Kumamoto J. Math.
Vol.**22** (2009), 35–41

Analogue of Ward correspondence for a degenerated Schlesinger system

Hironobu Kimura and Yoshikatsu Nakamura*

(Received April 21, 2009)

Abstract. A general Schlesinger system (GSS) is treated from the twistor theoretic point of view. We establish a correspondence between the set of solutions of GSS and the set of vector bundles on an open subset of twistor space $\mathbb{P}^N$, trivial on twistor lines and equivariant under the infinitesimal action of a maximal abelian subgroup of $PGL_{N+1}(\mathbb{C})$.

1 Introduction

In this short communication we discuss a degenerated Schlesinger system of simplest degeneracy from the twistor theoretic point of view. This point of view was introduced in [2] by Mason and Woodhouse.

Let N be a positive integer. Then the Schlesinger system is a nonlinear system of equations for the $N+1$ unknowns $A_0, A_1, \dots, A_N$ of $r \times r$ matrices satisfying $A_0 + A_1 + \cdots + A_N = 0$:

$$dA_j = \sum_{i=0, i\neq j}^{N} [A_i, A_j] d\log(t_i - t_j) \quad (j = 0, 1, \dots, N), \tag{1}$$

which is derived by L.Schlesinger [4] in 1912. The equation (1) describes the isomonodromic deformation of a Fuchsian differential equation on the projective line $\mathbb{P}^1$:

$$\frac{dy}{d\zeta} = \sum_{j=0}^{N} \frac{A_j(t)}{\zeta + t_j} y. \tag{2}$$

Instead of the equation (2), we consider a linear differential equation with $N-1$ regular singular points and one irregular singular point of Poincare rank 1:

$$\frac{dy}{d\zeta} = \left(\frac{A_0(t)}{\zeta + t_0} - \frac{t_1 A_1(t)}{(\zeta + t_0)^2} + \sum_{j=2}^{N} \frac{A_j(t)}{\zeta + t_j} \right) y \tag{3}$$

Mathematical Subject Classification (2000): 34M55, 34M15.
Key words: Schlesinger system, Ward correspondence, isomonodromic deformation.
*The second author was partially supported by Grand-in-Aid for Scientific Research (No.19340041), Japan Society of the Promotion of Science

with

$$A_0 + A_2 + \cdots + A_N = 0. \tag{4}$$

Note that the equation (3) is derived from the Fuchsian system (2) by the confluence; we consider a replacement

$$t_1 \to t_0 - \varepsilon t_1', \quad A_0 \to \varepsilon^{-1} A_1', \quad A_0 + A_1 \to A_0',$$

take a limit $\varepsilon \to 0$ and then denote again t_1', A_0', A_1' as t_1, A_0, A_1.

We describe the isomonodromic deformation of the system (3) from the twistor theoretic point of view following the idea of Mason and Woodhouse [2]. The idea is to establish an analogue of Ward correspondence between the set of holomorphic vector bundles of rank r on (some open set of) $\mathbb{P}^N$ with certain specific properties and the set of solutions to the nonlinear system of differential equations describing the isomonodromic deformation of (3). This nonlinear system will be called the general Schlesinger system (GSS, for short), see Section 2. The details will be published elsewhere.

2 From a bundle on $\mathbb{P}^N$ to a solution of GSS

Let H be a maximal abelian subgroup of $\mathrm{GL}_{N+1}(\mathbb{C})$:

$$H = \left\{ \begin{pmatrix} h_0 & h_1 & & & & \\ & h_0 & & & & \\ & & h_2 & & & \\ & & & h_3 & & \\ & & & & \ddots & \\ & & & & & h_N \end{pmatrix} \;\middle|\; h_i \neq 0 \ (i \neq 1) \right\},$$

which is a centralizer of a regular element in $\mathrm{GL}_{N+1}(\mathbb{C})$ of Jordan normal form:

$$a = \begin{pmatrix} a_0 & 1 & & & \\ & a_0 & & & \\ & & a_2 & & \\ & & & \ddots & \\ & & & & a_N \end{pmatrix}, \qquad a_i \neq a_j \quad (i \neq j).$$

Let $\mathbb{P}^N$ be the complex projective space with the homogeneous coordinates $x = (x_0, \ldots, x_N)$, which we call the twistor space. Consider the right action of H on $\mathbb{P}^N$ defined by

$$\mathbb{P}^N \times H_\lambda \ni ([x], h) \mapsto [xh] \in \mathbb{P}^N,$$

where $[x]$ denotes the point with the homogeneous coordinates x.

We know the following reult.

Theorem 2.1 *Let U be an open subset of the twistor space $\mathbb{P}^N$ and E be a holomorphic vector bundle E of rank r on U. Assume that*

1. U contains a line and is invariant under the action of H on $\mathbb{P}^N$,

2. the restriction of E to any line $L \subset U$ is trivial,

3. the action of H on U can be lifted infinitesimally on E.

Then the infinitesimal action of H determines a flat connection on E locally and it describes the isomonodromic deformation of the linear equation (3).

Remark 2.2 *Theorem 2.1 is stated in [2] with the group H_λ, instead of H, which is a centralizer of a regular element whose Jordan structure is specified by a partition λ of $N+1$. But the proof is given only in the case $\lambda = (1, \dots, 1)$, the case where the corresponding nonlinear system is the Schlesinger system (1). For general λ, the proof is given in [5]. The group H in the above theorem corresponds to the case $\lambda = (2, 1, \dots, 1)$.*

Let us give the explicit form of the flat connection, the nonlinear system of differential equations describing the isomonodromic deformation for (3) in Theorem 2.1.

1. An infinitesimal action of H on E determines a flat connection $\tilde{\nabla}$ which is described in a local trivialization as

$$\tilde{\nabla} = d - \left(\tilde{A}_0(x) d\log x_0 + \tilde{A}_1(x) d\theta_1(x) + \sum_{j=2}^{N} \tilde{A}_j(x) d\log x_j \right),$$

   where $\tilde{A}_j(x) \in \mathrm{Mat}_r(\mathbb{C})$ such that $\tilde{A}_0 + \tilde{A}_2 + \cdots + \tilde{A}_N = 0$ and $\theta_1 = x_1/x_0$.

2. Put

$$T = \left\{ t = \begin{pmatrix} t_0 & t_1 & t_2 & \cdots & t_N \\ 1 & 0 & 1 & \cdots & 1 \end{pmatrix} \;\middle|\; t_1 \neq 0, t_i \neq t_j \ (i \neq j; i, j \neq 1) \right\}. \quad (5)$$

   The pull back of $\tilde{\nabla}$ by the map $\Phi : \mathbb{P}^1 \times T \ni ([\vec{\zeta}], t) \mapsto [\vec{\zeta} t] \in \mathbb{P}^N$ gives a flat connection ∇:

$$\nabla = d - \left(A_0(t) d\log(\zeta + t_0) + A_1(t) d\varphi(\zeta) + \sum_{j=2}^{N} A_j(t) d\log(\zeta + t_j) \right), \quad (6)$$

   where $A_j(t) = \tilde{A}_j|_{\Phi(\mathbb{P}^1 \times \{t\})}$ and $\varphi(\zeta) = \theta_1(\zeta + t_0, t_1)$.

3. The flatness of ∇ is equivalent to the nonlinear differential equation:

$$dA_0 = \sum_{k=2}^{N} [A_k, A_0] d\log(t_k - t_0) + \sum_{k=2}^{N} [A_k, A_1] d\varphi(t_k), \quad (7)$$

$$dA_1 = [A_0, A_1] d\log t_1 + \sum_{k=2}^{N} ([A_k, A_1] d\log(t_k - t_0), \quad (8)$$

$$dA_j = \sum_{\substack{k=0 \\ k \neq 1, j}}^{N} [A_k, A_j] d\log(t_j - t_k) + [A_1, A_j] d\varphi(t_j) \quad (j \geq 2). \quad (9)$$

The system (7), (8) and (9) is called *the general Schlesinger system* (GSS for short).

Remark 2.3 *1) T is a realization of the quotient space Z/H, where $Z \subset \mathrm{Mat}_{2,N+1}(\mathbb{C})$ consists of matrices $z = (z_{ij})$ satisfying the conditions*

$$\begin{aligned} &\det \begin{pmatrix} z_{00} & z_{0j} \\ z_{10} & z_{1j} \end{pmatrix} \neq 0 \quad (j = 1, \ldots, N), \\ &\det \begin{pmatrix} z_{0i} & z_{0j} \\ z_{1i} & z_{1j} \end{pmatrix} \neq 0 \quad (i \neq j; i, j = 2, \ldots, N), \\ &z_{1j} \neq 0 \quad (j \neq 1). \end{aligned}$$

2) The equation $\nabla y = 0$ is the equation (3) together with

$$\begin{aligned} \frac{\partial y}{\partial t_0} &= \left(\frac{A_0(t)}{\zeta + t_0} - \frac{t_1 A_1(t)}{(\zeta + t_0)^2} \right) y, \\ \frac{\partial y}{\partial t_1} &= \frac{A_1(t)}{\zeta + t_0} y, \\ \frac{\partial y}{\partial t_j} &= \frac{A_j(t)}{\zeta + t_j} y \quad (j = 2, \ldots, N). \end{aligned}$$

The compatibility condition of these equations gives the system of equations (7), (8) and (9).

Proposition 2.4 *The equations (7), (8) and (9) are completely integrable.*

Proposition 2.5 *Let $A_j(t)$ $(j = 0, \ldots, N)$ be any solution to (7), (8) and (9). Then we have the following.*

1. *There hold the relations*

$$\frac{\partial A_j}{\partial t_0} + \sum_{k=2}^{N} \frac{\partial A_j}{\partial t_k} = 0 \ , \quad \sum_{k=0}^{N} t_k \frac{\partial A_j}{\partial t_k} = 0 \ , \quad (j = 0, 1, \cdots, N). \tag{10}$$

2. *For any constants $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq 0$, we have*

$$A_j(\lambda t + \mu q) = A_j(t), \tag{11}$$

where $t = {}^t(t_0, \ldots, t_N), q = {}^t(1, 0, 1, \ldots, 1)$ and

$$\lambda t + \mu q = (\lambda t_0 + \mu, \lambda t_1, \lambda t_2 + \mu, \cdots, \lambda t_N + \mu)$$

3 From a solution of GSS to a vector bundle

We can show the converse of Theorem 2.1. Namely, for any solution $(A_j(t))$ of GSS, we can construct a vector bundle E of rank r on an open subset U of twistor space $\mathbb{P}^N$ enjoying the properties in Theorem 2.1. We follow the idea of [2] in the construction.

Let Z and Z/H be as in Remark 2.3 and an element of Z be denoted as

$$(u,v) = \begin{pmatrix} u_0 & u_1 & \dots & u_N \\ v_0 & v_1 & \dots & v_N \end{pmatrix}.$$

Let T be a realization of Z/H given by (5). We identify T as an open subset of $\mathbb{C}^{N+1}$ in obvious manner:

$$\{t = (t_0, \dots, t_N) \mid t_1 \neq 0, t_i \neq t_j \ (i \neq j; i, j = 0, 2, \dots, N)\}.$$

Then the projection map $\pi_1 : Z \to Z/H = T$ is

$$(u,v) \mapsto \left(\frac{u_0}{v_0}, \frac{-u_0 v_1 + u_1 v_0}{v_0^2}, \frac{u_2}{v_2}, \dots, \frac{u_N}{v_N}, \right).$$

Define also $\pi_2 : Z \to \mathbb{P}^N$ by $\pi_2(u,v) = [u]$.

Suppose that a solution $(A_j(t))$ of GSS is defined on an open subset $R \subset T$. Taking account of Proposition 2.5, we may assume that if $t \in R$ then $\lambda t + \mu q \in R$ for any $\lambda \neq 0$ and $\mu \in \mathbb{C}$. Take a ball $B \subset \{v \in \mathbb{C}^{N+1} \mid v_j \neq 0 \ (j = 0, 2, \dots, N)\}$ and put

$$W = \{(u,v) \in \mathbb{C}^{2N+2} \mid \pi_1(u,v) \in R, v \in B\}, \quad U = \pi_2(W) \subset \mathbb{P}^N.$$

We construct a vector bundle E on U using a solution $(A_j(t))$ of GSS. Note that if $(u,v) \in W$ then U contains a projective line joining $[u]$ and $[v]$. Put $B_j(u,v) := A_j(\pi_1(u,v))$, which are defined on W.

Lemma 3.1 *For any $\lambda, \mu \in \mathbb{C}$ such that $\lambda \neq 0$, we have*

$$B_j(\lambda u + \mu v, v) = B_j(u,v), \quad (j = 0, 1, \dots, N). \tag{12}$$

Using $B_j(u,v)$, we define $2N+2$ linear differential operators on W as

$$\begin{aligned}
D_0 &= u_0 \frac{\partial}{\partial u_0} + u_1 \frac{\partial}{\partial u_1} - B_0(u,v), \\
D_1 &= u_0 \frac{\partial}{\partial u_1} - B_1(u,v), \\
D_j &= u_j \frac{\partial}{\partial u_j} - B_j(u,v) \qquad (2 \leq j \leq N), \\
D_0' &= v_0 \frac{\partial}{\partial v_0} + v_1 \frac{\partial}{\partial v_1} + B_0(u,v), \\
D_1' &= v_0 \frac{\partial}{\partial v_1} + B_1(u,v), \\
D_j' &= v_j \frac{\partial}{\partial v_j} + B_j(u,v) \qquad (2 \leq j \leq N).
\end{aligned}$$

Note that these operators commute each other since $B_j(u,v)$ comes from a solution $A_j(t)$ of GSS and that

$$\sum_{j\neq 1} D_j = \sum_{j=0}^{N} u_j \frac{\partial}{\partial u_j}\ , \quad \sum_{j\neq 1} D'_j = \sum_{j=0}^{N} v_j \frac{\partial}{\partial v_j} \tag{13}$$

hold because of the fact $B_0 + B_2 + \cdots + B_N = 0$. The commutativity of the operators D_j, D'_j implies that the system

$$\sum_{k\neq 1} D_k y = 0,\ \ D'_j y = 0 \tag{14}$$

is integrable. Using this fact we define the fiber $E_{[a]}$ on $[a] \in U$ of the vector bundle, which we want to construct, by

$$E_{[a]} = \left\{ y(u,v) \;\middle|\; y(u,v) \text{ is a solution of (14) holomorphic on } \pi_2^{-1}([a]) \right\}. \tag{15}$$

It is seen that $E_{[a]}$ is a r-dimensional $\mathbb{C}$-vector space and that $E = \cup_{[a]\in U} E_{[a]}$ is a vector bundle of rank r by virtue of holomorphy of solutions of (14) with respect to parameters.

We have the following result.

Theorem 3.2 *The holomorphic vector bundle E of rank r, constructed above from a solution $(A_j(t))$ of GSS, has the properties in Theorem 2.1.*

Remark 3.3 *If we apply Theorem 2.1 to the vector bundle E in Theorem 3.2, then we recover the original solution of GSS.*

References

[1] H. Kimura, T. Koitabashi, Normalizer of maximal abelian subgroup of GL(n) and general hypergeometric functions. Kumamoto J. Math. **9** (1996), 13–43.

[2] L.J.Mason and N.M.J.Woodhouse, *Twistor theory and the Schlesinger equations*, Applications of analytic and geometric methods to nonlinear differential equations (Exeter,1992),17-25, NATO Adv. Sci. Inst. Ser.C Math. Phys. Sci., 413, Kluwer Acad.Publ., Dordrecht (1993).

[3] L.J.Mason, N.M.J. Woodhouse; Integrability, Self-Duality and Twistor Theory, Oxford Univ. Press, (1996)

[4] L. Schlesinger, Über eine Klasse von Differentialsystemem beliebiliger Ordnung mit festen kritischen Punkten, J. für Math., **141** (1912), 96–145.

[5] N. Tseveene, The general Schlesinger systems from the twistor theoretic point of view, Master thesis of Graduate school of Scinence and Technology, Kumamoto University (2005).

[6] R.S.Ward and R.O.Wells, *Twistor geometry and field theory*, Cambridge University Press (1990).

Hironobu Kimura
Department of Mathematics
Kumamoto University
Kumamoto 860-8555, Japan
e-mail: hiro@sci.kumamoto-u.ac.jp

Yoshikatsu Nakamura
Nankan Senior High School
Tamanagun, 861-0892, Japan