $O_{10}^+(3)$. But we know that 41 divides the orders of $O_{11}(3)$, $S_{10}(3)$ and $O_{10}^+(3)$. Similarly 13 divides the orders of $L_5(3)$ and $L_6(3)$. Also $3 \sim 5$ in the prime graph of $U_5(2)$ and $U_6(2)$. Therefore if K/H is a simple group of Lie type, then K/H is isomorphic to $L_2(11)$. If K/H is an alternating group or a sporadic simple group which is a $C_{11,11}$-group, then K/H is A_{11}, A_{12}, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, McL, HS, Sz, $O'N$, Co_2 or J_1. Also $\Gamma(K/H)$ is a subgraph of $\Gamma(G)$. Therefore $3 \nsim 5$ in $\Gamma(K/H)$ and $\pi(K/H) \subseteq \{2, 3, 5, 7, 11\}$, which implies that the only possibilities for K/H are $L_2(11)$, M_{11}, M_{12} and M_{22}. If $K/H \cong M_{11}$, M_{12} or $L_2(11)$, then K/H has a $11 : 5$ subgroup by [6]. Also in these cases $7 \notin \pi(K/H)$ and hence $7 \in \pi(H)$. Now consider the $\{5, 7, 11\}$-subgroup T of G which is solvable and hence $t(T) \leq 2$, a contradiction. Therefore $K/H \cong M_{22}$ and since $Out(M_{22}) \cong \mathbb{Z}_2$ it follows that $G/H \cong M_{22}$ or $Aut(M_{22})$. Also H is a nilpotent π_1-group and so $\pi(H) \subseteq \{2, 3, 5, 7\}$. By using [6] we know that M_{22} has a $11 : 5$ subgroup. If $3 \in \pi(H)$, then let T be a $\{3, 5, 11\}$-subgroup of G which is solvable and hence $t(T) \leq 2$, which is a contradiction, since there exists no edge between 3, 5 and 11 in $\Gamma(G)$. Therefore $3 \notin \pi(H)$. Similarly it follows that $7 \notin \pi(H)$. Let $5 \in \pi(H)$ and $Q \in Syl_5(H)$. Also let $P \in Syl_3(K)$. We know that H is nilpotent and hence Q *char* H. Since $H \lhd K$ it follows that $Q \lhd K$. Therefore P acts by conjugation on Q and since $3 \nsim 5$ in $\Gamma(G)$ it follows that P acts fixed point freely on Q. Hence QP is a Frobenius group with Frobenius kernel Q and Frobenius complement P. Now by using Lemma 2.1 it follows that P is a cyclic group which implies that a Sylow 3-subgroup of M_{22} is cyclic. But this is a contradiction since a 3-Sylow subgroup of M_{22} are elementary abelian by [6]. Therefore H is a 2-group. Hence

$G/O_2(G) \cong M_{22}$ where $O_2(G) \neq 1$ or $G/O_2(G) \cong Aut(M_{22})$.
(b) If $M(G) = M(Aut(M_{22}))$, then $\Gamma(G) = \Gamma(Aut(M_{22}))$. Therefore by using (a) we conclude that $G/H \cong M_{22}$ or $Aut(M_{22})$ where H is a 2-group. Let Z be the center of H. Then Z is a normal subgroup of G and so $11 \mid (|Z| - 1)$, by Lemma 2.8. On the other hand we know that $|Z| \mid 2^8$. Therefore the only possibility is $|Z| = 1$, which implies that $H = 1$. Thus $G \cong Aut(M_{22})$ and so $Aut(M_{22})$ is characterizable by the set of orders of maximal abelian subgroups. □

Theorem 3.8. The automorphism group of the sporadic simple group J_3 is uniquely determined by its prime graph. Therefore $Aut(J_3)$ is characterizable by the set of orders of maximal abelian subgroups.
Proof. Let G be a finite group such that $\Gamma(G) = \Gamma(Aut(J_3))$. By considering 5, 17 and 19 we conclude that G is not solvable, and so G is not a 2-Frobenius group. Since $3 \nsim 17$, $3 \nsim 19$ and $17 \nsim 19$ in $\Gamma(G)$, it follows that G is neither a Frobenius group nor a 2-Frobenius group. So by using Lemma 2.4, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a $C_{19,19}$ simple group. By using Lemma 2.3, we have a list of $C_{19,19}$-simple groups. Since $\pi(K/H) \subseteq \pi(J_3)$ and $\pi(J_3) \cap \{7, 11, 13, 31\} = \emptyset$, it follows that the only possibilities for K/H are J_3 and $L_2(19^n)$, where $n \geq 1$.
Case 1. Let $K/H \cong L_2(19^n)$, where $n \geq 1$.
We know that $\pi_e(L_2(q))$ consists of all divisors of q, $(q+1)/d$ and $(q-1)/d$ where $d = (2, q-1)$ (see [22]). If $q = 19^n$, then $3 \mid (19^n - 1)/2$ and since $3 \sim 5$ and $3 \nsim 17$ in $\Gamma(G)$, it follows that if 5 divides $|G|$, then $5 \mid (19^n - 1)$ and if 17 is a divisor of $|G|$, then $17 \mid (19^n + 1)$. Note that $\pi(19 - 1) = \{2, 3\}$, $\pi(19^2 - 1) = \{2, 3, 5\}$ and $17 \mid (19^4 + 1)$. Now by using the Zsigmondy's Theorem, Lemmas 2.11 and 2.12 it follows that the only possibility is $n = 1$. Since $Out(L_2(19)) \cong \mathbb{Z}_2$, it follows that $G/H \cong L_2(19)$ or $L_2(19).2$. But in this case $\pi(K/H) = \{2, 3, 5, 19\}$ and so $17 \mid |H|$. We know that $L_2(19)$ contains a $19 : 9$ subgroup and hence G has a $\{3, 17, 19\}$-subgroup T which is solvable and so $t(T) \leq 2$. But this is a contradiction, since $t(T) = 3$. Therefore $K/H \ncong L_2(19)$.
Case 2. Let $K/H \cong J_3$.
Similar to the above discussions, we have $G/H \cong J_3$ or $Aut(J_3)$. Also H is a nilpotent π_1-group. Hence $\pi(H) \subseteq \{2, 3, 5, 17\}$. If $17 \in \pi(H)$, then let T be a $\{3, 17, 19\}$ subgroup of G, since J_3 has a $19 : 9$ subgroup. Obviously T is solvable and hence $t(T) \leq 2$, which is a contradiction. Therefore $\pi(H) \subseteq \{2, 3, 5\}$ and $G/H \cong J_3$ or $Aut(J_3)$.

Now similar to the proof of the last theorems we prove that $H = 1$. Let $2 \mid |H|$ and x be an element of order 19 in $K/H \cong J_3$ and $X = \langle x \rangle$. By using GAP we can see that for every irreducible character φ (mod 2) of J_3, we have $\langle \varphi|_X, 1|_X \rangle > 0$. Therefore we have $2 \sim 19$, which is a contradiction. If $3 \mid |H|$, then let x be an element of order 17 and use the irreducible characters of J_3 (mod 3), to conclude that $3 \sim 17$, which is a contradiction. If $5 \mid |H|$, then let x be an element of order 19 in J_3 and use the irreducible characters of J_3 (mod 5), which implies that $5 \sim 19$, and this is a contradiction. Therefore $H = 1$, and so $G \cong Aut(J_3)$, since $2 \nsim 17$ in $\Gamma(J_3)$ and $2 \sim 17$ in $\Gamma(G)$. □

Theorem 3.9. (a) Let G be a finite group satisfying $\Gamma(G) = \Gamma(Aut(HS))$. Then $G \cong McL$, $G/O_2(G) \cong U_6(2)$ or HS, where $O_2(G) \neq 1$ or $G/O_2(G) \cong Aut(HS)$ or $U_6(2).2$.
(b) The finite group $Aut(HS)$ is uniquely determined by the set of orders of maximal abelian subgroups.
Proof. **(a)** Let $\Gamma(G) = \Gamma(Aut(HS))$. Since there exists no edge between 3, 7 and 11 in $\Gamma(G)$, we conclude that G is not solvable and so G is not a 2-Frobenius group. Since $5 \nsim 7$, $5 \nsim 11$ and $7 \nsim 11$ in $\Gamma(G)$, it follows that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is one of the following groups: M_{11}, M_{12}, M_{22}, McL, HS, $U_5(2)$, $U_6(2)$ and $L_2(11)$.
Case 1. Let $K/H \cong M_{11}$, M_{12}, $U_5(2)$ or $L_2(11)$.
By using [6] we see that in each case $|Out(K/H)|$ is a divisor of 2 and $7 \notin \pi(G/H)$. Therefore $7 \in \pi(H)$. Also in each case, K/H has a $11:5$ subgroup. Hence it follows that G has a $\{5, 7, 11\}$ subgroup T, which is solvable and hence $t(T) \leq 2$. But this is a contradiction and so this case is impossible.
Case 2. Let $K/H \cong M_{22}$.
Since $Out(M_{22}) \cong \mathbb{Z}_2$, we have $G/H \cong M_{22}$ or $Aut(M_{22})$. First let $G/H \cong M_{22}$, where H is a π_1-group and $\pi_1 = \{2, 3, 5, 7\}$. We know that M_{22} has a $11:5$ subgroup (see [6]). Let $p \in \{2, 3, 7\}$. We know that there exists no edge between p, 5 and 11 in $\Gamma(G)$. Now since M_{22} has a $11:5$ subgroup we conclude that $\pi(H) \cap \{2, 3, 7\} = \emptyset$. If $5 \in \pi(H)$, then let P be a Sylow 5-subgroup of H. If $Q \in Syl_3(G)$, then Q acts fixed point freely on P, since $3 \nsim 5$ in $\Gamma(G)$. Therefore PQ is a Frobenius group which implies that Q be a cyclic group and this is a contradiction. Hence $H = 1$ and so $G = M_{22}$. But $\Gamma(M_{22}) \neq \Gamma(Aut(HS))$, since $2 \nsim 5$ in $\Gamma(M_{22})$. Therefore this case is impossible.

Now let $G/H \cong Aut(M_{22})$. By using [6], M_{22} has a $11:5$ subgroup. Similar to the above discussion we conclude that $\{3, 5, 7\} \cap \pi(H) = \emptyset$, and hence H is a 2-group. But in this case 3 and 5 are not joined which is a contradiction. Therefore Case 2 is impossible, too.
Case 3. Let $K/H \cong U_6(2)$.
We know that $U_6(2)$ has a $11:5$ subgroup and so $7 \notin \pi(H)$. Also $\pi(H) \subseteq \{2, 3, 5\}$. If $3 \in \pi(H)$, then let x be an element of $U_6(2)$ of order 7 and $X = \langle x \rangle$. Then for every irreducible character φ of $U_6(2)$ (mod 3) we have $\langle \varphi|_X, 1|_X \rangle > 0$. Therefore $3 \sim 7$ in $\Gamma(G)$, which is a contradiction. Similarly since $5 \nsim 7$ in $\Gamma(G)$, it follows that $5 \notin \pi(H)$. By using [6], we know that $Out(K/H) \cong S_3$. We know that $U_6(2).3$ has an element of order 21. Therefore $G/H \cong U_6(2)$ or $U_6(2).2$. As we mentioned above if $G/H \cong U_6(2)$, then $H \neq 1$ is a 2-group and so $G/O_2(G) \cong U_6(2)$, where $O_2(G) \neq 1$. Similarly if $G/H \cong U_6(2).2$, then $G/O_2(G) \cong U_6(2).2$.
Case 4. Let $K/H \cong HS$.
This case is similar to Case 3 and we omit the details of the proof. There exists a $11:5$ subgroup in HS. Similar to Case 3, it follows that $G/O_2(G) \cong HS$, where $O_2(G) \neq 1$, or $G/O_2(G) \cong Aut(HS)$.
Case 5. Let $K/H \cong McL$.
Again if $p \in \{2, 3, 5\}$ and $p \mid |H|$, then by using the Brauer character table of McL (mod p), we conclude that G has an element of order $11p$, which is impossible. Also McL has a $11:5$ subgroup, which implies that $7 \nmid |H|$. Therefore $H = 1$.

We know that $Out(McL) = 2$, and since $Aut(McL)$ has an element of order 22, we conclude that $G \ncong Aut(McL)$. Therefore $G \cong McL$.
(b) If $M(G) = M(Aut(HS))$, then $\Gamma(G) = \Gamma(Aut(HS))$. Therefore by using (a) we conclude that $G/H \cong HS$ or $U_6(2)$, where $H \neq 1$ is a 2-group; or $G/H \cong Aut(HS)$ or $U_6(2).2$, where H is a 2-group; or $G \cong McL$. We know that $5 \| |U_6(2)|$, but $5^2 \mid |Aut(HS)|$. Therefore there is an abelian subgroup of order 5^2 and hence 5^2 divides an element of $M(Aut(HS))$. Obviously if $G/H \cong U_6(2)$ or $U_6(2).2$, where H is a 2-group, then 5^2 does not divide any element of $M(G)$, which is a contradiction. Also McL is uniquely determined by $M(McL)$ by Theorem 2.1 in [12]. Therefore K/H is isomorphic to HS or $Aut(HS)$, where H is a 2-group. Let Z be the center of H. Then Z is a normal subgroup of G and so $11 \mid (|Z|-1)$. On the other hand we know that $|Z| \mid 2^{10}$. Therefore the only possibility is $|Z| = 1$ or $|Z| = 2^{10}$. If $|Z| = 2^{10}$, then 2^{10} divides an element of $M(Aut(HS))$. Therefore $Aut(HS)$ has an abelian subgroup of order 2^{10}. We note that $2^{10} \| |Aut(HS)|$. But we know that $Aut(HS)$ has a subgroup which is isomorphic to M_{22} and we know that the 2-Sylow subgroup of M_{22} is not abelian, which implies that $H = 1$. Thus $G \cong HS$ or $Aut(HS)$. Now by Theorem 2.1 in [12] the simple group HS is characterizable by the set of orders of maximal abelian subgroups. Therefore $G \cong Aut(HS)$ and so $Aut(HS)$ is characterizable by the set of orders of maximal abelian subgroups. □

Theorem 3.10. Let G be a finite group.

(a) If $\Gamma(G) = \Gamma(Aut(Fi'_{24}))$, then $G/O_2(G) \cong Fi'_{24}$, where $O_2(G) \neq 1$ or $G/O_2(G) \cong Aut(Fi'_{24})$.

(b) The finite group $Aut(Fi'_{24})$ is uniquely determined by the set of orders of maximal abelian subgroups.

Proof. **(a)** Let $\Gamma(G) = \Gamma(Aut(Fi'_{24}))$. By considering $\{7, 17, 23\}$-subgroup of G we conclude that G is not solvable. If G is a non-solvable Frobenius group, then $\{11, 13, 17, 23, 29\} \subseteq \pi(K)$, where K is the Frobenius kernel of G, which is a contradiction since $11 \nsim 13$. Hence by using Lemma 2.4, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is a $C_{29,29}$-simple group and $\pi(K/H) \subseteq \pi(G)$. Therefore K/H is $L_2(29)$, Ru or Fi'_{24}. If $K/H \cong L_2(29)$ or Ru, then $\{17, 23\} \subseteq \pi(H)$, which is a contradiction, since H is nilpotent and $17 \nsim 23$ in $\Gamma(G)$. Therefore $K/H \cong Fi'_{24}$ and so $G/H \cong Fi'_{24}$ or $Aut(Fi'_{24})$. By using [6], we know that Fi'_{24} has a $23:11$ subgroup. Therefore $\pi(H) \cap \{5, 7, 13, 17\} = \emptyset$. Also Fi'_{24} has a $29:7$ subgroup, and hence $\pi(H) \cap \{11, 13\} = \emptyset$. Also we know that Fi'_{24} has a subgroup isomorphic to He. If $3 \mid |H|$ and $N/H \cong He$, then for every irreducible character φ of He (mod 3) we have $\langle \varphi|_X, 1|_X \rangle > 0$, where $X = \langle x \rangle$ and $o(x) = 17$. Therefore $3 \sim 17$, which is a contradiction. Hence $G/O_2(G) \cong Fi'_{24}$ where $O_2(G) \neq 1$; or $G/O_2(G) \cong Aut(Fi'_{24})$.
(b) By assumptions we know that $\Gamma(G) = \Gamma(Fi'_{24})$ and $2^{22} \| |Aut(Fi'_{24})|$. Hence if Z is the center of H, then $Z \trianglelefteq G$. On the other hand $29 \mid (|Z|-1)$, which implies that $H = 1$ and by Theorem 2.1 in [12] we get the result. □

Theorem 3.11. Let G be a finite group.

(a) If $\Gamma(G) = \Gamma(Aut(Suz))$, then $G \cong Aut(Suz)$, Fi_{22} or $Aut(Fi_{22})$.

(b) The finite group $Aut(Suz)$ is uniquely determined by the set of orders of maximal abelian subgroups.

Proof. **(a)** Since there is no edge between 7, 11 and 13 in $\Gamma(G)$, it follows that G is not a solvable group. If G is a non-solvable Frobenius group, then $\{11, 13\} \subseteq \pi(K)$, where K is the Frobenius kernel of G which is a contradiction. Therefore G is neither a Frobenius group nor a 2-Frobenius group. Hence there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a $C_{13,13}$ simple group and $\pi(K/H) \subseteq \pi(G)$. Therefore K/H is $Sz(8)$, $U_3(4)$, ${}^3D_4(2)$, Suz, Fi_{22}, ${}^2F_4(2)'$, $L_2(27)$, $L_2(25)$, $L_2(13)$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $O_8^+(3)$, $S_6(3)$, $G_2(4)$, $S_4(5)$ or $G_2(3)$.

Let $K/H \cong Sz(8)$. It is known that $Out(Sz(8)) \cong \mathbb{Z}_3$ and so $G/H \cong Sz(8)$ or $Sz(8).3$. If $G/H \cong Sz(8)$, then $\{3, 11\} \subseteq \pi(H)$ which is a contradiction, since $3 \nsim 11$. If $G/H \cong Sz(8).3$, then let T be $\{3, 7, 11\}$-subgroup of G, since $Sz(8)$ has a $7 : 6$ subgroup. Then $t(T) = 3$, which is a contradiction.

If $K/H \cong {}^2F_4(2)'$, $U_3(4)$, $L_2(25)$, $L_4(3)$, $S_4(5)$ or $G_2(3)$, then $\{7, 11\} \subseteq \pi(H)$, which implies that $7 \sim 11$, since H is nilpotent, and this is a contradiction. If $K/H \cong {}^3D_4(2)$, $L_2(27)$, $L_2(13)$ or $L_3(3)$, then $\{5, 11\} \subseteq \pi(H)$ and we get a contradiction similarly, since $5 \nsim 11$.

If $K/H \cong G_2(4)$, $S_6(3)$, $O_7(3)$ or $O_8^+(3)$, then $11 \in \pi(H)$ and K/H has a $13 : 3$ subgroup by [6]. Now by considering T as a $\{3, 11, 13\}$-subgroup of G, it follows that $t(T) = 3$, which is a contradiction since T is solvable.

If $K/H \cong Fi_{22}$, then $G/H \cong Fi_{22}$ or $Aut(Fi_{22})$, where $\pi(H) \subseteq \{2, 3, 5, 7, 11\}$. Since Fi_{22} has a $11 : 5$ subgroup and a $13 : 3$ subgroup it follows that $\{7, 11\} \cap \pi(H) = \emptyset$. Therefore $G/H \cong Fi_{22}$ or $Aut(Fi_{22})$, where $\pi(H) \subseteq \{2, 3, 5\}$. Similar to the last cases we can prove that if $5 \mid |H|$ and $K/H \cong Fi_{22}$, then for every irreducible character φ of Fi_{22} (mod 5) we have $\langle \varphi|_X, 1|_X \rangle > 0$, where $X = \langle x \rangle$ and $o(x) = 13$. Therefore $5 \sim 13$, which is a contradiction. We know that ${}^2F_4'(2)$ is isomorphic to a subgroup of Fi_{22}. Now by using the Brauer character table of ${}^2F_4'(2)$ mod 2 or 3 we can prove that $H = 1$. Therefore $G \cong Fi_{22}$ or $Aut(Fi_{22})$.

If $K/H \cong Suz$, then $G/H \cong Suz$ or $Aut(Suz)$. If $G/H \cong Suz$, then $\pi(H) \subseteq \{2, 3, 5, 7, 11\}$. Since Suz has a $11 : 5$ and $13 : 3$ subgroups it follows that $7, 11 \notin \pi(H)$. Similar to previous cases if $p \in \{2, 3, 5\}$, then by using the irreducible characters of Suz (mod p) we can see that $p \sim 13$, which is a contradiction. Therefore $G \cong Aut(Suz)$, since $2 \nsim 11$ in $\Gamma(Suz)$.

(b) By using (a) we see that by assumption it follows that G is isomorphic to $Aut(Suz)$, Fi_{22} or $Aut(Fi_{22})$. Now by using Theorem 2.1 in [12] and Theorem 3.3 we get the result. □

We know that the prime graph of the automorphism group of M_{22}, J_3, HS, Suz, $O'N$ and Fi_{24}' are disconnected and so we can use Theorem A in [31] and Lemma 2.4. But the prime graph of $Aut(McL)$ is connected and so we need a completely different method for this group.

Theorem 3.12. Let G be a finite group such that $\Gamma(G) = \Gamma(Aut(McL))$. Then $G/O_2(G)$ is isomorphic to HS, $Aut(HS)$, McL, $Aut(McL)$, $U_6(2)$ or $U_6(2).2$.

Proof. We note that the prime graph of $Aut(McL)$ is connected and 2 is joined to 3, 5, 7 and 11. Also $3 \sim 5$ in $\Gamma(G)$ and there is not any other edge in $\Gamma(G)$.
Step 1. *If N is a maximal normal subgroup of G and $A = \pi(N) \cap \{5, 7, 11\}$, then A has at most one element.*

First we prove that G is not solvable. If G is a solvable group, then let H be a Hall $\{5, 7, 11\}$-subgroup G. Then H is solvable and $t(H) = 3$, which is a contradiction by Lemma 2.5.

Let N be a maximal normal solvable subgroup of G and obviously $N \neq G$. Also by Lemma 2.5, it follows that $|\pi(N)| \leq 2$, so $|\pi(A)| \leq 2$. If $|A| = 2$, then let $A = \{p_1, p_2\}$, $p \in \{5, 7, 11\} \setminus A$ and H be a Hall A-subgroup of N. Now N is a normal subgroup of G and H is a Hall subgroup of N. Therefore $G = NN_G(H)$, by the Frattini argument. Since $p \notin \pi(N)$, it follows that $p \in \pi(N_G(H))$ and so there is an element $y \in N_G(H)$ of order p. It is obvious that y acts fixed point freely on H and $o(y) = p$. Therefore H is nilpotent by Thompson's Theorem [9, Theorem 10.2.1], which implies that $p_1 \sim p_2$, a contradiction. Similarly we can prove that $\pi(N) \cap \{3, 7, 11\}$ has at most one element.

As a consequence of this result we conclude that $\pi(\overline{G}) \cap A$ has at least two elements and so there exists $p \in \{7, 11\}$ such that $p \in \pi(\overline{G})$.
Step 2. *Let $\overline{G} = G/N$. The subgroup $S = Socle(\overline{G})$ is a nonabelian simple group.*

We know that $C_{\overline{G}}(S) = 1$ and $N_{\overline{G}}(S) = \overline{G}$, which implies that $S \leq \overline{G} \leq Aut(S)$. The socle of a group is a direct product of minimal normal subgroups and so $S = M_1 \times M_2 \times \cdots \times M_r$, where M_i, $1 \leq i \leq r$, are minimal normal subgroups. Also every minimal normal subgroup is characteristically simple and so is a direct product of isomorphic simple groups. Therefore $S = P_1 \times \cdots \times P_k$, where each P_i, $1 \leq i \leq k$, is a non-abelian simple group. Also note that $\pi(S) \subseteq \pi(G) = \{2, 3, 5, 7, 11\}$ and so $\pi(P_i) \subseteq \{2, 3, 5, 7, 11\}$, for every $1 \leq i \leq k$. There exist only finitely many nonabelian simple groups P such that $\pi(P) \subseteq \{2, 3, 5, 7, 11\}$ and if P is a nonabelian simple group such that $\pi(P) \subseteq \{2, 3, 5, 7, 11\}$, then we can see that $2, 3 \in \pi(P)$ and $\pi(Out(P)) \subseteq \{2, 3\}$ (see [26]).

We claim that $k = 1$. Let $k \geq 2$. Then $7, 11 \notin \pi(S)$, since $3 \in \pi(P_i)$, for every $1 \leq i \leq k$, and $3 \nsim 7$ and $3 \nsim 11$ in $\Gamma(G)$. Hence $\pi(P_i) \subseteq \{2, 3, 5\}$ and by using [26] we see that for every $1 \leq i \leq k$, P_i is isomorphic to A_5, A_6 or $U_4(2)$. On the other hand, $7, 11 \in \pi(Out(S))$, since $Z(S) = 1$. We note that $\{7, 11\} \cap \pi(N)$ has at most one element. So let $p \in \{7, 11\} \cap \pi(\overline{G})$ and let $\varphi \in \overline{G}$ be an element of order p. Obviously $\varphi \in Aut(S)$. Let $Q = P_1^{\varphi}$ and $f_i : Q \to P_i$, $1 \leq i \leq k$, be the natural projection of Q to P_i. Also P_1 is a normal subgroup of S and so Q is a normal subgroup of S. Therefore $Im\ f_i \trianglelefteq P_i$ and P_i is a simple group, which implies that $Im\ f_i = 1$ or $Im\ f_i = P_i$, for every $1 \leq i \leq k$. On the other hand, P_1 is a simple group, and so Q is a simple group. Therefore $\ker f_i = 1$ or $\ker f_i = Q$. If $\ker f_i = 1$, then $Im\ f_i = P_i$, which implies that $Q \cong P_i$. Also if $\ker f_i = Q$, then $Im\ f_i = 1$. Hence there exists a unique j, $1 \leq j \leq k$, such that $P_1^{\varphi} = P_j$. Now if $j \neq 1$, then there exists a φ-orbit of length p. Without loss of generality let $\{P_1, \ldots, P_p\}$ be a φ-orbit. As we mentioned above $3 \in \pi(P_1)$. Let $g_1 \in P_1$ be an element of order 3 and let $g_{i+1} = g_i^{\varphi}$, where $1 \leq i \leq p - 1$. Now let x be the element of S whose projections x_i to P_i are defined as follows: $x_i = g_i$ for $i = 1, \ldots, p$ and $x_i = 1$ otherwise. Obviously x is of order 3 and so $x\varphi \in \overline{G}$

is of order $3p$, which is a contradiction since $3 \nsim p$ in $\Gamma(G)$. Therefore for every $1 \le i \le k$, we have $P_i^{\varphi} = P_i$. Since $\varphi \neq 1$, there exists $1 \le i \le k$ such that φ acts nontrivially on P_i. Therefore φ induces an outer automorphism of P_i of order p. Hence p is a divisor of $|Out(P_i)|$, which is a contradiction. Therefore $k = 1$ and S is a nonabelian simple group.

Step 3. *The subgroup S is isomorphic to McL, HS or $U_6(2)$.*

Up to now we prove that there is a nonabelian simple group S such that $S \le G/N \le Aut(S)$. Also we know that $\pi(S) \subseteq \{2, 3, 5, 7, 11\}$. Now we consider each possibility for S, separately.

If $S \cong A_5$, then $\pi(S) = \pi(Aut(S)) = \{2, 3, 5\}$ and so $\{7, 11\} \subseteq \pi(N)$, which is a contradiction by Step 1. Similarly it follows that S is not isomorphic to $L_2(7)$, $L_2(8)$, $A_6 \cong L_2(9)$, $U_3(3)$, $U_4(2)$.

If $S \cong L_2(11)$, then $\pi(S) = \{2, 3, 5, 11\}$ and so $7 \in \pi(N)$. Also $S \le G/N$ contains a Frobenius subgroup $11 : 5$ of order 55. Now by using Lemma 2.9, G contains an element of order 35, which is a contradiction. Similarly if $S \cong M_{11}$, M_{12}, $U_5(2)$, then $L_2(11) < S$ and $7 \in \pi(N)$. Therefore similarly follows that $5 \sim 7$ in $\Gamma(G)$, which is a contradiction.

If $S \cong A_7$, $A_8 \cong L_4(2)$, $L_3(4)$, $L_2(49)$, $U_3(5)$, A_9, J_2, $S_6(2)$, $U_4(3)$, $O_8^+(2)$, then $L_2(7) < S$ and $\pi(S) = \{2, 3, 5, 7\}$. Therefore $11 \in \pi(N)$ and also $L_2(7)$ contains a Frobenius subgroup $7 : 3$ of order 21. Now Lemma 2.9 implies that G contains an element of order 33 and so $3 \sim 11$ in $\Gamma(G)$, which is a contradiction.

If $S \cong A_{10}$, A_{11}, A_{12}, $S_4(7)$, then $3 \sim 7$ in $\Gamma(S)$, which is a contradiction.

If $S \cong M_{22}$, then since $3 \nsim 5$ in $\Gamma(S)$ it follows that $3 \in \pi(N)$ or $5 \in \pi(N)$.

Let $5 \in \pi(N)$. Let $x \in G/N$, $X = \langle x \rangle$ and $o(x) = 11$. Now by using [16] about irreducible characters of $M_{22} \pmod 5$, we can see that for every irreducible character φ of $M_{22} \pmod 5$ we have $\langle \varphi|_X, 1|_X \rangle > 0$. Now by using Lemma 2.10, it follows that $55 \in \pi_e(G)$, which is a contradiction. Therefore $5 \notin \pi(N)$. Similarly we can prove that $3 \notin \pi(N)$ and so $S \ncong M_{22}$.

If $S \cong HS$, then $HS \le G/N \le Aut(HS)$. Therefore $G/N \cong HS$ or $G/N \cong Aut(HS)$. In each case there exists a subgroup H of G such that $H/N \cong HS$. If $\{3, 5, 11\} \cap \pi(N) \neq \emptyset$, then let $p \in \{3, 5, 11\} \cap \pi(N)$, x be an element of order 7 in H/N and $X = \langle x \rangle$. Similar to the last case by using [16] we can see that for every irreducible character φ of $HS \pmod p$ we have $\langle \varphi|_X, 1|_X \rangle > 0$, and so G has an element of order $7p$, by Lemma 2.10, which is a contradiction. Similarly it follows that $7 \notin \pi(N)$. Therefore N is a 2-group.

Similar to the above discussion it follows that $G/O_2(G) \cong McL$. With the same method we conclude that $G/O_2(G) \cong Aut(McL)$, $U_6(2)$ or $U_6(2).2$. We omit the details of the proof for convenience. Now the proof of this theorem is completed. □

We note that if k is a natural number, then obviously

$$\begin{aligned}\Gamma(Aut(McL)) &= \Gamma(\mathbb{Z}_{2^k} \times Aut(HS)) = \Gamma(\mathbb{Z}_{2^k} \times HS) = \Gamma(\mathbb{Z}_{2^k} \times McL) \\ &= \Gamma(\mathbb{Z}_{2^k} \times Aut(McL)) = \Gamma(\mathbb{Z}_{2^k} \times U_6(2)) = \Gamma(\mathbb{Z}_{2^k} \times U_6(2).2).\end{aligned}$$

Remark. W. Shi and J. Bi in [29] put forward the following conjecture:

Conjecture. Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if (i) $|G| = |M|$, (ii) $\pi_e(G) = \pi_e(M)$.

This conjecture is valid for sporadic simple groups, alternating groups and some simple groups of Lie type [29]. As a consequence of the main results, we prove the validity of this conjecture for the groups under discussion.

Theorem 3.13. Let G be a finite group and A be an almost sporadic simple group, except $Aut(J_2)$ and $Aut(McL)$. If $|G| = |A|$ and $\pi_e(G) = \pi_e(A)$, then $G \cong A$.

We note that Theorem 3.13 was proved in [18] by using the characterization of almost sporadic simple groups with their order components. Now we give a new proof for this theorem. In fact we prove the following result which is a generalization of Shi-Bi Conjecture and so Theorem 3.13 is an immediate consequence of Theorem 3.14. Also note that Theorem 3.13 is a generalization of a result in [4].

Theorem 3.14. Let A be an almost sporadic simple group, except $Aut(J_2)$ and $Aut(McL)$. If G is a finite group satisfying $|G| = |A|$ and $\Gamma(G) = \Gamma(A)$, then $G \cong A$.
Proof. Obviously $Aut(O'N)$ and $Aut(J_3)$ are uniquely determined by their prime graphs. So let $A = Aut(M_{22})$. By using Theorem 3.7, it follows that $G/O_2(G) \cong M_{22}$ or $G/O_2(G) \cong Aut(M_{22})$. If $G/O_2(G) \cong M_{22}$, then $|O_2(G)| = 2$ and hence $O_2(G) \subseteq Z(G)$ which is a contradiction, since G has more than one component and hence $Z(G) = 1$. Therefore $G/O_2(G) \cong Aut(M_{22})$, which implies that $O_2(G) = 1$ and hence $G \cong Aut(M_{22})$. Let $A = Aut(HS)$. By using Theorem 3.9, it follows that $G/O_2(G) \cong U_6(2)$ or HS, where $O_2(G) \neq 1$; or $G/O_2(G) \cong U_6(2).2$, McL or $HS.2$. By using [6], it follows that 3^6 divides the orders of $U_6(2)$, $U_6(2).2$ and McL, but $3^6 \nmid |G|$. Therefore $G/O_2(G) \cong HS$ or $Aut(HS)$. Now we get the result similarly to the last case. For convenience we omit the details of the proof of other cases. $\square$

Remark. We know that $\Gamma(Aut(J_2))$ is connected and so we can not use Theorem A in [31]. Also we can not use the method of the proof of Theorem 3.12. Therefore we put forward the following question:

Question. Let G be a finite group satisfying $\Gamma(G) = \Gamma(Aut(J_2))$. What we can say about the structure of G?

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