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Monodromy preserving deformation of linear differential equations on a rational nodal curve [1]

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Abstract

We formulate a monodromy preserving deformation (MPD) of Fuchsian differential equations on an irreducible rational curve with one node (which we call *a rational nodal curve*) and derive systems of differential equations that govern the MPD on the rational nodal curve. We also show that the MPD systems on a rational nodal curve are solved in terms of a solution to the sixth Painlevé equation and a τ-quotient associated with it. The results in this paper provide a geometric background for the asymptotic analysis on the system of differential equations that governs the MPD on elliptic curves around the boundary in the moduli space of elliptic curves.

1 Introduction

Monodromy preserving deformations (MPDs) of linear differential equations with rational coefficients yield many interesting non-linear special functions such as the Painlevé transcendents. It is a natural problem to extend the theory to the cases of non-rational algebraic curves. In his papers [11] and [12], K. Okamoto began to study the MPD of linear differential equations on an elliptic curve, and several authors treated this subject ([7, 8, 9, 16, 17]). Our original motivation is to investigate analytic properties of solutions to the system of differential equations that governs the MPD of Fuchsian differential equations on elliptic curves. The MPD system on elliptic curves naturally has two types of independent variables, namely configurations of points on an elliptic curve and moduli of elliptic curves. The fiber of the boundary point in the moduli space of elliptic curves forms an irreducible rational curve with one node (which we call *a rational nodal curve*). In [10] the author proved that, given a solution to the sixth Painlevé equation (a P_{VI}-function) and a τ-quotient associated with it, there exists a unique solution to the MPD system on elliptic curves whose "boundary value at the boundary point" coincides with the given datum. A main purpose in the present paper is to provide a geometric interpretation of this result, that is we formulate an MPD of Fuchsian differential equations on a rational nodal curve and show that the datum consisting of a P_{VI}-function and a τ-quotient governs the MPD on the rational nodal curve.

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This paper is constructed as follows. In Section 2, we briefly review the MPD theories of Fuchsian differential equations with two regular singularities on an elliptic curve. Firstly, we follow Okamoto's formulation, which treats a second-order single differential equation. Secondly, we follow Korotkin-Samtleben's formulation, which treats a rank-two system of differential equations. Our MPD theory on a rational nodal curve models these theories on elliptic curves. In Section 3, we formulate an MPD of a second-order single Fuchsian differential equation on a rational nodal curve. First of all, we present a definition of the monodromy representation of solutions to a Fuchsian differential equation on the rational nodal curve. Then we consider a deformation problem of the Fuchsian differential equation and derive a Hamiltonian system as the isomonodromic condition (Theorem 3.1). In Section 4, we consider to take the limit $\tau \to +i\infty$ in the MPD equation on elliptic curves. In this paper we regard the period τ of elliptic curves as just a parameter (not as an independent variable). We show that the Hamiltonian system in Section 2 becomes the one in Section 3 in the limit $\tau \to +i\infty$. In Section 5, we formulate an MPD of a rank-two system of Fuchsian differential equations on the rational nodal curve and derive a system of differential equations that governs the MPD on the rational nodal curve (Theorem 5.1). We prove that the MPD system is solved in terms of a P_{VI}-function and a τ-quotient associated with it (Theorem 5.2). Lastly, we show that the Hamiltonian system obtained in Section 3 is generically equivalent to the MPD system obtained in this section (Theorem 5.3). From these results, we can conclude that the Hamiltonian system in Section 3 is solved in terms of a P_{VI}-function and a τ-quotient associated with it. On the other hand, it is known that the sixth Painlevé equation also can be written as a Hamiltonian system. Corollary 5.1 describes a relationship between the two Hamiltonian systems. This result seems to suggest a new and interesting relationship between τ-quotients and characteristic exponents.

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2 Review on the MPD theories on elliptic curves

In this section, we briefly review the MPD theories of Fuchsian differential equations with two regular singularities on an elliptic curve. Firstly, we follow Okamoto's formulation, which treats a second-order single differential equation. Secondly, we follow Korotkin-Samtleben's formulation, which treats a rank-two system of differential equations. This section is basically a summary of known results. See original papers [11, 13, 7, 9] for details.

Notation for elliptic functions. In this paper, we basically follow standard notations for elliptic functions. One can consult e.g. [2, 18]. For $\tau \in \mathbb{H}$, let $E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ be the complex torus with fundamental periods 1 and τ. We define Weierstrass' elliptic functions by

$$\zeta(z) = \frac{1}{z} + \sum{}' \left(\frac{1}{z-(m+n\tau)} + \frac{1}{m+n\tau} + \frac{z}{(m+n\tau)^2} \right),$$
$$\wp(z) = -\zeta'(z),$$

and we introduce the function

$$\mathfrak{z}(z;w) := \zeta(z-w) - \zeta(z) + \zeta(w).$$

We define Jacobi's theta function by

$$\vartheta_1(z) = \sqrt{-1} \sum_{n=-\infty}^{+\infty} (-1)^n e^{\pi\sqrt{-1}(n-1/2)^2\tau + 2\pi\sqrt{-1}(n-1/2)z},$$

which is an odd function of z. We introduce the following functions:

$$\rho(z) := \vartheta_1'(z)/\vartheta_1(z),$$
$$\mathfrak{s}(z;\lambda) := \frac{\vartheta_1(z-\lambda)\vartheta_1'}{\vartheta_1(z)\vartheta_1(-\lambda)}.$$

The correspondence between Weierstrass and Jacobi's pictures is given by

$$\zeta(z) = \rho(z) + \eta_1 z, \ \wp(z) = -\rho'(z) - \eta_1,$$

where

$$\eta_1 = -\frac{1}{3}\frac{\vartheta_1'''}{\vartheta_1'}.$$

We consider the following second-order Fuchsian differential equation on E_τ:

$$\frac{d^2 w}{dz^2} = Q(z;t)w, \tag{1}$$

where

$$\begin{aligned} Q(z;t) =& \nu + a_1\wp(z) + a_2\wp(z-t) + \frac{3}{4}\wp(z-\lambda_1) + \frac{3}{4}\wp(z-\lambda_2) \\ &+ H\mathfrak{z}(z;t) - \mu_1\mathfrak{z}(z;\lambda_1) - \mu_2\mathfrak{z}(z;\lambda_2). \end{aligned} \tag{2}$$

The Riemann scheme of (1) reads

$$\left\{ \begin{array}{cccc} [0] & [t] & [\lambda_k] \ (k=1,2) & \\ \frac{1}{2}(1+c_1) & \frac{1}{2}(1+c_2) & \frac{3}{2} & ; \ z \\ \frac{1}{2}(1-c_1) & \frac{1}{2}(1-c_2) & -\frac{1}{2} & \end{array} \right\},$$

where $a_i = (c_i^2 - 1)/4$, $i = 1, 2$.

In what follows, we put the following assumptions on the equation (1):

A1 At λ_k $(k = 1, 2)$, no solution have logarithmic singularities.

A2 The differential equation (1) is not reducible.

A3 Neither c_1 nor c_2 is an integer.

The coefficients H and ν are expressible in terms of the other parameters by the assumption **A1**:

$$H = M[\mu_1^2 - \mu_2^2 + N(\mu_1 + \mu_2) - \Delta_1 + \Delta_2], \tag{3}$$

$$\begin{aligned}\nu =& M[\mu_2^2 \mathfrak{z}(\lambda_1; t) - \mu_1^2 \mathfrak{z}(\lambda_2; t) - N(\mu_1 \mathfrak{z}(\lambda_1; t) + \mu_2 \mathfrak{z}(\lambda_2; t)) \\ &+ \Delta_1 \mathfrak{z}(\lambda_2; t) - \Delta_2 \mathfrak{z}(\lambda_1; t)],\end{aligned} \tag{4}$$

where $M = 1/(\mathfrak{z}(\lambda_1; t) - \mathfrak{z}(\lambda_2; t))$, $N = \mathfrak{z}(\lambda_1; \lambda_2)$, and $\Delta_k = a_1\wp(\lambda_k) + a_2\wp(\lambda_k - t) + 3\wp(\lambda_1 - \lambda_2)/4$, $k = 1, 2$. Let $W(z) = (w_1(z), w_2(z))$ be a fundamental system of solutions to (1) defined near a base point, we define the monodromy matrices of $W(z)$ as follows:

$$W^{l_i}(z) = W(z)M_i, \quad i = 0, 1, 2, \infty,$$

where $W^{l_i}(z)$ stands for the analytic continuation of $W(z)$ along the loop l_i drawn in Figure 1: l_1 and l_2 are loops starting from the base point and turning anticlockwise around $[0]$ and $[t]$, respectively. The loops l_0 and l_∞ are basic periods of the elliptic curve. Note that $M_i \in SL(2, \mathbb{C})$. We call the set of matrices $\{M_\infty, M_0, M_1, M_2\}$ the monodromy datum associated with $W(z)$. Note that there exists the following unique relation among M_i's:

$$M_\infty^{-1} M_0^{-1} M_\infty M_0 = M_1 M_2, \tag{5}$$

which comes form the homotopy equivalence relation among the loops:

$$l_\infty^{-1} \cdot l_0^{-1} \cdot l_\infty \cdot l_0 \sim l_1 \cdot l_2.$$

Theorem 2.1 (Okamoto [13])**.** *The monodromy preserving deformation of the Fuchsian equation* (1) *with an independent variable t is governed by the Hamiltonian system with the Hamiltonian function H:*

$$\begin{cases} \frac{\partial \lambda_k}{\partial t} = \frac{\partial H}{\partial \mu_k}, \\ \frac{\partial \mu_k}{\partial t} = -\frac{\partial H}{\partial \lambda_k}, \end{cases} \quad k = 1, 2. \tag{6}$$

In fact, we can also take the period τ as an independent variable:

Theorem 2.2 (Kawai [7])**.** *We introduce another function by*

$$K := \frac{1}{2\pi i}[\nu + H\rho(t) - \mu_1\rho(\lambda_1) - \mu_2\rho(\lambda_2)], \tag{7}$$

then the MPD of (1) *with an independent variable τ is governed by the Hamiltonian system with the Hamiltonian function K:*

$$\begin{cases} \frac{\partial \lambda_k}{\partial \tau} = \frac{\partial K}{\partial \mu_k}, \\ \frac{\partial \mu_k}{\partial \tau} = -\frac{\partial K}{\partial \lambda_k}, \end{cases} \quad k = 1, 2. \tag{8}$$

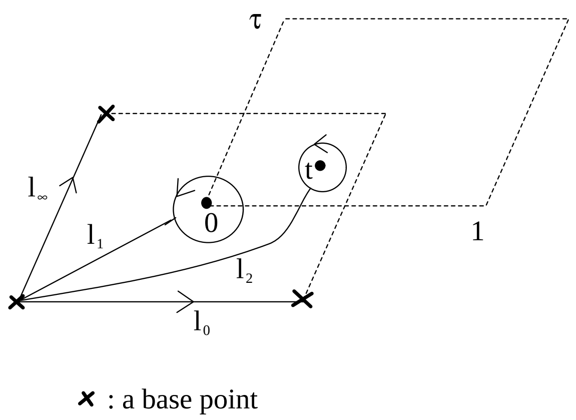

Figure 1: Loops on E_τ.

However, in this paper, we regard τ as just a parameter not as an independent variable.

Next, we consider the following system of differential equations:

$$\frac{dY}{dz} = A(z)Y, \tag{9}$$

where

$$A(z) = \begin{pmatrix} \alpha_0 + \alpha_1\rho(z) - \alpha_1\rho(z-t) & \beta_1\mathfrak{s}(z;\lambda) + \beta_2\mathfrak{s}(z-t;\lambda) \\ \gamma_1\mathfrak{s}(z;-\lambda) + \gamma_2\mathfrak{s}(z-t;-\lambda) & -\alpha_0 - \alpha_1\rho(z) + \alpha_1\rho(z-t) \end{pmatrix}$$

with the relations

$$-\alpha_1^2 - \beta_1\gamma_1 = -\frac{c_1^2}{4},\ -\alpha_1^2 - \beta_2\gamma_2 = -\frac{c_2^2}{4}, \tag{10}$$

for some constants c_1, c_2. We make the assumptions **A2**, **A3** also on the system (9). The matrix $A(z)$ has the following quasi-periodicities:

$$A(z+1) = A(z),\ A(z+\tau) = \begin{pmatrix} e^{\pi i\lambda} & 0 \\ 0 & e^{-\pi i\lambda} \end{pmatrix} A(z) \begin{pmatrix} e^{-\pi i\lambda} & 0 \\ 0 & e^{\pi i\lambda} \end{pmatrix}.$$

Remark 2.1. *These quasi-periodicities of $A(z)$ suggests that we should regard $A(z)$ as a holomorphic connection on a vector bundle on E_τ, which is the direct sum of two line bundles parameterized by λ.*

Let $Y(z)$ be a fundamental system of solutions to (9). We define the monodromy matrices of $Y(z)$ as follows:

$$Y^{l_i}(z) = Y(z)M_i, \quad i = 0, 1, 2,$$
$$Y^{l_\infty}(z) = \begin{pmatrix} e^{\pi i\lambda} & 0 \\ 0 & e^{-\pi i\lambda} \end{pmatrix} Y(z)M_\infty.$$

Proposition 2.1 (Korotkin-Samtleben [9]). *Let $Y(z;t)$ be a family of fundamental systems of solutions to* (9)*. Then the monodromy matrices of $Y(z;t)$ are independent of t, if and only if $Y(z;t)$ and $\lambda=\lambda(t)$ satisfy the following system of differential equations:*

$$\frac{\partial Y}{\partial t}(z;t)=B(z;t)Y(z;t),\quad \frac{\partial\lambda}{\partial t}=-2\alpha_1, \tag{11}$$

where

$$B(z;t)=\begin{pmatrix} \varepsilon+\alpha_1\rho(z-t) & -\beta_2\mathfrak{s}(z-t;\lambda) \\ -\gamma_2\mathfrak{s}(z-t;-\lambda) & -\varepsilon-\alpha_1\rho(z-t) \end{pmatrix} \tag{12}$$

and ε is some function of t and independent of z.

Remark 2.2. *The parameter ε is not essential because it comes from ambiguity of the normalization of $Y(z;t)$.*

Proposition 2.2 ([9]). *Let $Y(z;\tau)$ be a family of fundamental systems of solutions to* (9)*. Then the monodromy matrices of $Y(z;\tau)$ are independent of τ, if and only if $Y(z;\tau)$ and $\lambda=\lambda(\tau)$ satisfy the following system of differential equations:*

$$\frac{\partial Y}{\partial \tau}(z;\tau)=C(z;\tau)Y(z;\tau),\quad \pi i\frac{\partial\lambda}{\partial \tau}=\alpha_0, \tag{13}$$

where

$$C(z;\tau)=\begin{pmatrix} \delta+\frac{\alpha_1}{4\pi i}(\rho(z)^2+\rho'(z)-\rho(z-t)^2-\rho'(z-t)) & -\frac{\beta_1}{2\pi i}\frac{\partial\mathfrak{s}}{\partial\lambda}(z;\lambda)-\frac{\beta_2}{2\pi i}\frac{\partial\mathfrak{s}}{\partial\lambda}(z-t;\lambda) \\ -\frac{\gamma_1}{2\pi i}\frac{\partial\mathfrak{s}}{\partial\lambda}(z;-\lambda)-\frac{\gamma_2}{2\pi i}\frac{\partial\mathfrak{s}}{\partial\lambda}(z-t;-\lambda) & -\delta-\frac{\alpha_1}{4\pi i}(\rho(z)^2+\rho'(z)-\rho(z-t)^2-\rho'(z-t)) \end{pmatrix} \tag{14}$$

and δ is some function of τ and independent of z.

From the integrability conditions

$$\frac{\partial A}{\partial t}-\frac{\partial B}{\partial z}+[A,B]=0,\ \frac{\partial A}{\partial \tau}-\frac{\partial C}{\partial z}+[A,C]=0$$

between (9), (11) and (13), we obtain the following system of partial differential equations for the coefficients of $A(z)$ and λ ([9]):

$$\frac{\partial\lambda}{\partial t}=-2\alpha_1, \tag{15}$$

$$\frac{\partial\alpha_0}{\partial t}=-\beta_1\gamma_2\frac{\partial\mathfrak{s}}{\partial\lambda}(t;\lambda)+\beta_2\gamma_1\frac{\partial\mathfrak{s}}{\partial\lambda}(t;-\lambda), \tag{16}$$

$$\frac{\partial\alpha_1}{\partial t}=-\beta_1\gamma_2\mathfrak{s}(t;\lambda)+\beta_2\gamma_1\mathfrak{s}(t;-\lambda), \tag{17}$$

$$\frac{\partial\beta_1}{\partial t}=-2\pi i\alpha_1\beta_1-2\alpha_1\beta_1\rho(t)-2\alpha_1\beta_2\mathfrak{s}(t;-\lambda), \tag{18}$$

$$\frac{\partial\beta_2}{\partial t}=-2\pi i\alpha_1\beta_2+2\alpha_0\beta_2+2\alpha_1\beta_2\rho(t)+2\alpha_1\beta_1\mathfrak{s}(t;\lambda), \tag{19}$$

$$\frac{\partial \gamma_1}{\partial t} = 2\pi i \alpha_1 \gamma_1 + 2\alpha_1 \gamma_1 \rho(t) + 2\alpha_1 \gamma_2 \mathfrak{s}(t;\lambda), \tag{20}$$

$$\frac{\partial \gamma_2}{\partial t} = 2\pi i \alpha_1 \gamma_2 - 2\alpha_0 \gamma_2 - 2\alpha_1 \gamma_2 \rho(t) - 2\alpha_1 \gamma_1 \mathfrak{s}(t;-\lambda), \tag{21}$$

and

$$2\pi i \frac{\partial \lambda}{\partial \tau} = 2\alpha_0, \tag{22}$$

$$2\pi i \frac{\partial \alpha_0}{\partial \tau} = -(\beta_1\gamma_1 + \beta_2\gamma_2)\rho''(\lambda) + \beta_1\gamma_2 \frac{\partial^2 \mathfrak{s}}{\partial \lambda^2}(t;\lambda) - \beta_2\gamma_1 \frac{\partial^2 \mathfrak{s}}{\partial \lambda^2}(t;-\lambda), \tag{23}$$

$$2\pi i \frac{\partial \alpha_1}{\partial \tau} = \beta_1\gamma_2 \frac{\partial \mathfrak{s}}{\partial \lambda}(t;\lambda) - \beta_2\gamma_1 \frac{\partial \mathfrak{s}}{\partial \lambda}(t;-\lambda), \tag{24}$$

$$\begin{aligned} 2\pi i \frac{\partial \beta_1}{\partial \tau} =& 2\pi i(\alpha_0 - \pi i c_0)\beta_1 + 2\alpha_1\beta_2 \frac{\partial \mathfrak{s}}{\partial \lambda}(t;-\lambda) \\ &+ \alpha_1\beta_1(2\wp(\lambda) - \rho(t)^2 + \wp(t)), \end{aligned} \tag{25}$$

$$\begin{aligned} 2\pi i \frac{\partial \beta_2}{\partial \tau} =& 2\pi i(\alpha_0 - \pi i c_0)\beta_2 - 2\alpha_1\beta_1 \frac{\partial \mathfrak{s}}{\partial \lambda}(t;\lambda) \\ &- \alpha_1\beta_2(2\wp(\lambda) - \rho(t)^2 + \wp(t)), \end{aligned} \tag{26}$$

$$\begin{aligned} 2\pi i \frac{\partial \gamma_1}{\partial \tau} =& -2\pi i(\alpha_0 - \pi i c_0)\gamma_1 - 2\alpha_1\gamma_2 \frac{\partial \mathfrak{s}}{\partial \lambda}(t;\lambda) \\ &- \alpha_1\gamma_1(2\wp(\lambda) - \rho(t)^2 + \wp(t)), \end{aligned} \tag{27}$$

$$\begin{aligned} 2\pi i \frac{\partial \gamma_2}{\partial \tau} =& -2\pi i(\alpha_0 - \pi i c_0)\gamma_2 + 2\alpha_1\gamma_1 \frac{\partial \mathfrak{s}}{\partial \lambda}(t;-\lambda) \\ &+ \alpha_1\gamma_2(2\wp(\lambda) - \rho(t)^2 + \wp(t)). \end{aligned} \tag{28}$$

Proposition 2.3. *The system of partial differential equations* (15)-(28) *is left invariant by the change of the dependent variables:*

$$\begin{aligned} &(\overline{\lambda}, \overline{\alpha}_0, \overline{\alpha}_1, \overline{\beta}_1, \overline{\beta}_2, \overline{\gamma}_1, \overline{\gamma}_2) \\ =&(\lambda + m\tau + n, \alpha_0 + m\pi i, \alpha_1, \beta_1, e^{2\pi i m\tau}\beta_2, \gamma_1, e^{-2\pi i m\tau}\gamma_2), \end{aligned} \tag{29}$$

for any $m, n \in \mathbb{Z}$.

Proof. By the definition of $\mathfrak{s}(z;\lambda)$, the following equality holds:

$$\mathfrak{s}(z; \lambda + m\tau + n) = e^{2\pi i m z}\mathfrak{s}(z;\lambda). \tag{30}$$

Let $Y(z)$ be a fundamental system of solutions to (9). We set

$$\overline{Y}(z) = \begin{pmatrix} e^{\pi i z} & 0 \\ 0 & e^{-\pi i z} \end{pmatrix} Y(z), \tag{31}$$

then we can easily check that $\overline{Y}(z)$ becomes a fundamental system of solutions to the equation

$$\frac{d\overline{Y}}{dz}(z) = \overline{A}(z)\overline{Y}(z),$$

where

$$\overline{A}(z) = \begin{pmatrix} \overline{\alpha}_0 + \overline{\alpha}_1\rho(z) - \overline{\alpha}_1\rho(z-t) & \overline{\beta}_1\mathfrak{s}(z;\overline{\lambda}) + \overline{\beta}_2\mathfrak{s}(z-t;\overline{\lambda}) \\ \overline{\gamma}_1\mathfrak{s}(z;-\overline{\lambda}) + \overline{\gamma}_2\mathfrak{s}(z-t;-\overline{\lambda}) & -\overline{\alpha}_0 - \overline{\alpha}_1\rho(z) + \overline{\alpha}_1\rho(z-t) \end{pmatrix},$$

by making use of (30) and the relation (29). The transformation of the unknown functions (31) has no effect on the monodromy. Therefore, if $Y(z;t,\tau)$ is a monodromy-invariant family of fundamental solutions, so is $\overline{Y}(z;t,\tau)$. □

Remark 2.3. *Remark 2.1 and Proposition 2.3 say that the variable λ runs over the Jacobi variety of E_τ.*

3 MPD of a second-order single differential equation on a rational nodal curve

Let C be a rational nodal curve. Around the double point P, C is defined locally analytically by $xy = 0$. The dualizing sheaf ω_C is generated by $\mathcal{O}_C(dx/x)$ at $x \neq 0$ and $\mathcal{O}_C(dy/y)$ at $y \neq 0$, and we have the relation $(dx/x) + (dy/y) = 0$. Let $\widetilde{C}$ be a normalization of C, which is a non-singular rational curve, and we suppose that two points $\{\infty, 0\}$ on $\widetilde{C}$ are mapped to the double point P on C (see Figure 2). Since we need to take a coordinate, we pull back all objects on C to $\widetilde{C}$ and work on $\widetilde{C}$. For example, a vector bundle E on C is defined by a pair $(\widetilde{E}, \iota)$, where $\widetilde{E}$ is a vector bundle on $\widetilde{C}$ and ι is an isomorphism between the fiber $\widetilde{E}_\infty$ over $\{\infty\}$ and the fiber $\widetilde{E}_0$ over $\{0\}$. Similarly, a (holomorphic) connection on E is defined by a connection on $\widetilde{E}$ which has regular singular points only at ∞ and 0, and the sum of whose residue matrices vanishes.

In this section, we shall give a formulation of an MPD of a Fuchsian differential equation on the rational nodal curve C. From the above consideration, a second-order Fuchsian differential equation with two regular singularities on C should be defined by that on $\widetilde{C}$ with four regular singular points (namely $0, \infty$ and the inverse image of the prescribed two singular points), and the sum of whose exponents at ∞ and 0 is equal to 0. Hence, it is natural that we consider the following linear differential equation:

$$\frac{d^2\varphi}{dx^2} = Q(x;t)\varphi, \tag{32}$$

where

$$\begin{aligned} Q(x;t) =& \frac{a_0}{x^2} + \frac{a_1}{(x-1)^2} + \frac{a_2}{(x-t)^2} + \sum_{k=1,2} \frac{3}{4(x-\lambda_k)^2} - \frac{a_1 + a_2 + 3/2}{x(x-1)} \\ &+ \frac{t(t-1)L}{x(x-1)(x-t)} - \sum_{k=1,2} \frac{\lambda_k(\lambda_k - 1)\eta_k}{x(x-1)(x-\lambda_k)}. \end{aligned}$$

The Riemann scheme of the equation (32) reads

$$\left\{\begin{array}{cccccc} 0 & 1 & t & \lambda_k(k=1,2) & \infty & \\ \frac{1}{2}(1+c_0) & \frac{1}{2}(1+c_1) & \frac{1}{2}(1+c_2) & \frac{3}{2} & -\frac{1}{2}(1+c_0) & ;x \\ \frac{1}{2}(1-c_0) & \frac{1}{2}(1-c_1) & \frac{1}{2}(1-c_2) & -\frac{1}{2} & -\frac{1}{2}(1-c_0) & \end{array}\right\},$$

where $a_i = (c_i^2 - 1)/4$ $(i = 0, 1, 2)$. Note that the sum of the characteristic exponents at $x = 0$ and ∞ equals zero. We make the assumptions **A1**, **A2**, and **A3** on (32).

Remark 3.1. *For each t (which we regard as a deformation parameter), the differential equation (32) has three local parameters a_0, a_1, a_2 and five global parameters $\lambda_1, \lambda_2, \eta_1, \eta_2, L$. There exist two algebraic relations among these parameters by the assumption* **A1**. *Therefore the equation (32) contains essentially three independent global parameters. The reason why we let (32) carry two apparent singularities, while we put one apparent singularity in the standard MPD theory ([4, 15]), shall be explained later (Remark 3.2).*

Let $\Phi(x) = (\varphi_1(x), \varphi_2(x))$ be a fundamental system of solutions to (32), then we need to give a definition of the monodromy representation of $\Phi(x)$ on C. For this aim, we take a base point b_0 on $\tilde{C}$ as a sufficiently large positive real number. Let l_1 and l_2 be loops on $\tilde{C}$ starting from the base point b_0 and turning anticlockwise around 1 and t respectively. Let l_0 be a loop on $\tilde{C}$ starting from b_0 and turning clockwise around ∞. Let l_0' be a loop on $\tilde{C}$ starting from b_0^{-1} and turning anticlockwise around 0. Let l_∞ be a path on $\tilde{C}$ joining b_0 and b_0^{-1} along the real axis avoiding 1 into the direction of the lower half plane. We remark that $l_1, l_2, l_0, l_0', l_\infty$ should be considered loops on the rational nodal curve C, particularly l_0 and l_0' should be identified on C (see Figure 2). We are going to assign matrices $M_i \in SL(2, \mathbb{C})$ to the loops l_i $(i = 0, 1, 2, \infty)$ and l_0', respectively, satisfying the relation

$$M_\infty^{-1} M_0^{-1} M_\infty M_0 = M_1 M_2. \tag{33}$$

The relation (33) comes from the homotopy equivalence relation among the paths on $\tilde{C}$:

$$l_\infty^{-1} \cdot {l'}_0^{-1} \cdot l_\infty \cdot l_0 \sim l_1 \cdot l_2$$

(we should assign the same matrix M_0 to the two loops l_0 and l_0'). For $i = 0, 1, 2$, we assign the ordinary monodromy matrix of $\Phi(x)$ along the loop l_i:

$$\Phi^{l_i}(x) = \Phi(x) M_i,$$

where $\Phi^{l_i}(x)$ stands for the analytic continuation of $\Phi(x)$ along the loop l_i. In order to define the monodromy matrix M_∞, we take a fundamental system of solutions to (32) around $x = \infty$ with the following form:

$$\Phi_\infty(x) = \left(x^{(1+c_0)/2} f_1(x^{-1}), x^{(1-c_0)/2} f_2(x^{-1})\right), \tag{34}$$

where $f_1(x^{-1})$ and $f_2(x^{-1})$ are holomorphic in x^{-1} and never vanish at $x = \infty$. Here we take branches of the multi-valued functions $x^{(1+c_0)/2}$ and $x^{(1-c_0)/2}$ around $x = \infty$ as $\arg x = 0$ at $x = b_0$. We take another fundamental system of solutions around $x = 0$ with the following form:

$$\Phi_0(x) = \left(x^{(1+c_0)/2} g_1(x), x^{(1-c_0)/2} g_2(x)\right), \tag{35}$$

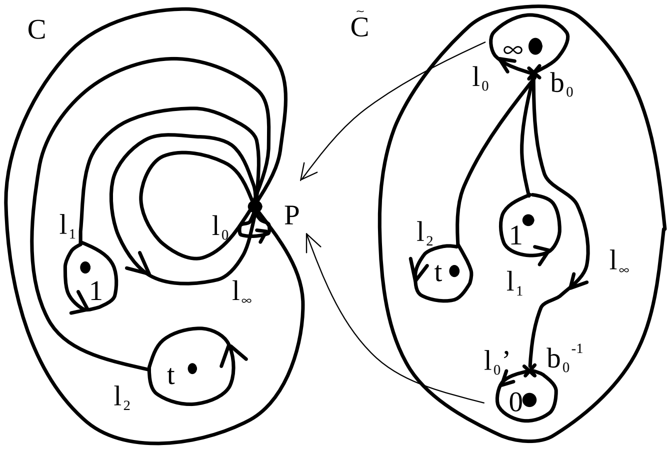

Figure 2: Loops on C.

where $g_1(x)$ and $g_2(x)$ are holomorphic in x such that $g_1(0) = f_1(0)$, $g_2(0) = f_2(0)$. Here we take branches of the multi-valued functions $x^{(1+c_0)/2}$ and $x^{(1-c_0)/2}$ around $x = 0$ as $\arg x = 0$ at $x = b_0^{-1}$. We introduce a matrix $C_{\infty 0} \in GL(2, \mathbb{C})$ by

$$\Phi_\infty(x) = \Phi_0(x) C_{\infty 0}, \tag{36}$$

where the analytic continuation of $\Phi_\infty(x)$ is done along l_∞. And, let $C \in GL(2, \mathbb{C})$ be the matrix determined by the relation $\Phi(x) = \Phi_\infty(x) C$, then we define M_∞ by $M_\infty = C^{-1} C_{\infty 0} C$. The following two lemmas justify us in defining the monodromy datum associated with $\Phi(x)$ by the set of the matrices $\{M_0, M_1, M_2, M_\infty\}$.

Lemma 3.1. *The definition of M_∞ does not depend on the choice of the fundamental solutions $\Phi_\infty(x)$ and $\Phi_0(x)$ as long as they satisfy the conditions $g_1(0) = f_1(0)$ and $g_2(0) = f_2(0)$. Here we remark that the conditions $g_i(0) = f_i(0)$ $(i = 1, 2)$ correspond to the identification between the two points ∞ and 0 on C.*

Proof. We make another choice of fundamental solutions $\Phi'_\infty(x)$ and $\Phi'_0(x)$ such that

$$\Phi'_\infty(x) = \left(x^{(1+c_0)/2} f'_1(x^{-1}), x^{(1-c_0)/2} f'_2(x^{-1})\right),$$

$$\Phi'_0(x) = \left(x^{(1+c_0)/2} g'_1(x), x^{(1-c_0)/2} g'_2(x)\right),$$

and $g'_1(0) = f'_1(0)$, $g'_2(0) = f'_2(0)$. Then $\Phi'_\infty(x)$ and $\Phi'_0(x)$ are related to $\Phi_\infty(x)$ and $\Phi_0(x)$ respectively by

$$\Phi'_\infty(x) = \Phi_\infty(x) D, \quad \Phi'_0(x) = \Phi_0(x) D,$$

for some matrix $D \in GL(2, \mathbb{C})$, so that we have $C'_{\infty 0} = D^{-1} C_{\infty 0} D$ and $C' = D^{-1} C$. Hence we obtain

$$C'^{-1} C'_{\infty 0} C' = (C^{-1} D)(D^{-1} C_{\infty 0} D)(D^{-1} C) = C^{-1} C_{\infty 0} C.$$

□

Lemma 3.2. *The determinant of the matrix M_∞ is equal to* 1*, namely $M_\infty \in SL(2,\mathbb{C})$. And M_∞ satisfies the relation* (33)*.*

Proof. Firstly we prove the first assertion. Let $W_\infty(x) = W_\infty$ (and $W_0(x) = W_0$) be the Wronskian of the fundamental solution $\Phi_\infty(x)$ (and $\Phi_0(x)$ respectively). Note that the Wronskian of any fundamental solution to (32) is independent of x because the equation (32) is of SL-type. By the relation (36), we have $W_\infty = W_0 \det C_{\infty 0}$. On the other hand, by (34) and (35), we have

$$W_\infty = W_\infty(\infty) = c_0 f_1(0) f_2(0) = c_0 g_1(0) g_2(0) = W_0(0) = W_0.$$

Hence we see that $\det M_\infty = \det C_{\infty 0} = 1$. Next we prove the second assertion. We chase the analytic continuation of $\Phi(x) = \Phi_\infty(x) C$ along the loops starting from b_0:

$$\Phi^{l_0}(x) = \Phi(x) C^{-1} \begin{pmatrix} e^{\pi i(1+c_0)} & 0 \\ 0 & e^{\pi i(1-c_0)} \end{pmatrix} C = \Phi(x) M_0,$$

$$\begin{aligned} \Phi^{l_\infty^{-1} \cdot l_0'^{-1} \cdot l_\infty}(x) &= \Phi(x) M_\infty^{-1} C^{-1} \begin{pmatrix} e^{-\pi i(1+c_0)} & 0 \\ 0 & e^{-\pi i(1-c_0)} \end{pmatrix} C M_\infty \\ &= \Phi(x) M_\infty^{-1} M_0^{-1} M_\infty, \end{aligned}$$

$$\begin{aligned} \Phi^{l_1}(x) &= \Phi(x) M_1, \\ \Phi^{l_2}(x) &= \Phi(x) M_2. \end{aligned}$$

Therefore we can conclude that the monodromy matrices defined here satisfy the relation (33) from the homotopy equivalence relation among the loops on $\tilde{C}$. □

Remark 3.2. *In the standard theory of the MPD on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, t, \infty\}$, the monodromy datum is defined by the set of matrices*

$$\{N_0 := M_\infty^{-1} M_0 M_\infty, N_1 := M_1, N_2 := M_2, N_\infty = (N_0 N_1 N_2)^{-1}\}$$

with the relation $N_0 N_1 N_2 N_\infty = I$. As is well known, the space of monodromy representations on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, t, \infty\}$ (with fixed characteristic exponents) is two-dimensional. The monodromy datum $\{M_0, M_1, M_2, M_\infty\}$ defined here has one more degree of freedom (i.e. indefiniteness of M_∞) than the ordinary monodromy datum $\{N_0, N_1, N_2, N_\infty\}$. So that the space of our monodromy representations on $C \setminus \{1, t\}$ is three-dimensional, which coincides with the number of the global parameters contained in the differential equation (32). This is the reason why we consider the Fuchsian equation (32) with two apparent singularities. In other words, if we consider the ordinary MPD of (32) (taking $\{N_0, N_1, N_2, N_\infty\}$ as its monodromy data), the two apparent singularities add a superfluous degree of freedom. We can necessarily reduce the two apparent singularities to one apparent singularity with keeping the monodromy matrices invariant. (This problem is studied in Ishikawa [3].)

We shall find the condition for admitting a monodromy-invariant family of solutions to the equations (32). By the assumption **A1**, we have the following relations:

$$\eta_1^2 = U_1 + \frac{t(t-1)L}{\lambda_1(\lambda_1-1)(\lambda_1-t)} - \frac{\lambda_2(\lambda_2-1)\eta_2}{\lambda_1(\lambda_1-1)(\lambda_1-\lambda_2)} + \frac{2\lambda_1-1}{\lambda_1(\lambda_1-1)}\eta_1, \tag{37}$$

$$\eta_2^2 = U_2 + \frac{t(t-1)L}{\lambda_2(\lambda_2-1)(\lambda_2-t)} - \frac{\lambda_1(\lambda_1-1)\eta_1}{\lambda_2(\lambda_2-1)(\lambda_2-\lambda_1)} + \frac{2\lambda_2-1}{\lambda_2(\lambda_2-1)}\eta_2, \tag{38}$$

where we put

$$U_k = \frac{a_0}{\lambda_k^2} + \frac{a_1}{(\lambda_k-1)^2} + \frac{-a_1-a_2-3/2}{\lambda_k(\lambda_k-1)} + \frac{a_2}{(\lambda_k-t)^2} + \frac{3}{4(\lambda_k-\lambda_{k+1})^2}. \tag{39}$$

We denote the right hand sides of (37) and (38) by $\mathcal{U}_k$, $k=1,2$, respectively. Then (37) and (38) are rewritten as $\eta_k^2 = \mathcal{U}_k$, $k=1,2$. Here we summarize the situation: we consider t a deformation parameter and $\lambda_1, \lambda_2, \eta_1, \eta_2, L$ dependent variables. There exist two algebraic relations (37),(38) among $\lambda_1, \lambda_2, \eta_1, \eta_2, L$. We should like to regard L as a Hamiltonian function on the analogy of the standard MPD theory. But a problem is that we do not have any canonical choice of representation for L by the other parameters $\lambda_1, \lambda_2, \eta_1, \eta_2$. In any case, the following lemma can be proved by similar discussions to the standard monodromy preserving deformation (see, for example [15, 4]):

Lemma 3.3. *Let $\Phi(x;t)$ be a family of fundamental solutions to* (32)*. The ordinary monodromy matrices $\{N_0, N_1, N_2, N_\infty\}$ of $\Phi(x;t)$ are independent of t, if and only if there exists a rational function $A(x;t)$ in x such that*

$$\begin{cases} \frac{\partial^2}{\partial x^2}\Phi(x;t) = Q(x;t)\Phi(x;t) \\ \frac{\partial}{\partial t}\Phi(x;t) = A(x;t)\frac{\partial \Phi}{\partial x} - \frac{1}{2}\frac{\partial A}{\partial x}\Phi(x;t). \end{cases} \tag{40}$$

The system (40) *is called the extended system.*

The integrability condition of (40) yields a differential equation for A:

$$\frac{\partial^3 A}{\partial x^3} - 4Q\frac{\partial A}{\partial x} - 2\frac{\partial Q}{\partial x}A + 2\frac{\partial Q}{\partial t} = 0. \tag{41}$$

We find an explicit form of A.

Lemma 3.4. *For any fixed t, $A(x;t)$ has the following properties:*

(*i*) *It is holomorphic in x outside the set $\{\lambda_1, \lambda_2, \infty\}$,*

(*ii*) *$x = \lambda_k$ $(k = 1,2)$ and $x=\infty$ are poles of order at most one,*

(*iii*) *$x = 0,1$ are zeros of order at least one.*

In addition, the monodromy matrix M_∞ is independent of t, if and only if the following equality holds:

$$\lim_{x\to\infty} x^{-1}A(x;t) = \lim_{x\to 0} x^{-1}A(x;t). \tag{42}$$

Proof. The first assertion can be proved in a way similar to the standard monodromy preserving deformation. So we prove only the second assertion. Firstly we assume that the extended system (40) admits a monodromy-invariant family of solutions $\Phi(x;t)$ in our sense (namely M_∞ is also invariant). Let $W(t)$ be the Wronskian of $\Phi(x;t)$. Applying Cramer's formula to the extended system, we have the following representation of $A(x;t)$:

$$W(t)A(x;t) = \det\begin{pmatrix} \Phi(x;t) \\ \partial\Phi(x;t)/\partial t \end{pmatrix}. \tag{43}$$

On the other hand, we have

$$\Phi(x;t) = \Phi_\infty(x;t)C(t) = \Phi_0(x;t)C_{\infty 0}(t)C(t) = \Phi_0(x;t)C(t)M_\infty.$$

Noting that $\det M_\infty = 1$, we have

$$\begin{aligned} W(t)A(x;t) &= \det\begin{pmatrix} \Phi_\infty(x;t)C(t) \\ (\partial\Phi_\infty(x;t)/\partial t)C(t) + \Phi_\infty(x;t)(\partial C(t)/\partial t) \end{pmatrix} \\ &= \det\begin{pmatrix} \Phi_0(x;t)C(t) \\ (\partial\Phi_0(x;t)/\partial t)C(t) + \Phi_0(x;t)(\partial C(t)/\partial t) \end{pmatrix}. \end{aligned} \tag{44}$$

We put

$$C(t) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad \frac{\partial C(t)}{\partial t} = \begin{pmatrix} \dot{c}_{11} & \dot{c}_{12} \\ \dot{c}_{21} & \dot{c}_{22} \end{pmatrix}.$$

Then, around $x = \infty$, we have

$$\begin{aligned} &\det\begin{pmatrix} \Phi_\infty(x;t)C(t) \\ (\partial\Phi_\infty(x;t)/\partial t)C(t) + \Phi_\infty(x;t)(\partial C(t)/\partial t) \end{pmatrix} \\ &= \det\begin{pmatrix} \Phi_\infty(x;t) \\ (\partial\Phi_\infty(x;t)/\partial t) \end{pmatrix}\det C(t) + \det\begin{pmatrix} \Phi_\infty(x;t)C(t) \\ \Phi_\infty(x;t)(\partial C(t)/\partial t) \end{pmatrix} \\ &= x\left(f_1(x^{-1};t)\frac{\partial f_2(x^{-1};t)}{\partial t} - \frac{\partial f_1(x^{-1};t)}{\partial t}f_2(x^{-1};t)\right)(c_{11}c_{22} - c_{12}c_{21}) \\ &\quad + x^{1+c_0}(c_{11}\dot{c}_{12} - \dot{c}_{11}c_{12})f_1(x^{-1};t)^2 + x^{1-c_0}(c_{21}\dot{c}_{22} - \dot{c}_{21}c_{22})f_2(x^{-1};t)^2 \\ &\quad + x(c_{11}\dot{c}_{22} - \dot{c}_{11}c_{22} + c_{21}\dot{c}_{12} - \dot{c}_{21}c_{12})f_1(x^{-1};t)f_2(x^{-1};t), \end{aligned}$$

Because $A(x;t)$ is rational in x, the coefficients $c_{11}\dot{c}_{12} - \dot{c}_{11}c_{12}$ and $c_{21}\dot{c}_{22} - \dot{c}_{21}c_{22}$ must vanish. Therefore we have

$$\begin{aligned} &W(t)\lim_{x\to\infty} x^{-1}A(x;t) \\ &= \left(f_1(0;t)\frac{\partial f_2(0;t)}{\partial t} - \frac{\partial f_1(0;t)}{\partial t}f_2(0;t)\right)(c_{11}c_{22} - c_{12}c_{21}) \\ &\quad + (c_{11}\dot{c}_{22} - \dot{c}_{11}c_{22} + c_{21}\dot{c}_{12} - \dot{c}_{21}c_{12})f_1(0;t)f_2(0;t). \end{aligned}$$

In a similar way, we have

$$\begin{aligned} &W(t)\lim_{x\to 0} x^{-1}A(x;t) \\ &= \left(g_1(0;t)\frac{\partial g_2(0;t)}{\partial t} - \frac{\partial g_1(0;t)}{\partial t}g_2(0;t)\right)(c_{11}c_{22} - c_{12}c_{21}) \\ &\quad + (c_{11}\dot{c}_{22} - \dot{c}_{11}c_{22} + c_{21}\dot{c}_{12} - \dot{c}_{21}c_{12})g_1(0;t)g_2(0;t). \end{aligned}$$

Since we have assumed that $f_1(0;t) = g_1(0;t)$ and $f_2(0;t) = g_2(0;t)$ hold for any t, we obtain the equality (42).

Conversely, we assume that the equality (42) holds. Then we should like to conclude that, for any solution $\Phi(x;t)$ of (40), the monodromy matrix $M_\infty(t)$ of $\Phi(x;t)$ is invariant, i.e., $\partial M_\infty(t)/\partial t = 0$. Substitute $\Phi(x;t) = \Phi_\infty(x;t)C(t) = \Phi_0(x;t)C(t)M_\infty(t)$ into the second equation of (40), then we have

$$\begin{aligned}&\frac{\partial \Phi_\infty(x;t)}{\partial t} + \Phi_\infty(x;t)\frac{\partial C(t)}{\partial t}C(t)^{-1}\\ =&A(x;t)\frac{\partial \Phi_\infty(x;t)}{\partial x} - \frac{1}{2}\frac{\partial A(x;t)}{\partial x}\Phi_\infty(x;t)\end{aligned} \tag{45}$$

and

$$\begin{aligned}&\frac{\partial \Phi_0(x;t)}{\partial t} + \Phi_0(x;t)\left(\frac{\partial C(t)}{\partial t}C(t)^{-1} + C(t)\frac{\partial M_\infty(t)}{\partial t}M_\infty(t)^{-1}C(t)^{-1}\right)\\ =&A(x;t)\frac{\partial \Phi_0(x;t)}{\partial x} - \frac{1}{2}\frac{\partial A(x;t)}{\partial x}\Phi_0(x;t).\end{aligned} \tag{46}$$

We put

$$\frac{\partial C(t)}{\partial t}C(t)^{-1} = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix},$$
$$C(t)\frac{\partial M_\infty(t)}{\partial t}M_\infty(t)^{-1}C(t)^{-1} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

Substitute (34) into (45), then we have the following equalities:

$$\begin{aligned}&\frac{\partial f_1(x^{-1};t)}{\partial t} + n_{11}f_1(x^{-1};t) + n_{21}x^{-c_0}f_2(x^{-1};t)\\ =&x^{-1}A(x;t)\left(\frac{1+c_0}{2}f_1(x^{-1};t) + x\frac{\partial f_1(x^{-1};t)}{\partial x}\right) - \frac{1}{2}\frac{\partial A(x;t)}{\partial x}f_1(x^{-1};t),\end{aligned} \tag{47}$$

$$\begin{aligned}&\frac{\partial f_2(x^{-1};t)}{\partial t} + n_{22}f_2(x^{-1};t) + n_{12}x^{c_0}f_1(x^{-1};t)\\ =&x^{-1}A(x;t)\left(\frac{1-c_0}{2}f_2(x^{-1};t) + x\frac{\partial f_2(x^{-1};t)}{\partial x}\right) - \frac{1}{2}\frac{\partial A(x;t)}{\partial x}f_2(x^{-1};t).\end{aligned} \tag{48}$$

Since the right hand sides of these equalities are meromorphic functions of x^{-1}, we must have

$$n_{21} = n_{12} = 0.$$

In a similar way, from (46) we have

$$m_{21} = n_{21} + m_{21} = 0, \quad m_{12} = n_{12} + m_{12} = 0$$

and

$$\begin{aligned}
&\frac{\partial g_1(x;t)}{\partial t} + (n_{11} + m_{11}) g_1(x;t) \\
=&x^{-1} A(x;t) \left(\frac{1+c_0}{2} g_1(x;t) + x \frac{\partial g_1(x;t)}{\partial x} \right) - \frac{1}{2} \frac{\partial A(x;t)}{\partial x} g_1(x;t),
\end{aligned} \tag{49}$$

$$\begin{aligned}
&\frac{\partial g_2(x;t)}{\partial t} + (n_{22} + m_{22}) g_2(x;t) \\
=&x^{-1} A(x;t) \left(\frac{1-c_0}{2} g_2(x;t) + x \frac{\partial g_2(x;t)}{\partial x} \right) - \frac{1}{2} \frac{\partial A(x;t)}{\partial x} g_2(x;t).
\end{aligned} \tag{50}$$

Take the limit $x \to \infty$ in (47) and (48), then we have

$$\frac{\partial f_1(0;t)}{\partial t} + n_{11} f_1(0;t) = \frac{c_0}{2} f_1(0;t) \lim_{x \to \infty} x^{-1} A(x;t)$$

and

$$\frac{\partial f_2(0;t)}{\partial t} + n_{22} f_2(0;t) = -\frac{c_0}{2} f_1(0;t) \lim_{x \to \infty} x^{-1} A(x;t),$$

respectively. Take the limit $x \to 0$ in (49) and (50), then we have

$$\frac{\partial g_1(0;t)}{\partial t} + (n_{11} + m_{11}) g_1(0;t) = \frac{c_0}{2} g_1(0;t) \lim_{x \to 0} x^{-1} A(x;t)$$

and

$$\frac{\partial g_2(0;t)}{\partial t} + (n_{22} + m_{22}) g_1(0;t) = -\frac{c_0}{2} g_2(0;t) \lim_{x \to 0} x^{-1} A(x;t)$$

respectively. Noting the equality (42) and $f_i(0;t) = g_i(0;t)$ $(i = 1,2)$, we obtain

$$m_{11} = m_{12} = m_{21} = m_{22} = 0.$$

It concludes that $\partial M_\infty(t)/\partial t = 0$. □

By the first assertion of Lemma 3.4, we have

$$A(x;t) = M \frac{x(x-1)(ax+b)}{(x-\lambda_1)(x-\lambda_2)}, \tag{51}$$

where M, a, b are independent of x. Substitute (51) into (41) and compare the coefficients of the term $(x-t)^{-3}$, then we have $A(t;t) = -1$. From this equality and (42), we have

$$A(x;t) = -\frac{x(x-1)(x-\lambda_1\lambda_2)(t-\lambda_1)(t-\lambda_2)}{(x-\lambda_1)(x-\lambda_2)t(t-1)(t-\lambda_1\lambda_2)}. \tag{52}$$

We put

$$\mathcal{A}(x) := \frac{1}{2} A \frac{\partial^3 A}{\partial x^3} - \frac{\partial}{\partial x}(QA^2) + A \frac{\partial Q}{\partial t}. \tag{53}$$

Lemma 3.5. *$\mathcal{A}(x)$ is expanded in a partial fraction as follows:*

$$\mathcal{A}(x) = \frac{v}{(x-t)} + \sum_{k=1}^{2}\sum_{m=1}^{4} \frac{w_m^k}{(x-\lambda_k)^m}. \tag{54}$$

Furthermore, the equation $\mathcal{A}(x) = 0$ is equivalent to $w_m^k = 0$ $(m = 1, 2, 3, 4, k = 1, 2)$.

Proof. Note that the poles of $\mathcal{A}(x)$ are included in $\{0, 1, t, \infty, \lambda_1, \lambda_2\}$. We can check that $\mathcal{A}(x)$ has the following properties:

(i) It is holomorphic at $x = 0, 1$,

(ii) $x = t$ is a pole of order at most 1,

(iii) $x = \lambda_k$ $(k = 1, 2)$ are poles of order at most 4,

(iv) $x = \infty$ is a zero of order at least 2.

The lemma follows from these properties. □

Lemma 3.6. *Put $M = -\frac{(t-\lambda_1)(t-\lambda_2)}{t(t-1)(t-\lambda_1\lambda_2)}$. We expand $A(x;t)$ and $Q(x;t)$ in Laurent series at $x = \lambda_k$ $(k = 1, 2)$:*

$$A(x;t) = M\left[\frac{M^k}{x-\lambda_k} + \sum_{n=0}^{\infty} M^{k,n}(x-\lambda_k)^n\right],$$

$$Q(x;t) = \frac{3}{4(x-\lambda_k)^2} - \frac{\eta_k}{x-\lambda_k} + \sum_{n=0}^{\infty} \mathcal{U}_{k,n}(x-\lambda_k)^n,$$

respectively. Then we have

$$M^1 = \frac{\lambda_1(\lambda_1-1)(\lambda_1-\lambda_1\lambda_2)}{\lambda_1-\lambda_2},\ M^2 = \frac{\lambda_2(\lambda_2-1)(\lambda_2-\lambda_1\lambda_2)}{\lambda_2-\lambda_1}, \tag{55}$$

$$M^{1,0} = \frac{(2\lambda_1-1)(\lambda_1-\lambda_1\lambda_2)}{\lambda_1-\lambda_2} - \frac{\lambda_1(\lambda_1-1)(\lambda_2-\lambda_1\lambda_2)}{(\lambda_1-\lambda_2)^2}, \tag{56}$$

$$M^{2,0} = \frac{(2\lambda_2-1)(\lambda_2-\lambda_1\lambda_2)}{\lambda_2-\lambda_1} - \frac{\lambda_2(\lambda_2-1)(\lambda_1-\lambda_1\lambda_2)}{(\lambda_2-\lambda_1)^2}, \tag{57}$$

$$M^{1,1} = \frac{\lambda_1-\lambda_1\lambda_2}{\lambda_1-\lambda_2} - \frac{(\lambda_1^2-2\lambda_1\lambda_2+\lambda_2)(\lambda_2-\lambda_1\lambda_2)}{(\lambda_1-\lambda_2)^3}, \tag{58}$$

$$M^{2,1} = \frac{\lambda_2-\lambda_1\lambda_2}{\lambda_2-\lambda_1} - \frac{(\lambda_2^2-2\lambda_1\lambda_2+\lambda_1)(\lambda_1-\lambda_1\lambda_2)}{(\lambda_2-\lambda_1)^3}, \tag{59}$$

$$M^{1,2} = -\frac{\lambda_2(\lambda_2-1)(\lambda_2-\lambda_1\lambda_2)}{(\lambda_1-\lambda_2)^4} = \frac{M^2}{(\lambda_1-\lambda_2)^3}, \tag{60}$$

$$M^{2,2} = -\frac{\lambda_1(\lambda_1-1)(\lambda_1-\lambda_1\lambda_2)}{(\lambda_2-\lambda_1)^4} = \frac{M^1}{(\lambda_2-\lambda_1)^3}, \tag{61}$$

$$\mathcal{U}_{k,0} = \mathcal{U}_k, \;\; k=1,2, \tag{62}$$

$$\mathcal{U}_{k,1} = \left(\frac{\partial}{\partial\lambda_k}\right)\mathcal{U}_{k,0} - \frac{\eta_k}{\lambda_k(\lambda_k-1)}, \;\; k=1,2, \tag{63}$$

for the coefficients of the expansions, where the symbol $\left(\frac{\partial}{\partial\lambda_k}\right)\mathcal{U}_{k,0}$ *denotes the partial differentiation of* $\mathcal{U}_{k,0}$ *with respect to* λ_k *regarding the other letters (especially* L*) as constants.*

Proof. We prove only the equalities (62) and (63). The other equalities are obtained by direct computations. By the definition of $Q(x;t)$, we have

$$\begin{aligned}
&Q(x;t) - \frac{3}{4(x-\lambda_1)^2} + \frac{\eta_1}{x-\lambda_1} \\
=&\frac{a_0}{x^2} + \frac{a_1}{(x-1)^2} + \frac{a_2}{(x-t)^2} + \frac{3}{4(x-\lambda_2)^2} - \frac{a_1+a_2+3/2}{x(x-1)} + \frac{t(t-1)L}{x(x-1)(x-t)} \\
&- \frac{\lambda_2(\lambda_2-1)\eta_2}{x(x-1)(x-\lambda_2)} + \frac{2x-1}{x(x-1)}\eta_1 - \frac{x-\lambda_1}{x(x-1)}\eta_1.
\end{aligned}$$

Therefore we have

$$\mathcal{U}_{1,0} = \left(Q(x;t) - \frac{3}{4(x-\lambda_1)^2} + \frac{\eta_1}{x-\lambda_1}\right)\Bigg|_{x=\lambda_1} = \mathcal{U}_1,$$

and

$$\begin{aligned}
\mathcal{U}_{1,1} &= \frac{\partial}{\partial x}\left(Q(x;t) - \frac{3}{4(x-\lambda_1)^2} + \frac{\eta_1}{x-\lambda_1}\right)\Bigg|_{x=\lambda_1} \\
&= \left(\frac{\partial}{\partial\lambda_1}\right)\mathcal{U}_{1,0} - \frac{\eta_1}{\lambda_1(\lambda_1-1)}.
\end{aligned}$$

□

Substitute these expansions into (53), then we see that the condition $w_k^4 = 0$ implies

$$\frac{\partial\lambda_k}{\partial t} = M\left[2M^k\eta_k - M^{k,0}\right]. \tag{64}$$

The condition $w_k^2 = 0$ implies

$$\frac{\partial\eta_k}{\partial t} = M\left[M^k\mathcal{U}_{k,1} + M^{k,1}\eta_k - \frac{3}{2}M^{k,2}\right]. \tag{65}$$

The conditions $w_k^3 = 0, w_k^1 = 0$ are dependent on the non-logarithmic conditions (37),(38) and yield no more new relation.

Theorem 3.1. *Introduce a function $L(\lambda_k, \eta_k, t)$ of λ_k, η_k and t by*

$$L(\lambda_k, \eta_k, t) = M \sum_{k=1}^{2} (M^k \eta_k^2 - M^{k,0} \eta_k - M^k U_k), \tag{66}$$

which is obtained from (37) $\times M^1 +$ (38) $\times M^2$. *Then the MPD of the Fuchsian equation* (32) *on the rational nodal curve C is governed by the following Hamiltonian system with the Hamiltonian function $L = L(\lambda_k, \eta_k, t)$:*

$$\begin{cases} \frac{d\lambda_k}{dt} = \frac{\partial L}{\partial \eta_k} \\ \frac{d\eta_k}{dt} = -\frac{\partial L}{\partial \lambda_k} \end{cases} \tag{67}$$

with a constraint

$$a_0 = \frac{\lambda_1 \lambda_2}{(\lambda_1 \lambda_2 - t)(\lambda_1 - \lambda_2)} [\lambda_1(\lambda_1 - 1)(\lambda_1 - t)(\eta_1^2 - \mathcal{V}_1) - \lambda_2(\lambda_2 - 1)(\lambda_2 - t)(\eta_2^2 - \mathcal{V}_2)], \tag{68}$$

where we put $\mathcal{V}_k = \mathcal{U}_k - a_0/\lambda_k^2$, $k = 1, 2$. Here we remark that the constraint (68) is obtained by eliminating L from (37) and (38).

Proof. We denote the right hand side of (68) by $a_0(\lambda_k, \eta_k, t)$. Then the constraint (68) is written as $a_0 = a_0(\lambda_k, \eta_k, t)$. We prove that the differential equation (67) under the expression (66) of L coincides with the equations (64) and (65). Noting that M, M^k and $M^{k,0}$ are independent of η_k, we have

$$\frac{\partial L(\lambda_k, \eta_k, t)}{\partial \eta_k} = M(2M^k \eta_k - M^{k,0}).$$

So the first equation in (67) coincides with (64). We have only to prove that the equality (65) coincides with the second equation in (67) under the relations $L = L(\lambda_k, \eta_k, t)$ and $a_0 = a_0(\lambda_k, \eta_k, t)$. Differentiating (37) and (38) with respect to λ_1, we have

$$0 = \left(\frac{\partial}{\partial \lambda}\right) \mathcal{U}_1 + \frac{t(t-1)}{\lambda_1(\lambda_1 - 1)(\lambda_1 - t)} \frac{\partial L}{\partial \lambda_1} + \frac{1}{\lambda_1^2} \frac{\partial a_0}{\partial \lambda_1} \tag{69}$$

and

$$\begin{aligned} 0 =& \frac{3}{2(\lambda_2 - \lambda_1)^3} + \frac{t(t-1)}{\lambda_2(\lambda_2 - 1)(\lambda_2 - t)} \frac{\partial L}{\partial \lambda_1} - \frac{(2\lambda_1 - 1)\eta_1}{\lambda_2(\lambda_2 - 1)(\lambda_2 - \lambda_1)} \\ & - \frac{\lambda_1(\lambda_1 - 1)\eta_1}{\lambda_2(\lambda_2 - 1)(\lambda_2 - \lambda_1)^2} + \frac{1}{\lambda_2^2} \frac{\partial a_0}{\partial \lambda_1}, \end{aligned} \tag{70}$$

respectively. Eliminating $\partial a_0 / \partial \lambda_1$ from (69) and (70), we have

$$-\frac{\partial L(\lambda_k, \eta_k, t)}{\partial \lambda_1} = M \left[M^1 \mathcal{U}_{1,1} + M^{1,1} \eta_1 - \frac{3}{2} M^{1,2} \right].$$

Hence we can obtain the desired coincidence. □

Remark 3.3. *Since the constraint (68) has been derived from the non-logarithmic conditions, it is obvious that (68) holds along the monodromy preserving deformation of (32). In fact, we can prove straightforwardly that* $a_0(\lambda_k, \eta_k, t)$ *is constant along any solution of the differential equation (67) by computing*

$$\begin{aligned}&\frac{d}{dt}a_0(\lambda_k, \eta_k, t)\\=&\frac{\partial a_0}{\partial t}(\lambda_k, \eta_k, t) + \sum_{k=1}^{2}\left(\frac{d\lambda_k}{dt}\frac{\partial a_0}{\partial \lambda_k}(\lambda_k, \eta_k, t) + \frac{d\eta_k}{dt}\frac{\partial a_0}{\partial \eta_k}(\lambda_k, \eta_k, t)\right) = 0.\end{aligned}$$

Remark 3.4. *We now consider the Hamiltonian system (67) without the constraint forgetting its derivation from the MPD. Since the Hamiltonian* $L(\lambda_k, \eta_k, t)$ *does not contain the parameter* a_0 *(*a_0 *is eliminated in the process of* $(37) \times M^1 + (38) \times M^2$*),* $a_0(\lambda_k, \eta_k, t)$ *gives a first integral of the system (67). In this picture, we should regard* a_0 *as a function of the dynamical variables (not a parameter). We shall see that this view point is natural (Remark 4.1 and Corollary 5.1).*

4 Degeneration of elliptic curves and MPD on a rational nodal curve

We set $q = e^{2\pi i\tau}$. We identify the complex torus E_τ with $\mathbf{C}^*/ < q >$ via the mapping $z \mapsto x = e^{2\pi iz}$, where $< q >$ denotes the multiplicative group generated by q. We shall consider to take the limit $q \to 0$. We use the following formulas in later calculations: the elliptic functions $\mathfrak{z}(z; w)$ and $\wp(z)$ converge to

$$\mathfrak{z}(z; w) \to \pi i\left(\frac{e^{2\pi iz} + e^{2\pi iw}}{e^{2\pi iz} - e^{2\pi iw}} - \frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1} + \frac{e^{2\pi iw} + 1}{e^{2\pi iw} - 1}\right), \tag{71}$$

$$\wp(z) \to (\pi i)^2\left(\frac{4e^{2\pi iz}}{(e^{2\pi iz} - 1)^2} + \frac{1}{3}\right), \tag{72}$$

as $q \to 0$, respectively. In what follows, by abuse of notation, we denote $e^{2\pi i\lambda_k}$ and $e^{2\pi it}$ by the symbols λ_k and t respectively. Since the differential equation (1) contains the parameter q in its coefficients, we can take the limit $q \to 0$, then (1) converges to

$$\begin{aligned}&\frac{d^2w}{dx^2} + \frac{1}{x}\frac{dw}{dx} + \left\{-\frac{b_0}{x^2} - \frac{a_1}{(x-1)^2} - \frac{a_2}{(x-t)^2} - \frac{3}{4(x-\lambda_1)^2} - \frac{3}{4(x-\lambda_2)^2}\right.\\&\quad + \frac{a_1 + a_2 + 3/2}{x(x-1)} + \frac{t(t-1)}{x(x-1)(x-t)}\left(\frac{a_2}{t} - \frac{H}{2\pi it}\right)\\&\quad\left. + \sum_{k=1}^{2}\frac{\lambda_k(\lambda_k - 1)}{x(x-1)(x-\lambda_k)}\left(\frac{\mu_k}{2\pi i\lambda_k} + \frac{3}{4\lambda_k}\right)\right\}w = 0,\end{aligned} \tag{73}$$

where

$$b_0 = \frac{\nu}{(2\pi i)^2} + \frac{a_1 + a_2 + 3/2}{12} + \frac{H}{4\pi i}\frac{t+1}{t-1} - \sum_{k=1}^{2}\frac{\mu_k}{4\pi i}\frac{\lambda_k + 1}{\lambda_k - 1}. \tag{74}$$

On the other hand, we can take the limit of the Hamiltonian (3) as $q \to 0$:

$$H = M[\mu_1^2 - \mu_2^2 + N(\mu_1 + \mu_2) - \Delta_1 + \Delta_2], \tag{75}$$

with

$$M \to \frac{(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 - t)(\lambda_2 - t)}{2\pi i(t-1)(t - \lambda_1\lambda_2)(\lambda_1 - \lambda_2)},$$
$$N \to \pi i \left(\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} + \frac{2(\lambda_1 - \lambda_2)}{(\lambda_1 - 1)(\lambda_2 - 1)} \right),$$
$$\Delta_k \to (\pi i)^2 \left(\frac{4a_1\lambda_k}{(\lambda_k - 1)^2} + \frac{4a_2\lambda_k t}{(\lambda_k - t)^2} + \frac{a_1 + a_2}{3} + \frac{3\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^2} + \frac{1}{4} \right).$$

Then, the Hamiltonian system (6) converges to the following system of differential equations:

$$\frac{d\mu_k}{dt} = -\frac{\lambda_k}{t}\frac{\partial H}{\partial \lambda_k}, \quad \frac{d\lambda_k}{dt} = \frac{\lambda_k}{t}\frac{\partial H}{\partial \mu_k}, \quad k = 1, 2. \tag{76}$$

The equation (73) is transformed into a differential equation of the type (32) by the change of the unknown function $w = x^{-1/2}\varphi$. Then the correspondence between the coefficients of the two differential equations (73) and (32) are written as follows:

$$a_0 = b_0 - \frac{1}{4}, \tag{77}$$

$$L = \frac{H}{2\pi i t} - \frac{a_2}{t}, \tag{78}$$

$$\eta_k = \frac{\mu_k}{2\pi i \lambda_k} + \frac{3}{4\lambda_k}. \tag{79}$$

Proposition 4.1. *The change of variables* (78), (79) *take the differential system* (76) *into the Hamiltonian system* (67) *with the Hamiltonian function* $L(\lambda_k, \eta_k, t)$. *In other words, the Hamiltonian system (6) becomes the Hamiltonian system (67) in the limit* $q \to 0$.

Proof. It can be checked directly. □

Remark 4.1. *From (74) and (77), we see that the variable* ν *in (2) (see also (4)) essentially reduces to the characteristic exponent* a_0 *in the limit* $q \to 0$. *In this context, it is natural that* a_0 *is considered a function of the dynamical variables. We can geometrically interpret this fact as follows. The variable* ν *on a non-singular elliptic curve has global nature, namely we can not describe it explicitly in terms of the local data. But, in the limit of* $q \to 0$, *it becomes the characteristic exponent as a result of vanishing of one of the basic periods.*

5 MPD of a system of linear differential equations on a rational nodal curve

In this section, we formulate an MPD of a rank-two system of Fuchsian differential equations on the rational nodal curve C and we derive a system of differential equations that governs the MPD on C (Theorem 5.1). Then we solve it by relating our MPD theory on C to the standard MPD theory on $\tilde{C}$ (Theorem 5.2). We also show that the Hamiltonian system (67) in Section 3 is generically equivalent to the MPD system derived in this section (Theorem 5.3). From these results, we can conclude that the Hamiltonian system in Section 3 is solved in terms of a P_{VI}-function and a τ-quotient associated with it. It is known that the sixth Painlevé equation can be written as a Hamiltonian system. We observe that we may regard the characteristic exponent at the double point on the rational nodal curve as a dynamical variable. And we show that the τ-quotient should be essentially a canonically conjugate variable to the characteristic exponent (Corollary 5.1).

First of all, we recall the standard MPD theory on $\tilde{C} = \mathbb{P}^1(\mathbb{C})$ with four regular singular points $\{0, 1, t, \infty\}$: consider

$$\frac{dZ}{dx} = P(x;t)Z, \tag{80}$$

where

$$P(x;t) = \frac{P_0}{x} + \frac{P_1}{x-1} + \frac{P_2}{x-t}, \tag{81}$$

$$P_i = \begin{pmatrix} p_i & q_i \\ r_i & -p_i \end{pmatrix}, \quad i = 0, 1, 2,$$

and assume that

$$P_\infty := -P_0 - P_1 - P_2 = \begin{pmatrix} -c_0/2 & 0 \\ 0 & c_0/2 \end{pmatrix}, \tag{82}$$

$$-p_i^2 - q_i r_i = -\frac{c_i^2}{4}, \quad i = 0, 1, 2, \tag{83}$$

for some constants c_0, c_1, c_2. We take a fundamental system of solutions $Z(x)$ normalized at the base point b_0:

$$\begin{aligned} Z(x) &= (I + O(x^{-1}))x^{T_0} && \text{around } x = \infty \\ &= K_1(I + O(x-1))(x-1)^{T_1}C_1 && \text{around } x = 1 \\ &= K_2(I + O(x-t))(x-t)^{T_2}C_2 && \text{around } x = t \\ &= K_0(I + O(x))x^{T_0}C_0 && \text{around } x = 0, \end{aligned} \tag{84}$$

where $T_i = \begin{pmatrix} c_i/2 & 0 \\ 0 & -c_i/2 \end{pmatrix}$, $i = 0, 1, 2$ and the analytic continuation of $Z(x)$ is done along the same paths as in Section 3. We define the monodromy datum associated with $Z(x)$ by the following set of matrices:

$$\left\{ N_\infty = e^{-2\pi\sqrt{-1}T_0}, N_i = C_i^{-1} e^{2\pi\sqrt{-1}T_i} C_i \right\}_{i=0,1,2}. \tag{85}$$

We note that these matrices are subject to the unique relation

$$N_\infty N_0 N_1 N_2 = I. \tag{86}$$

The following fact is well-known:

Proposition 5.1 (Jimbo-Miwa-Ueno [5]). *Let $Z(x;t)$ be a family of fundamental solutions to* (80)*. Then the monodromy datum associated with $Z(x;t)$ is independent of t, if and only if the coefficients $\{P_i\}_{i=0,1,2}$ satisfy the following system of differential equations, which is called the Schlesinger system:*

$$\begin{cases} \frac{dP_0}{dt} = \frac{1}{t}[P_2, P_0] \\ \frac{dP_1}{dt} = \frac{1}{t-1}[P_2, P_1] \\ \frac{dP_2}{dt} = -\frac{1}{t}[P_2, P_0] - \frac{1}{t-1}[P_2, P_1]. \end{cases} \tag{87}$$

Moreover, if the connection matrix C_0 is also independent of t, then K_0 in (84) satisfies the following differential equation:

$$\frac{dK_0}{dt} = \Theta_0 K_0, \tag{88}$$

where $\Theta_0 = \frac{1}{t}P_2$.

Remark 5.1. *While the monodromy datum $\{N_\infty, N_0, N_1, N_2\}$ is an invariant of the Fuchsian equation (80), the connection matrix C_0 is not an invariant of (80) itself. The connection matrix C_0 depends on choice of the gauge matrix K_0 at $x = 0$. Therefore we can not determine C_0 uniquely even if the solution $Z(x)$ is normalized. This remark is essential to our formulation.*

We introduce the τ-function associated with a solution $\{P_0, P_1, P_2\}$ to (87) by

$$\frac{d}{dt}\log\tau(t) = \mathrm{tr}\left(\frac{P_0}{t} + \frac{P_1}{t-1}\right)P_2.$$

Proposition 5.2 (Jimbo-Miwa [6]). *The components $(K_0)_{ab}$ $(a, b \in \{1,2\})$ of the solution matrix K_0 to* (88) *can be written in terms of τ-quotients associated with $\{P_0, P_1, P_2\}$:*

$$(K_0)_{ab} = const.\, q\left\{\begin{array}{cc} \infty & 0 \\ a & b \end{array}; P_0, P_1, P_2\right\},$$

where $q\left\{\begin{array}{cc} \infty & 0 \\ a & b \end{array}; P_0, P_1, P_2\right\} = \tau\left\{\begin{array}{cc} \infty & 0 \\ a & b \end{array}\right\}/\tau$, and $\tau\left\{\begin{array}{cc} \infty & 0 \\ a & b \end{array}\right\}$ stands for the elementary Schlesinger transformation from τ of the type $\left\{\begin{array}{cc} \infty & 0 \\ a & b \end{array}\right\}$ (see [6] for details).

Noting that $K_0^{-1}P_0K_0 = T_0$ and $\det K_0 = 1$, we have the following expression of K_0:

$$K_0 = \begin{pmatrix} k & -q_0(c_0k)^{-1} \\ -q_0^{-1}(p_0 - \frac{c_0}{2})k & (c_0k)^{-1}(p_0 + \frac{c_0}{2}) \end{pmatrix} \tag{89}$$

and

$$k = const. q \left\{ \begin{array}{cc} \infty & 0 \\ 1 & 1 \end{array} ; P_0, P_1, P_2 \right\}. \tag{90}$$

We are now going to give a formulation of MPD of a system of differential equations on C with regular singular points $\{1, t\}$. As was explained in Section 3, a monodromy representation on C consists of a set of matrices $\{M_\infty, M_0, M_1, M_2 \in SL(2, \mathbb{C})\}$ with a unique relation, which should be a purely topological object. Since the degree of freedom of monodromy representations on C is greater than that on $\tilde{C}$, we have to add an extra parameter to the differential system (80). For $\lambda \in \mathbb{C}^*$, we consider the following system of differential equations:

$$\frac{dY(x)}{dx} = A_\lambda(x;t)Y(x), \quad A_\lambda(x;t) = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_2}{x-t}, \tag{91}$$

where the coefficients are square matrices: $A_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & -\alpha_i \end{pmatrix}$ with $\det A_i = -c_i^2/4$ $(i = 0, 1, 2)$, and assume that, for $A_\infty := -A_0 - A_1 - A_2$, A_∞ and A_0 have the relation

$$A_\infty + \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} A_0 \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = 0. \tag{92}$$

Note that A_∞ is not diagonalized in general.

Remark 5.2. *We explain a geometric meaning of (91) and (92). We define a non-trivial rank-two vector bundle E_λ on the rational nodal curve C parameterized by $\lambda \in \mathbb{C}^*$ in the following manner: We take the trivial bundle $\tilde{E} = \mathbb{C}^2$ on $\tilde{C}$, and define an isomorphism ι_λ from $\tilde{E}_\infty = \mathbb{C}^2$ to $\tilde{E}_0 = \mathbb{C}^2$ by*

$$\iota_\lambda : \begin{pmatrix} y_1^{(\infty)} \\ y_2^{(\infty)} \end{pmatrix} \in \tilde{E}_\infty \mapsto \begin{pmatrix} y_1^{(0)} \\ y_2^{(0)} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} y_1^{(\infty)} \\ y_2^{(\infty)} \end{pmatrix} \in \tilde{E}_0.$$

Then we define E_λ by the pair $(\tilde{E}, \iota_\lambda)$ (see the beginning of Section 3). The differential system (91) *naturally induces a connection on E_λ. The multiplicative group $\mathbb{C}^*$ over which λ runs may be identified with the generalized Jacobian of C (cf. Remark 2.3).*

Let $G \in SL(2, \mathbb{C})$ be a matrix such as $G^{-1} A_\infty G = \begin{pmatrix} -c_0/2 & 0 \\ 0 & c_0/2 \end{pmatrix}$. We note that such a matrix G is determined up to multiplication from the right by diagonal matrices. Then we have a fundamental solution to the system (91) with the following form:

$$\begin{aligned} Y(x) &= G(I + O(x^{-1}))x^{T_0} \quad \text{around } x = \infty \\ &= G_1(I + O(x-1))(x-1)^{T_1}C_1 \quad \text{around } x = 1 \\ &= G_2(I + O(x-t))(x-t)^{T_2}C_2 \quad \text{around } x = t \\ &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} G(I + O(x))x^{T_0}M_\infty \quad \text{around } x = 0, \end{aligned} \tag{93}$$

where the analytic continuation of $Y(x)$ is done along the same paths as in Section 3.

Definition 5.1. *We define the monodromy datum associated with the fundamental solution $Y(x)$ by the set of matrices*

$$\left\{M_\infty, M_0 = e^{2\pi i T_0}, M_k = C_k^{-1} e^{2\pi i T_k} C_k \ (k = 1, 2)\right\}.$$

We can check that these matrices satisfy the relation

$$M_\infty^{-1} M_0^{-1} M_\infty M_0 = M_1 M_2$$

in a similar way in Section 3. Here we remark that the matrix M_∞ is uniquely determined by the solution $Y(x)$ (cf. Remark 5.1).

Proposition 5.3. *Let $Y(x;t)$ be a family of fundamental solutions to* (91)*. The monodromy datum associated with $Y(x;t)$ is independent of t, if and only if $Y(x;t)$ and $\lambda = \lambda(t)$ satisfy the following system of differential equations:*

$$\frac{\partial Y}{\partial t}(x;t) = B(x;t)Y(x;t), \ \frac{d\lambda}{dt} = \frac{\alpha_2}{t}\lambda, \tag{94}$$

where

$$B(x;t) = -\frac{A_2}{x-t} + \begin{pmatrix} \varepsilon & \frac{\beta_2}{t(\lambda^2-1)} \\ \frac{\lambda^2\gamma_2}{t(1-\lambda^2)} & -\varepsilon \end{pmatrix}$$

and ε is some function of t and independent of x. The indeterminateness of ε comes from ambiguity of the normalization of G.

Proof. Firstly we assume that the monodromy matrices M_∞, M_0, M_1, M_2 are independent of t. By the assumption, we can assume that C_1 and C_2 are also independent of t by suitably retaking G_1 and G_2 respectively. Put $B(x;t) := (\partial Y(x;t)/\partial t)Y(x;t)^{-1}$, then we immediately see that $B(x;t)$ is a single-valued function of x. We investigate behaviors of $B(x;t)$ at each singular point. Around $x = \infty$, we have

$$B(x;t) = \frac{\partial G}{\partial t}G^{-1} + O(x^{-1}).$$

In similar ways, we see that $B(x;t)$ is holomorphic in x at $x = 1$ and has a pole at $x = t$, where the principal part is given by $-\frac{A_2}{x-t}$. Around $x = 0$, we have

$$B(x;t) = \begin{pmatrix} \frac{d\lambda}{dt}\lambda^{-1} & 0 \\ 0 & -\frac{d\lambda}{dt}\lambda^{-1} \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \frac{\partial G}{\partial t}G^{-1} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} + O(x).$$

In particular, $B(x;t)$ is holomorphic in x at $x = 0$, from which we have

$$B(x;t) = -\frac{A_2}{x-t} + \frac{\partial G}{\partial t}G^{-1}$$

and

$$\begin{aligned} B(0;t) &= \frac{A_2}{t} + \frac{\partial G}{\partial t}G^{-1} \\ &= \begin{pmatrix} \frac{d\lambda}{dt}\lambda^{-1} & 0 \\ 0 & -\frac{d\lambda}{dt}\lambda^{-1} \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \frac{\partial G}{\partial t}G^{-1} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}. \end{aligned} \tag{95}$$

Put

$$\frac{\partial G}{\partial t}G^{-1} = \begin{pmatrix} \varepsilon & g_{12} \\ g_{21} & -\varepsilon \end{pmatrix},$$

then we have the following equations from (95):

$$\frac{d\lambda}{dt} = \frac{\alpha_2}{t},\ g_{12} = \frac{\beta_2}{t(\lambda^2-1)},\ g_{21} = \frac{\lambda^2\gamma_2}{t(1-\lambda^2)}.$$

In order to prove the inverse assertion, it is enough to trace the above discussion in the opposite direction. □

Theorem 5.1. *A family of the differential systems*

$$\frac{\partial Y}{\partial x}(x;t) = A_\lambda(x;t)Y(x;t) \tag{96}$$

admits a monodromy-invariant family of fundamental solutions, if and only if the coefficients of $A_\lambda(x;t)$ satisfy the following system of differential equations:

$$\begin{cases} \frac{\partial A_0}{\partial t} = \frac{1}{t}[A_2, A_0] + [\mathcal{G}, A_0] \\ \frac{\partial A_1}{\partial t} = \frac{1}{t-1}[A_2, A_1] + [\mathcal{G}, A_1] \\ \frac{\partial A_2}{\partial t} = \frac{1}{t}[A_0, A_2] + \frac{1}{t-1}[A_1, A_2] + [\mathcal{G}, A_2] \\ \frac{\partial \lambda}{\partial t} = \frac{\alpha_2}{t}\lambda, \end{cases} \tag{97}$$

where we put

$$\mathcal{G} = \begin{pmatrix} \varepsilon & \frac{\beta_2}{t(\lambda^2-1)} \\ \frac{\lambda^2\gamma_2}{t(1-\lambda^2)} & -\varepsilon \end{pmatrix}.$$

Proof. It is obtained from the integrability condition between (94) and (96). □

In order to solve the differential system (97), we relate our MPD theory on the rational nodal curve C to the standard MPD theory on $\tilde{C}$. For a monodromy-invariant fundamental solution $Y(x;t)$ to (96), put

$$Z(x;t) = G^{-1}Y(x;t), \tag{98}$$

then $Z(x;t)$ satisfies the differential system (80).

Theorem 5.2. *For a solution $\{A_0, A_1, A_2, \lambda\}$ to* (97)*, put*

$$P_i = G^{-1}A_iG, \quad i = 0, 1, 2,$$

and

$$K_0 = G^{-1}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}G,$$

then $\{P_0, P_1, P_2, K_0\}$ becomes a solution to the differential equations (87) and (88). Conversely, we can reproduce a solution $\{A_0, A_1, A_2, \lambda\}$ to (97) *from a solution $\{P_0, P_1, P_2, K_0\}$ to (87) and (88).*

Proof. The first assertion is obvious because the gauge transformation (98) has no effect on the monodromy matrices. Given a solution $\{P_0, P_1, P_2, K_0\}$, the solution $\{A_0, A_1, A_2, \lambda\}$ is reproduced as follows. Let λ be a solution to the quadratic equation for Λ:

$$\Lambda^2 - (\mathrm{tr}\, K_0)\Lambda + 1 = 0$$

(we can not distinguish λ from λ^{-1} in general). From the relations $K_0 = G^{-1}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} G$ and $\det G = 1$, we can express G in terms of the components of K_0. We can recover A_i $(i = 0, 1, 2)$ by $A_i = GP_iG^{-1}$. □

We investigate a relationship between the Hamiltonian system (67) and the differential system (97).

Proposition 5.4. *For a given solution* $\{A_0, A_1, A_2, \lambda\}$ *to* (97)*, take an associated monodromy-invariant fundamental solution* $Y(x;t) = \begin{pmatrix} y_1^{(1)} & y_1^{(2)} \\ y_2^{(1)} & y_2^{(2)} \end{pmatrix}$*:*

$$\begin{cases} \frac{\partial Y}{\partial x}(x;t) = A_\lambda(x;t)Y(x;t) \\ \frac{\partial Y}{\partial t}(x;t) = B(x;t)Y(x;t) \\ \frac{\partial \lambda}{\partial t} = \frac{\alpha_2}{t}\lambda. \end{cases} \tag{99}$$

Let λ_1*,* λ_2 *be two distinct solutions to the following quadratic equation for* x*:*

$$\begin{aligned} \lambda^2\beta_0^{-1}x(x-1)(x-t)A_\lambda(x;t)_{12} &= x^2 - \lambda^2(t + 1 + (t\beta_1 + \beta_2)\beta_0^{-1})x + t\lambda^2 \\ &= (x - \lambda_1)(x - \lambda_2) = 0. \end{aligned}$$

Then, (99) *is equivalent to the extended system*

$$\begin{cases} \frac{\partial^2}{\partial x^2}\Phi(x;t) = Q(x;t)\Phi(x;t) \\ \frac{\partial}{\partial t}\Phi(x;t) = A(x;t)\frac{\partial \Phi}{\partial x}(x;t) - \frac{1}{2}\frac{\partial A}{\partial x}(x;t)\Phi(x;t), \end{cases} \tag{100}$$

with

$$A(x;t) = -\frac{x(x-1)(x-\lambda_1\lambda_2)(t-\lambda_1)(t-\lambda_2)}{(x-\lambda_1)(x-\lambda_2)t(t-1)(t-\lambda_1\lambda_2)},$$

by the change of the unknown functions

$$\Phi(x;t) = x^{1/2}(x-1)^{1/2}(x-t)^{1/2}(x-\lambda_1)^{-1/2}(x-\lambda_2)^{-1/2}(y_1^{(1)}, y_1^{(2)}).$$

Proof. We can easily see that

$$Q(x;t) = \frac{f''}{f} + A'_{\lambda,11} - \frac{A'_{\lambda,12}}{A_{\lambda,12}}A_{\lambda,11} + A_{\lambda,12}A_{\lambda,21} + A^2_{\lambda,11}$$

and $A(x;t) = B(x;t)_{12}/A_\lambda(x;t)_{12}$, where

$$f = x^{1/2}(x-1)^{1/2}(x-t)^{1/2}(x-\lambda_1)^{-1/2}(x-\lambda_2)^{-1/2},$$

and $'$ stands for the differentiation with respect to x. On the other hand, we have

$$A_\lambda(x;t)_{12} = \frac{\beta_0(x-\lambda_1)(x-\lambda_2)}{\lambda^2 x(x-1)(x-t)},$$
$$B(x;t)_{12} = \frac{\beta_2(x-t\lambda^2)}{t(\lambda^2-1)(x-t)} = -\frac{\beta_2(x-\lambda_1\lambda_2)}{(t-\lambda_1\lambda_2)(x-t)},$$
$$\frac{\beta_2}{\beta_0} = \frac{(t-\lambda_1)(t-\lambda_2)}{t(t-1)\lambda^2}.$$

Hence we obtain

$$A(x;t) = -\frac{x(x-1)(x-\lambda_1\lambda_2)(t-\lambda_1)(t-\lambda_2)}{(x-\lambda_1)(x-\lambda_2)t(t-1)(t-\lambda_1\lambda_2)}.$$

□

Theorem 5.3. *The Hamiltonian system* (67) *and the differential system* (97) *are equivalent to each other by the following correspondence between the dependent variables: for a solution* $\{A_0, A_1, A_2, \lambda\}$ *to the system* (97), *we set*

$$\lambda_1 + \lambda_2 = \lambda^2(t+1+(t\beta_1+\beta_2)\beta_0^{-1}),$$
$$\lambda_1\lambda_2 = t\lambda^2,$$

$$\eta_1 = \frac{\alpha_0+1/2}{\lambda_1} + \frac{\alpha_1+1/2}{\lambda_1-1} + \frac{\alpha_2+1/2}{\lambda_1-t} - \frac{1}{2(\lambda_1-\lambda_2)},$$
$$\eta_2 = \frac{\alpha_0+1/2}{\lambda_2} + \frac{\alpha_1+1/2}{\lambda_2-1} + \frac{\alpha_2+1/2}{\lambda_2-t} - \frac{1}{2(\lambda_2-\lambda_1)},$$

$$L = \frac{1}{t}(\alpha_0+\alpha_2+\frac{1}{2}+tr(A_0A_2)) + \frac{1}{t-1}(\frac{1}{2}+tr(A_1A_2)) - \frac{\alpha_2+1/2}{t-\lambda_1} - \frac{\alpha_2+1/2}{t-\lambda_2},$$

then $(\lambda_1, \lambda_2, \eta_1, \eta_2)$ *satisfies the system* (67) *and* L *coincides with the Hamiltonian of* (67) *as a function of* t. *Conversely, for a solution* $(\lambda_1, \lambda_2, \eta_1, \eta_2)$ *to* (67), *we set*

$$\frac{\beta_1}{\beta_0} = -\frac{t(\lambda_1-1)(\lambda_2-1)}{(t-1)\lambda_1\lambda_2},$$
$$\frac{\beta_2}{\beta_0} = \frac{(t-\lambda_1)(t-\lambda_2)}{(t-1)\lambda_1\lambda_2},$$

$$\alpha_0 = \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)(\lambda_1\lambda_2-t)}((\lambda_1-1)(\lambda_1-t)n_1 - (\lambda_2-1)(\lambda_2-t)n_2),$$
$$\alpha_1 = \frac{(\lambda_1-1)(\lambda_1-t)(\lambda_2-1)(\lambda_2-t)}{(t-1)(\lambda_1-\lambda_2)(\lambda_1\lambda_2-t)}(\lambda_1 n_1 - \lambda_2 n_2),$$
$$\alpha_2 = -\alpha_1, \;\; \gamma_i = (\frac{c_i^2}{4} - \alpha_i^2)\beta_i^{-1}, \quad i = 0, 1, 2,$$

$$\lambda = (t^{-1}\lambda_1\lambda_2)^{1/2},$$

where we put

$$n_1 = \eta_1 - \frac{1}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_1 - 1} + \frac{1}{\lambda_1 - t} - \frac{1}{\lambda_1 - \lambda_2}\right),$$
$$n_2 = \eta_2 - \frac{1}{2}\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_2 - 1} + \frac{1}{\lambda_2 - t} + \frac{1}{\lambda_1 - \lambda_2}\right),$$

then $\{A_0, A_1, A_2, \lambda\}$ *satisfies* (97).

Proof. The time evolution of the solution $\{A_0, A_1, A_2, \lambda\}$ (and $\{\lambda_1, \lambda_2, \eta_1, \eta_2\}$) is determined by the integrability condition of the system (99) (and (100) respectively). However the systems (99) and (100) are equivalent to each other as was proved in Proposition 5.4. □

Putting Theorem 5.2 and Theorem 5.3 together, we can conclude that the Hamiltonian system (67) is solved in terms of solutions to (87) and (88), namely a P_{VI}-function and the τ-quotient associated with it. (Note that the Schlesinger system (87) is equivalent to P_{VI}.) It is known that the sixth Painlevé equation can be written as a Hamiltonian system: put

$$\begin{aligned} H_{VI} =& \frac{1}{t(t-1)}\{y(y-1)(y-1)z^2 \\ &- (c_0(y-1)(y-t) + c_1 y(y-t) + (c_2-1)y(y-1))z \\ &+ (c_0 + \frac{c_1+c_2}{2})(\frac{c_1+c_2}{2} - 1)(y-t) + c_0(c_2-1)(t-1)\}, \end{aligned}$$

then P_{VI} is equivalent to the Hamiltonian system

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{\partial H_{VI}}{\partial z} \\ \dfrac{dz}{dt} = -\dfrac{\partial H_{VI}}{\partial y}. \end{cases} \tag{101}$$

We have seen that we may consider the characteristic exponent at the double point a function of the dynamical variables. In that case, we ask a question: "Can we find a Hamiltonian system extending the H_{VI} system (101) in such a way that the obtained system is equivalent to the Hamiltonian system (67)?" We shall give a partial answer to this question (The word "partial" means that we shall prove the equivalence only as differential equations not as Hamiltonian systems.):

Corollary 5.1. *For the Hamiltonian function* H_{VI}*, we consider the following Hamiltonian system with canonical variables* (y, b_0, z, c_0) *(where* b_0 *is a cyclic coordinate and we regard* c_0 *as a canonical variable though it is a parameter in the standard theory):*

$$\frac{dy}{dt} = \frac{\partial H_{VI}}{\partial z} = \frac{y(y-1)(y-t)}{t(t-1)}\left(2z - \frac{c_0}{y} - \frac{c_1}{y-1} - \frac{c_2-1}{y-t}\right), \tag{102}$$

$$\frac{dz}{dt} = -\frac{\partial H_{VI}}{\partial y} = \frac{1}{t(t-1)}\{(-3y^2+2(1+t)y-t)z^2 + ((2y-1-t)c_0+(2y-t)c_1+(2y-1)(c_2-1))z + (c_0+\frac{c_1+c_2}{2})(\frac{c_1+c_2}{2}-1)\}, \tag{103}$$

$$\frac{db_0}{dt} = \frac{\partial H_{VI}}{\partial c_0} = \frac{1}{t(t-1)}\{-(y-1)(y-t)z + (\frac{c_1+c_2}{2}-1)(y-t)+(c_2-1)(t-1)\}, \tag{104}$$

$$\frac{dc_0}{dt} = -\frac{\partial H_{VI}}{\partial b_0} = 0. \tag{105}$$

Then the system of differential equations (102)-(105) is equivalent to the system (67). In particular, b_0 *is solved by* $b_0 = \log(y^{-1}kt^{c_2/2})$*, where* $k = const.$ $q\left\{\begin{array}{cc} \infty & 0 \\ 1 & 1 \end{array}\right\}$.

Proof. According to Appendix C in [6], the dependent variables y, z of the H_{VI} system are related to those of the Schlesinger system (87) as follows:

$$P_0 = \begin{pmatrix} z_0 + c_0/2 & -uz_0 \\ u^{-1}(z_0+c_0) & -z_0 - c_0/2 \end{pmatrix},$$
$$P_1 = \begin{pmatrix} z_1 + c_1/2 & -vz_1 \\ v^{-1}(z_1+c_1) & -z_1 - c_1/2 \end{pmatrix},$$
$$P_2 = \begin{pmatrix} z_t + c_2/2 & -wz_t \\ w^{-1}(z_t+c_2) & -z_t - c_2/2 \end{pmatrix},$$

where

$$u = \frac{Xy}{tz_0}, \quad v = -\frac{X(y-1)}{(t-1)z_1}, \quad w = \frac{X(y-t)}{t(t-1)z_t},$$

$$z_0 = \frac{y}{tc_0}\left\{y(y-1)(y-t)\tilde{z}^2 + (c_1(y-t)+tc_2(y-1) - (c_1+c_2)(y-1)(y-t))\tilde{z} + \frac{(c_1+c_2)^2}{4}(y-t-1) - \frac{c_1+c_2}{2}(c_1+tc_2)\right\},$$

$$z_1 = -\frac{y-1}{(t-1)c_0}\left\{y(y-1)(y-t)\tilde{z}^2 + ((c_0+c_1)(y-t)+tc_2(y-1) - (c_1+c_2)(y-1)(y-t))\tilde{z} + \frac{(c_1+c_2)^2}{4}(y-t) - \frac{c_1+c_2}{2}(c_1+tc_2) - \frac{c_1+c_2}{2}(c_0+\frac{c_1+c_2}{2})\right\},$$

$$
\begin{aligned}
z_t =&\frac{y-t}{t(t-1)c_0}\left\{y(y-1)(y-t)\tilde{z}^2+(c_1(y-t)+t(c_0+c_2)(y-1)\right.\\
&-(c_1+c_2)(y-1)(y-t))\tilde{z}+\frac{(c_1+c_2)^2}{4}(y-1)\\
&\left.-\frac{c_1+c_2}{2}(c_1+tc_2)-t\frac{c_1+c_2}{2}(c_0+\frac{c_1+c_2}{2})\right\},
\end{aligned}
$$

$$
\tilde{z}=z-\frac{c_0}{y}-\frac{c_1}{y-1}-\frac{c_2}{y-t}
$$

and X is an overall parameter. (Note that the notations in [6] differ from ours.) By (88) and (89), we see that $k = const.\, q\left\{\begin{array}{cc}\infty & 0\\ 1 & 1\end{array}\right\}$ is a general solution to the differential equation

$$
\begin{aligned}
\frac{dk}{dt}&=\frac{1}{t}\left(p_2-\frac{q_2}{q_0}(p_0-\frac{c_0}{2})\right)k \qquad (106)\\
&=\frac{1}{t}\left(z_t+\frac{c_2}{2}-\frac{wz_t}{u}\right)k\\
&=\left\{\frac{(y-1)(y-t)}{t(t-1)}\left(z-\frac{c_0}{y}-\frac{c_1}{y-1}-\frac{c_2}{y-t}\right)+\frac{(c_1+c_2)(y-t)}{2t(t-1)}+\frac{c_2}{2t}\right\}k.
\end{aligned}
$$

As is explained above, we have the following correspondence

$$
\begin{aligned}
&y=y(\lambda_1,\lambda_2,\eta_1,\eta_2,t),\\
&z=z(\lambda_1,\lambda_2,\eta_1,\eta_2,t),\\
&c_0=(1+4a_0)^{1/2}=c_0(\lambda_1,\lambda_2,\eta_1,\eta_2,t),\\
&k=k(\lambda_1,\lambda_2,\eta_1,\eta_2,t)
\end{aligned}
$$

such that, given a solution $\lambda_k(t),\eta_k(t)$ $(k=1,2)$ to the system (67), $y(t)=y(\lambda_k(t),\eta_k(t),t)$, $z(t)=z(\lambda_k(t),\eta_k(t),t)$ satisfy the equations (102) and (103), $c_0(t)=c_0(\lambda_k(t),\eta_k(t),t)$ is constant with respect to t and $k(t)=k(\lambda_k(t),\eta_k(t),t)$ satisfies the differential equation (106) with $y=y(t)$, $z=z(t)$, and vice versa. In order to prove the statement, we have only to verify that $b_0=\log(y^{-1}kt^{c_2/2})$ satisfies the differential equation (104). However we have

$$
\begin{aligned}
&\frac{d}{dt}\log(y^{-1}kt^{c_2/2})=k^{-1}\frac{dk}{dt}-y^{-1}\frac{dy}{dt}+\frac{c_2}{2t}\\
=&\frac{1}{t(t-1)}\{-(y-1)(y-t)z+(\frac{c_1+c_2}{2}-1)(y-t)+(c_2-1)(t-1)\}
\end{aligned}
$$

from (106) and (102), which coincides with (104). □

References

[1] R. Fuchs, Über lineare homogenen Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen, Math. Ann. 63 (1907), 301-321.

[2] A. Hurwitz, R. Courant, Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen, Springer-Verlag Berlin Göttingen Heidelberg New York, 1964.

[3] H. Ishikawa, On monodromy preserving deformation of Fuchsian differential equations with two apparent singularities (in Japanese), Master's thesis, the Univ. of Tokyo (1999).

[4] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, From Gauss to Painlevé, Aspects of Mathematics E16, Vieweg (1991).

[5] M. Jimbo, T. Miwa, K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. Physica 2D (1981), 306-352.

[6] M. Jimbo, T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II, Physica 2D (1981), 407-448.

[7] S. Kawai, Isomonodromic deformation of Fuchsian projective connections on elliptic curves, Nagoya Math. J. 171 (2003), 127-161.

[8] D. Korotkin, Isomonodromic deformations in genus zero and one: algebro-geometric solutions and Schlesinger transformations. Integrable systems: from classical to quantum, CRM Proc. Lecture Notes, 26, 2000, 87-104.

[9] D. Korotkin, H. Samtleben, On the quantization of isomonodromic deformations on the torus, Int. J. Mod. Phys. A 12 (1997), 2013-2029.

[10] T. Mano, Studies on monodromy preserving deformation of linear differential equations on elliptic curves, J. Math. Phys. 50 (2009), DOI: 10.1063/1.3204973 (21 pages).

[11] K. Okamoto, On Fuchs'problem on a torus.I, Funkcial. Ekvac. 14 (1971), 137-152.

[12] K. Okamoto, Sur le Problème de Fuchs sur un tore. II, J. Fac. Sci. Univ. Tokyo Sec.IA 24 (1977), 357-372.

[13] K. Okamoto, The Hamiltonian structure derived from the holonomic deformation of the linear ordinary differential equations on an elliptic curve, Sci. Papers Collegh Arts Sci. Univ. Tokyo 37 (1987), 1-11.

[14] K. Okamoto, On the holonomic deformation of linear ordinary differential equations on an elliptic curve, Kyushu J. Math. 49 (1995), 281-308.

[15] K. Okamoto, Introduction to the Painlevé equations (in Japanese), Sophia Kokyuroku in Mathematics No.19 (1985), Iwanami shoten (2009).

[16] Y. Sasaki, Special solutions of the Hamiltonian system on an elliptic curve, J. Math. Sci. Univ. Tokyo 9 (2002), 669-687.

[17] K. Takasaki, Gaudin model, KZ Equation, and isomonodromic problem on torus, Lett. Math. Phys. 44 (2) (1998), 143-156.

[18] E. T. Whittaker, G. N. Watson, A course of modern analysis, Cambridge University Press, London, 1902.

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