Kumamoto J. Math.
Vol.**25** (2012), 1–15

The Best Constant of Discrete Sobolev Inequality Corresponding to a Bending Problem of a String

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(Received May 5, 2011)
(Accepted October 17, 2011)

Abstract. The best constant of the discrete Sobolev inequality

$$\left(\max_{0 \le j \le N-1} |u(j)| \right)^2 \le C \sum_{i=0}^{N-1} \Big[\ |u(i) - u(i+1)|^2 + q|u(i)|^2 \ \Big],$$

where $\boldsymbol{u} = {}^t(u(0), \cdots, u(N-1)) \in \mathbf{C}^N$ satisfies $u(N) = u(0)$, is obtained, where q takes not only positive values but also zero or negative values. The best constant of above inequality is equal to a harmonic mean of positive eigenvalues of the symmetric second-order difference matrix. We stress that the best constant is obtained through the boundary value problem of second-order difference equation, which describes a bending phenomenon of a string. This boundary value problem is essentially solved by finding an inverse or its variant, which includes a Penrose-Moore generalized inverse.

1 Conclusion

Let $N = 2n + 1 + \varepsilon$ $(n = 1, 2, 3, \cdots,\ \varepsilon = 0, 1)$ if $N \ge 3$, q be a real number and $\omega = \exp(\sqrt{-1}\, 2\pi/N)$ the N-th root of 1. For $\boldsymbol{u} = {}^t(u(0), \cdots, u(N-1))$, we introduce Sobolev energy

$$E(\boldsymbol{u}) = \sum_{j=0}^{N-1} \Big[\ |u(j) - u(j+1)|^2 + q|u(j)|^2 \ \Big], \qquad u(N) = u(0).$$

With this setting, we have the following conclusions:

Mathematical Subject Classification (2010): 46E39.
Key words: Sobolev inequality, Best constant, Reproducing kernel, Discrete Fourier Transform.

Theorem 1.1. *Let q be $0 < q < \infty$. Then for any $\boldsymbol{u} \in \mathbf{C}^N$, there exists a positive constant C which is independent of $\boldsymbol{u}$ such that the discrete Sobolev inequality*

$$\left(\max_{0 \le j \le N-1} |u(j)| \right)^2 \le C E(\boldsymbol{u}) \tag{1.1}$$

holds. Among such C, the best constant C_0 is given by

$$C_0 = \frac{U_N\left(\dfrac{q+2}{2}\right)}{2\left(T_N\left(\dfrac{q+2}{2}\right) - 1\right)}, \tag{1.2}$$

where $T_N(x)$ defined by $T_N(\cos(\theta)) = \cos(N\theta)$ is Chebyshev polynomial of the first kind and $U_N(x)$ defined by $U_N(\cos(\theta)) = \sin(N\theta)/\sin(\theta)$ is Chebyshev polynomial of the second kind. Moreover, C_0 is equivallently expressed as

$$C_0 = \begin{vmatrix} q+2 & -1 & & \\ -1 & q+2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & q+2 \end{vmatrix} \Bigg/ \begin{vmatrix} q+2 & -1 & & -1 \\ -1 & q+2 & \ddots & \\ & \ddots & \ddots & -1 \\ -1 & & -1 & q+2 \end{vmatrix}, \tag{1.3}$$

where the numerator is a size of $(N-1)\times(N-1)$ determinant and the denominator is a $N \times N$ determinant. If we replace C by C_0 in the above inequality (1.1), then the equality holds for

$$\boldsymbol{u} = \frac{1}{N} \sum_{k=0}^{N-1} {}^t(\cdots, \omega^{(j-j_0)k}, \cdots) \frac{1}{4\sin^2(\pi k/N) + q}$$

with j_0 ($j_0 = 0, 1, 2, \cdots, N-1$) arbitrarily fixed.

Theorem 1.2. *Let q satisfies $-4\sin^2(\pi/N) < q \le 0$. Then for any $\boldsymbol{u} \in \mathbf{C}^N$ satisfying $\sum_{j=0}^{N-1} u(j) = 0$, there exists a positive constant C which is independent of $\boldsymbol{u}$ such that the discrete Sobolev inequality (1.1) holds. Among such C, the best constant C_0 is given by*

$$C_0 = \frac{1}{N} \sum_{k=1}^{N-1} \frac{1}{4\sin^2(\pi k/N) + q}.$$

If we replace C by C_0 in (1.1), then the equality holds for

$$\boldsymbol{u} = \frac{1}{N} \sum_{k=1}^{N-1} {}^t(\cdots, \omega^{(j-j_0)k}, \cdots) \frac{1}{4\sin^2(\pi k/N) + q}$$

with j_0 ($j_0 = 0, 1, 2, \cdots, N-1$) arbitrarily fixed.

Theorem 1.3. *For any fixed* m $(m = 1, 2, 3, \cdots, n-1+\varepsilon)$*, let us assume* $-4\sin^2(\pi(m+1)/N) < q \le -4\sin^2(\pi m/N)$*. Then for any* $\boldsymbol{u} \in \mathbf{C}^N$ *satisfying* $\sum_{j=0}^{N-1} u(j) = 0$ *and* $\sum_{j=0}^{N-1} \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} (2\pi jk) u(j) = 0 \quad (k = 1, 2, 3, \cdots, m)$*, there exists a positive constant* C *which is independent of* $\boldsymbol{u}$ *such that the discrete Sobolev inequality* (1.1) *holds. Among such* C*, the best constant* C_0 *is given by*

$$C_0 = \frac{1}{N} \sum_{k=m+1}^{N-m-1} \frac{1}{4\sin^2(\pi k/N) + q}.$$

If we replace C *by* C_0 *in* (1.1)*, then the equality holds for*

$$\boldsymbol{u} = \frac{1}{N} \sum_{k=m+1}^{N-m-1} {}^t(\cdots, \omega^{(j-j_0)k}, \cdots) \frac{1}{4\sin^2(\pi k/N) + q}$$

with j_0 $(j_0 = 0, 1, 2, \cdots, N-1)$ *arbitrarily fixed.*

The above theorems are a discrete version of [1, Theorem 1.3] and the constant C_0 is also regarded as a best constant of discrete Sobolev inequality on regular polygon. In [2], we have obtained the best constant of discrete Sobolev inequality on regular polyhedron (Tetra-, Hexa-, Octa-, Dodeca-, Icosa-hedron). It should be noted the cases $q > 0$ and $q = 0$ are essentially solved in [3] and [4], respectively. However, we also treat these cases for the sake of self-containedness. The engineering meaning of Sobolev inequality (1.1) is that the square of the maximum bending displacement of a string $u(i)$ is estimated from above by the constant multiple of its potential energy $E(\boldsymbol{u})$.

This paper is composed of five sections. In section 2, we explain about the bending problem of a string and prepare some basic tools which play important roles in this paper. In section 3, we present a reproducing relation. Section 4 is devoted to prove of Theorem 1.1∼1.3. Finally in section 5, we calculate the minimum of discrete Sobolev functional.

2 Discrete bending problem of a string

Let us consider a string which is supported by uniformly distributed springs with spring constant q on a fixed ceiling. Further, let $f(x)$ denote the load at x and $u(x)$ the bending displacement at x. It is well known that bending displacement $u(x)$ is governed by second order linear ordinary differential equation [1] as:

$$-u'' + qu = f(x). \tag{2.1}$$

In this paper, we consider the best constant of discrete Sobolev inequality (1.1) which is obtained through the construction of pseudo-inverse (an extension of

Penrose-Moore generalized inverse) of the discretization of (2.1) under periodinc boundary condition. That is

BVP

$$\begin{cases} -u(i-1)+(2+q)\,u(i)-u(i+1)=f(i) \qquad (0 \le i \le N-1), \\ u(-1)=u(N-1), \quad u(N)=u(0). \end{cases}$$

We introduce some $N \times N$ matrices which play important roles in this paper. A matrix $\boldsymbol{W}$ is defined by

$$\boldsymbol{W} = \Big(\ \omega^{ij} \ \Big),$$

which satisfies

$$\boldsymbol{W}^{-1} = \frac{1}{N}\,\boldsymbol{W}^* = \frac{1}{N}\Big(\ \omega^{-ij} \ \Big), \qquad \boldsymbol{W}\,\boldsymbol{W}^* = \boldsymbol{W}^*\,\boldsymbol{W} = N\,\boldsymbol{I}.$$

$\boldsymbol{E}_k$ $(k \in \mathbf{Z})$ are orthogonal projection matrices defined by

$$\boldsymbol{E}_k = \frac{1}{N}\Big(\ \omega^{(i-j)k} \ \Big)$$

which satisfy the following properties: $\boldsymbol{E}_k\boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k$, $\boldsymbol{E}_k^* = \boldsymbol{E}_k$, $\boldsymbol{E}_{-k} = \boldsymbol{E}_{N-k}$. Then $\delta(i)$ $(i \in \mathbf{Z})$ is Kronecker delta symbol. $\boldsymbol{L}$ is a rotate-left matrix defined by

$$\boldsymbol{L} = \Big(\ \delta(i-j+1) \ \Big) = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}.$$

$\boldsymbol{L}$ is a unitary matrix, that is, $\boldsymbol{L}^* = {}^t\boldsymbol{L} = \boldsymbol{L}^{-1} = \boldsymbol{L}^{N-1}$, and satisfies $\boldsymbol{L}^k = \Big(\ \delta(i-j+k) \ \Big)$, $\boldsymbol{L}^N = \Big(\ \delta(i-j) \ \Big) = \boldsymbol{I}$. Eigenvalues of $\boldsymbol{L}$ are ω^i $(i = 0, 1, \cdots, N-1)$. $\boldsymbol{L}$ is diagonalized by the matrix $\boldsymbol{W}$ as

$$\boldsymbol{L} = \boldsymbol{W}\widehat{\boldsymbol{L}}\boldsymbol{W}^{-1} = \frac{1}{N}\boldsymbol{W}\widehat{\boldsymbol{L}}\boldsymbol{W}^* \quad \text{where} \quad \widehat{\boldsymbol{L}} = \Big(\ \omega^i\delta(i-j) \ \Big). \tag{2.2}$$

Using $\boldsymbol{E}_k$, we have the spectral decomposition of $\boldsymbol{L}$ as

$$\boldsymbol{L} = \frac{1}{N}\Big(\ \omega^{ij} \ \Big)\Big(\omega^i\delta(i-j)\Big)\Big(\ \omega^{-ij} \ \Big) = \sum_{k=0}^{N-1} \omega^k \boldsymbol{E}_k. \tag{2.3}$$

The matrices $\boldsymbol{L}^{-i}$ and $\boldsymbol{E}_i$ satisfy

$$\boldsymbol{L}^{-i} = \sum_{k=0}^{N-1} \omega^{-ik}\boldsymbol{E}_k, \qquad \boldsymbol{E}_i = \frac{1}{N}\sum_{k=0}^{N-1} \omega^{ik}\boldsymbol{L}^{-k}.$$

In particular, if $i=0$, we have the spectral decomposition of an identity matrix $\boldsymbol{I}$

$$\boldsymbol{I}=\sum_{k=0}^{N-1}\boldsymbol{E}_k. \tag{2.4}$$

The following linear transformations $\widehat{\ }$ and $^{1[\text{-}1]\frown}$ are called DFT(Discrete Fourier Transform) and IDFT(Inverse Discrete Fourier Transform), respectively.

$$\begin{aligned}
&\mathbf{C}^N \ni \boldsymbol{u} \ \stackrel{\widehat{\ }}{\longrightarrow}\ \widehat{\boldsymbol{u}}=\boldsymbol{W}^*\boldsymbol{u}\in\mathbf{C}^N \qquad \Leftrightarrow \\
&\qquad\qquad \widehat{u}(i)=\sum_{k=0}^{N-1}\omega^{-ik}u(k)\quad (i=0,1,\cdots,N-1),\\
&\mathbf{C}^N \ni \boldsymbol{v} \ \stackrel{1[\text{-}1]\frown}{\longrightarrow}\ \boldsymbol{v}^{1[\text{-}1]\frown}=\frac{1}{N}\boldsymbol{W}\boldsymbol{v}\in\mathbf{C}^N \qquad \Leftrightarrow \\
&\qquad\qquad v^{1[\text{-}1]\frown}(i)=\frac{1}{N}\sum_{k=0}^{N-1}\omega^{ik}v(k)\quad (i=0,1,\cdots,N-1).
\end{aligned}$$

We have the following Lemma.

Lemma 2.1. *Let $\boldsymbol{A}$ be a symmetric second-order difference matrix given by*

$$\boldsymbol{A}=\Big(\ a(i-j)\ \Big)=\begin{cases}\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix} & (N=2),\\[2ex] \begin{pmatrix}2 & -1 & & -1\\ -1 & \ddots & \ddots & \\ & \ddots & & -1\\ -1 & & -1 & 2\end{pmatrix} & (N=3,4,5,\cdots),\end{cases}$$

$$a(i)=\begin{cases}2 & (\mathrm{Mod}(i,N)=0),\\ -1 & (\mathrm{Mod}(i,N)=1,\ N-1),\qquad -2\quad (\mathrm{Mod}(i,2)=1),\\ 0 & (else).\end{cases}$$

Then $\boldsymbol{A}$ is expressed in the following three equivalent forms:

(1) $\displaystyle \boldsymbol{A}=\sum_{k=0}^{N-1}a(k)\boldsymbol{L}^{-k}=2\boldsymbol{I}-\boldsymbol{L}-\boldsymbol{L}^{-1}=(\boldsymbol{I}-\boldsymbol{L})(\boldsymbol{I}-\boldsymbol{L})^*.$

(2) $\displaystyle \boldsymbol{A}=\sum_{k=0}^{N-1}\widehat{a}(k)\boldsymbol{E}_k=\sum_{k=1}^{n}\widehat{a}(k)(\boldsymbol{E}_k+\boldsymbol{E}_{-k})+\varepsilon\,\widehat{a}(N/2)\boldsymbol{E}_{N/2}.$

(3) $\displaystyle \boldsymbol{A}=\boldsymbol{W}\widehat{\boldsymbol{A}}\boldsymbol{W}^{-1},\qquad \widehat{\boldsymbol{A}}=\Big(\widehat{a}(i)\delta(i-j)\Big).$

From (3)*, we see that* $\widehat{a}(k)$ $(k = 0, 1, \cdots, N-1)$ *are eigenvalues of* $\boldsymbol{A}$ *and satisfy the following relations*

$$\widehat{a}(k) = 2 - \omega^k - \omega^{-k} = \left| 1 - \omega^k \right|^2 = 2 - 2\cos(2\pi k/N) = 4\sin^2(\pi k/N), \tag{2.5}$$

$$\widehat{a}(0) = 0 < \widehat{a}(1) = \widehat{a}(N-1) < \cdots < \widehat{a}(n) = \widehat{a}(N-n)$$
$$\begin{cases} < 4 & (\varepsilon = 0), \\ < \widehat{a}(n+1) = \widehat{a}(N/2) = 4 & (\varepsilon = 1). \end{cases}$$

Moreover, corresponding normalized orthogonal eigenvectors $\boldsymbol{\varphi}_k$ $(k = 0, 1, \cdots, N-1)$ *are given by*

$$\boldsymbol{\varphi}_k = \frac{1}{\sqrt{N}}\, {}^t(1, \omega^k, \omega^{2k}, \cdots, \omega^{(N-1)k}).$$

Proof of Lemma 2.1 Since the proof of (1) is standard and easy, we omit it. Using (2.3) and (2.4), we have

$$\boldsymbol{A} = 2\boldsymbol{I} - \boldsymbol{L} - \boldsymbol{L}^{-1} = \sum_{k=0}^{N-1} \left(2 - \omega^k - \omega^{-k}\right) \boldsymbol{E}_k = \sum_{k=0}^{N-1} \widehat{a}(k) \boldsymbol{E}_k =$$
$$\sum_{k=1}^{n} \widehat{a}(k)(\boldsymbol{E}_k + \boldsymbol{E}_{-k}) + \varepsilon\, \widehat{a}(N/2)\boldsymbol{E}_{N/2}.$$

We have obtained (2). Using (2.2), we have

$$\boldsymbol{A} = 2\boldsymbol{I} - \boldsymbol{L} - \boldsymbol{L}^{-1} = \boldsymbol{W}\left(2\boldsymbol{I} - \widehat{\boldsymbol{L}} - \widehat{\boldsymbol{L}}^{-1}\right)\boldsymbol{W}^{-1} =$$
$$\boldsymbol{W}\Big(\left(2 - \omega^i - \omega^{-i}\right)\delta(i-j)\Big)\boldsymbol{W}^{-1} =$$
$$\boldsymbol{W}\Big(\widehat{a}(i)\delta(i-j)\Big)\boldsymbol{W}^{-1} = \boldsymbol{W}\widehat{\boldsymbol{A}}\boldsymbol{W}^{-1}.$$

We have obtained (3). Thus we proved Lemma 2.1. ∎

It should be noted that from (2.2), $\boldsymbol{\varphi}_k$ $(k = 0, 1, \cdots, N-1)$ are also eigenvectors of $\boldsymbol{L}$ and that the relation $\boldsymbol{E}_k = \boldsymbol{\varphi}_k \boldsymbol{\varphi}_k^*$ holds. Introducing vectors

$$\boldsymbol{u} = {}^t(u(0), \cdots, u(N-1)), \qquad \boldsymbol{f} = {}^t(f(0), \cdots, f(N-1))$$

and a symmetric second order difference matrix $\boldsymbol{A}$, one can rewrite BVP as

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$$(\boldsymbol{A} + q\boldsymbol{I})\,\boldsymbol{u} = \boldsymbol{f}.$$

We assume the following three cases for q:

(I) $0 < q < \infty$.

(II) $-4\sin^2(\pi/N) < q \le 0$.
(III) For any fixed $m(m = 1, 2, 3, \cdots, n-1+\varepsilon)$, $-4\sin^2(\pi(m+1)/N) < q \le -4\sin^2(\pi m/N)$.

Correspondingly, we introduce Sobolev spaces

$$H = \left\{ \boldsymbol{u} \in \mathbf{C}^N \;\middle|\; \text{(I)} \quad \text{none}, \quad \text{(II)} \quad \sum_{j=0}^{N-1} u(j) = 0, \right.$$
$$\left. \text{(III)} \quad \text{(II) and} \quad \sum_{j=0}^{N-1} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (2\pi jk)\, u(j) = 0 \quad (1 \le k \le m) \right\}. \tag{2.6}$$

Using $\boldsymbol{E}_k$, Sobolev spaces H given by (2.6) is rewritten equivalently as

$$H = \left\{ \boldsymbol{u} \in \mathbf{C}^N \;\middle|\; \text{(I)} \quad \text{none}, \qquad \text{(II)} \quad \boldsymbol{E}_0 \boldsymbol{u} = \boldsymbol{0}, \right.$$
$$\left. \text{(III)} \quad \boldsymbol{E}_k \boldsymbol{u} = \boldsymbol{0} \quad (|k| \le m) \right\}. \tag{2.7}$$

Moreover, we use the following symbol:

$$K = \left\{ k \in \mathbf{N} \;\middle|\; \text{(I) } 0 \le k \le N-1, \quad \text{(II) } 1 \le k \le N-1, \right.$$
$$\left. \text{(III) } m+1 \le k \le N-m-1 \right\}.$$

Then, we have the following proposition concerning BVP.

Proposition 2.1. *For any $\boldsymbol{f} \in H$, the difference equation*

$$\begin{cases} (\boldsymbol{A} + q\boldsymbol{I})\boldsymbol{u} = \boldsymbol{f}, \\ \boldsymbol{u} \in H \end{cases}$$

has one and only one solution given by

$$\boldsymbol{u} = \boldsymbol{G}\boldsymbol{f}, \qquad \boldsymbol{G} = \Big(\; g(i-j) \; \Big),$$

where i, j satisfies $0 \le i, j \le N-1$ and

$$g(i) = \frac{1}{N} \sum_{k \in K} \omega^{ik} \widehat{g}(k), \qquad \widehat{g}(k) = \frac{1}{\widehat{a}(k) + q}.$$

We note that $g(i)$ is a periodic function, that is $g(i+N) = g(i)$. $\boldsymbol{G}$ is expressed in the following three equivalent forms:

(1) $\quad \boldsymbol{G} = \displaystyle\sum_{k=0}^{N-1} g(k) \boldsymbol{L}^{-k} \qquad \text{(I)} \sim \text{(III)}.$

(2) $$\boldsymbol{G} = \sum_{k\in K} \widehat{g}(k)\boldsymbol{E}_k =$$
$$\begin{cases} \widehat{g}(0)\boldsymbol{E}_0 + \displaystyle\sum_{k=1}^{n} \widehat{g}(k)(\boldsymbol{E}_k + \boldsymbol{E}_{-k}) + \varepsilon\, \widehat{g}(N/2)\boldsymbol{E}_{N/2} & \text{(I)}, \\ \displaystyle\sum_{k=1}^{n} \widehat{g}(k)(\boldsymbol{E}_k + \boldsymbol{E}_{-k}) + \varepsilon\, \widehat{g}(N/2)\boldsymbol{E}_{N/2} & \text{(II)}, \\ \displaystyle\sum_{k=m+1}^{n} \widehat{g}(k)(\boldsymbol{E}_k + \boldsymbol{E}_{-k}) + \varepsilon\, \widehat{g}(N/2)\boldsymbol{E}_{N/2} & \text{(III)}. \end{cases}$$

(3) $$\boldsymbol{G} = \boldsymbol{W}\widehat{\boldsymbol{G}}\boldsymbol{W}^{-1}, \quad \widehat{\boldsymbol{G}} = \Big(\widehat{g}(i)\delta(i-j)\Big) \qquad \text{(I)}.$$

The matrix $\boldsymbol{G}$ satisfies the following relation:

$$(\boldsymbol{A} + q\boldsymbol{I})\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{A} + q\boldsymbol{I}) = \begin{cases} \boldsymbol{I} & \text{(I)}, \\ \boldsymbol{I} - \boldsymbol{E}_0 & \text{(II)}, \\ \boldsymbol{I} - \displaystyle\sum_{|k|\le m} \boldsymbol{E}_k & \text{(III)}. \end{cases}$$

The above proposition is a discrete version of [1, Theorem 1.1].
Proof of Proposition 2.1 From (2.4) and (2.7), we have

$$\sum_{k\in K} \boldsymbol{E}_k\boldsymbol{f} = \left\{ \begin{array}{ll} \boldsymbol{I}\boldsymbol{f} & \text{(I)} \\ (\boldsymbol{I} - \boldsymbol{E}_0)\boldsymbol{f} & \text{(II)} \\ \left(\boldsymbol{I} - \displaystyle\sum_{|k|\le m} \boldsymbol{E}_k\right)\boldsymbol{f} & \text{(III)} \end{array} \right\} = \boldsymbol{f} =$$
$$(\boldsymbol{A} + q\boldsymbol{I})\boldsymbol{u} = \sum_{k\in K} \left(\widehat{a}(k) + q\right)\boldsymbol{E}_k\boldsymbol{u}.$$

Operating $\boldsymbol{E}_l$ from the left on both sides of the above relation and using the relation $\boldsymbol{E}_l\boldsymbol{E}_k = \delta(l-k)\boldsymbol{E}_l$, we obtain

$$\boldsymbol{E}_l\boldsymbol{u} = \frac{1}{\widehat{a}(l) + q}\boldsymbol{E}_l\boldsymbol{f} \qquad (l \in K).$$

Thus we have

$$\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u} = \sum_{k\in K} \boldsymbol{E}_k\boldsymbol{u} = \sum_{k\in K} \frac{1}{\widehat{a}(k) + q}\boldsymbol{E}_k\boldsymbol{f} = \boldsymbol{G}\boldsymbol{f},$$

where

$$\boldsymbol{G} = \sum_{k\in K} \frac{1}{\widehat{a}(k) + q}\boldsymbol{E}_k = \sum_{k\in K} \widehat{g}(k)\boldsymbol{E}_k,$$

which gives (2) in Proposition 2.1. In fact, we have

$$(\boldsymbol{A}+q\boldsymbol{I})\boldsymbol{G} = \sum_{k,l\in K} \frac{\widehat{a}(k)+q}{\widehat{a}(l)+q}\boldsymbol{E}_k\boldsymbol{E}_l = \sum_{k,l\in K} \frac{\widehat{a}(k)+q}{\widehat{a}(l)+q}\delta(k-l)\boldsymbol{E}_k =$$

$$\sum_{k\in K}\boldsymbol{E}_k = \left\{\begin{array}{ll} \boldsymbol{I} & \text{(I)} \\ \boldsymbol{I}-\boldsymbol{E}_0 & \text{(II)} \\ \boldsymbol{I}-\sum_{|k|\le m}\boldsymbol{E}_k & \text{(III)} \end{array}\right\}.$$

We note that $\boldsymbol{G}$ is an inverse matrix of $\boldsymbol{A}+q\boldsymbol{I}$ in the case (I) and a Penrose-Moore generalized inverse matrix in the case $q=0$. The matrix $\boldsymbol{G}$ in the forms (1) and (3) can be easily derived by using the facts in section 2, so we omit them. ■

3 Reproducing relation

In this section, we show that $\boldsymbol{G}$ is a reproducing matrix for H and inner product $(\cdot,\cdot)_H$. We introduce a standard inner product:

$$(\boldsymbol{u},\,\boldsymbol{v}) = \boldsymbol{v}^*\boldsymbol{u}, \qquad \|\boldsymbol{u}\|^2 = (\boldsymbol{u},\boldsymbol{u}) \qquad (\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{C}^N),$$

Sobolev inner product:

$$(\boldsymbol{u},\boldsymbol{v})_H = ((\boldsymbol{A}+q\boldsymbol{I})\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{v}^*(\boldsymbol{A}+q\boldsymbol{I})\boldsymbol{u},$$
$$\|\boldsymbol{u}\|_H^2 = (\boldsymbol{u},\boldsymbol{u})_H = E(\boldsymbol{u}) \qquad (\boldsymbol{u},\boldsymbol{v}\in H)$$

and N-dimensional vector:

$$\boldsymbol{\delta}_j = {}^t(0,\cdots,0,\overset{j}{\smile}\!\!\!\!1,0,\cdots,0).$$

At first, we show the positive definiteness of Sobolev inner product $(\cdot,\cdot)_H$.

Lemma 3.1. $(\cdot,\cdot)_H$ *is an inner product.*

Proof of Lemma 3.1 First, we treat the case (I). Since

$$\|\boldsymbol{u}\|_H^2 = (\boldsymbol{u},\boldsymbol{u})_H = ((\boldsymbol{A}+q\boldsymbol{I})\boldsymbol{u},\boldsymbol{u}) = (\boldsymbol{A}\boldsymbol{u},\boldsymbol{u}) + (q\boldsymbol{I}\boldsymbol{u},\boldsymbol{u}) =$$
$$((\boldsymbol{I}-\boldsymbol{L})^*(\boldsymbol{I}-\boldsymbol{L})\boldsymbol{u},\boldsymbol{u}) + q(\boldsymbol{u},\boldsymbol{u}) = ((\boldsymbol{I}-\boldsymbol{L})\boldsymbol{u},(\boldsymbol{I}-\boldsymbol{L})\boldsymbol{u}) + q(\boldsymbol{u},\boldsymbol{u}) =$$
$$\|(\boldsymbol{I}-\boldsymbol{L})\boldsymbol{u}\|^2 + q\|\boldsymbol{u}\|^2 \ge q\|\boldsymbol{u}\|^2,$$

we have $\|\boldsymbol{u}\|_H^2 \ge 0$ and $\|\boldsymbol{u}\|_H^2 = 0$ holds if and only if $\boldsymbol{u}=\boldsymbol{0}$. In the second place, we treat the case (III). The case (II) is proved in the same way. Since $\boldsymbol{E}_k\boldsymbol{u}=\boldsymbol{0}$ $(|k|\le m)$,

$$\boldsymbol{I} = \sum_{k=0}^{N-1}\boldsymbol{E}_k, \qquad \boldsymbol{u} = \sum_{k=m+1}^{N-m-1}\boldsymbol{E}_k\boldsymbol{u}, \qquad \|\boldsymbol{u}\|^2 = \sum_{k=m+1}^{N-m-1}\|\boldsymbol{E}_k\boldsymbol{u}\|^2,$$

we have

$$\|\boldsymbol{u}\|_H^2 = (\boldsymbol{u},\boldsymbol{u})_H = ((\boldsymbol{A}+q\boldsymbol{I})\boldsymbol{u},\boldsymbol{u}) = \left(\sum_{k=0}^{N-1} (\widehat{a}(k)+q)\, \boldsymbol{E}_k\boldsymbol{u}, \sum_{l=m+1}^{N-m-1} \boldsymbol{E}_l\boldsymbol{u} \right) =$$

$$\sum_{k=m+1}^{N-m-1} (\widehat{a}(k)+q)\, \|\boldsymbol{E}_k\boldsymbol{u}\|^2 \geq (\widehat{a}(m+1)+q) \sum_{k=m+1}^{N-m-1} \|\boldsymbol{E}_k\boldsymbol{u}\|^2 =$$

$$(\widehat{a}(m+1)+q)\, \|\boldsymbol{u}\|^2.$$

Since $\widehat{a}(m+1)+q > 0$, we have $\|\boldsymbol{u}\|_H^2 \geq 0$ and $\|\boldsymbol{u}\|_H^2 = 0$ yields $\boldsymbol{u} = \boldsymbol{0}$. This shows that $(\boldsymbol{u},\boldsymbol{v})_H$ is an inner product in H. ■

Lemma 3.2. *For any $\boldsymbol{u} \in H$ and fixed j $(0 \leq j \leq N-1)$, we have the following reproducing relations:*

(1) $u(j) = (\boldsymbol{u}, \boldsymbol{G}\boldsymbol{\delta}_j)_H$.

(2) $g(0) = \|\boldsymbol{G}\boldsymbol{\delta}_j\|_H^2$.

Proof of Lemma 3.2 For any $\boldsymbol{u} \in H$, using $\boldsymbol{G}^* = \boldsymbol{G}$, we have

$$(\boldsymbol{u},\, \boldsymbol{G}\,\boldsymbol{\delta}_j)_H = ((\boldsymbol{A}+q\boldsymbol{I})\boldsymbol{u},\, \boldsymbol{G}\boldsymbol{\delta}_j) = {}^t\boldsymbol{\delta}_j\boldsymbol{G}^*(\boldsymbol{A}+q\boldsymbol{I})\boldsymbol{u} =$$

$$\left\{ \begin{array}{ll} {}^t\boldsymbol{\delta}_j\boldsymbol{I}\boldsymbol{u} & \text{(I)} \\ {}^t\boldsymbol{\delta}_j(\boldsymbol{I}-\boldsymbol{E}_0)\boldsymbol{u} & \text{(II)} \\ {}^t\boldsymbol{\delta}_j\Big(\boldsymbol{I} - \sum\limits_{|k|\leq m} \boldsymbol{E}_k\Big)\boldsymbol{u} & \text{(III)} \end{array} \right\} = {}^t\boldsymbol{\delta}_j\boldsymbol{u} = u(j)$$

This shows (1). Applying (1) to $\boldsymbol{u} = \boldsymbol{G}\boldsymbol{\delta}_j = {}^t(\cdots, g(i-j), \cdots)$, we obtain (2). ■

4 Discrete Sobolev inequality

In this section, we give a proof of (1.1) in Threorem 1.1∼1.3.

Proof of (1.1) **in Theorem 1.1∼1.3** Applying Schwarz inequality to Lemma 3.2 (1), we have

$$|u(j)|^2 \leq \|\boldsymbol{u}\|_H^2 \|\boldsymbol{G}\boldsymbol{\delta}_j\|_H^2 = g(0)\|\boldsymbol{u}\|_H^2.$$

Taking the maximum with respect to j on both sides, we have the discrete Sobolev inequality

$$\left(\max_{0\leq j\leq N-1} |u(j)| \right)^2 \leq g(0)\|\boldsymbol{u}\|_H^2.$$

If we take $\boldsymbol{u} = \boldsymbol{G}\,\boldsymbol{\delta}_{j_0} = {}^t(\cdots, g(j-j_0), \cdots)$ in the above inequality, then we have

$$\left(\max_{0\leq j\leq N-1} |g(j-j_0)| \right)^2 \leq g(0)\|\boldsymbol{G}\boldsymbol{\delta}_{j_0}\|_H^2 = (g(0))^2,$$

where j_0 is fixed number satisfying $0 \le j_0 \le N-1$. Combining this and a trivial inequality

$$(g(0))^2 \le \left(\max_{0 \le j \le N-1} |g(j-j_0)| \right)^2,$$

we have

$$\left(\max_{0 \le j \le N-1} |g(j-j_0)| \right)^2 = g(0) \| \boldsymbol{G} \boldsymbol{\delta}_{j_0} \|_H^2.$$

The above equality shows that $C_0 = g(0)$, that is, $g(0)$ is the least constant of the discrete Sobolev inequality. ■

Proof of (1.2) **and** (1.3) **in Threorem 1.1** We start with

$$g(0) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{\widehat{a}(k)+q} = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{2-\omega^k-\omega^{-k}+q}.$$

If we put $q = a + a^{-1} - 2 > 0$ $(a > 0)$, then we have

$$Ng(0) = -\sum_{k=0}^{N-1} \frac{\omega^k}{(\omega^k - a)(\omega^k - a^{-1})} = \frac{1}{a-a^{-1}} \sum_{k=0}^{N-1} \left[\frac{a}{a-\omega^k} + \frac{a^{-1}}{\omega^k - a^{-1}} \right] =$$

$$\frac{1}{a-a^{-1}} \sum_{k=0}^{N-1} \left[\frac{1}{1-a^{-1}\omega^k} + (a^{-1}\omega^{-k}) \frac{1}{1-a^{-1}\omega^{-k}} \right] =$$

$$\frac{1}{a-a^{-1}} \sum_{k=0}^{N-1} \sum_{j=0}^{\infty} \left[\left(a^{-1}\omega^k\right)^j + \left(a^{-1}\omega^{-k}\right)^{j+1} \right] =$$

$$\frac{1}{a-a^{-1}} \left[\sum_{j=0}^{\infty} a^{-j} \sum_{k=0}^{N-1} \omega^{kj} + \sum_{j=0}^{\infty} a^{-(j+1)} \sum_{k=0}^{N-1} \omega^{-k(j+1)} \right]. \tag{4.1}$$

Using the relation

$$\sum_{k=0}^{N-1} \omega^{kj} = \begin{cases} N & (\mathrm{Mod}(j,N)=0) \\ 0 & (\mathrm{Mod}(i,N) \neq 0)), \end{cases}$$

$$\sum_{k=0}^{N-1} \omega^{-k(j+1)} = \begin{cases} N & (\mathrm{Mod}(-(j+1),N)=0) \\ 0 & (\mathrm{Mod}(-(j+1),N) \neq 0)), \end{cases}$$

and putting $j = Nl$ $(l = 0,1,2,\cdots)$ on the first term of (4.1) and $j+1 = Nl$ $(l = 1,2,3\cdots)$ on the second term of (4.1), then we have

$$Ng(0) = \frac{N}{a-a^{-1}} \left[\sum_{l=0}^{\infty} a^{-Nl} + \sum_{l=1}^{\infty} a^{-Nl} \right] = \frac{N}{a-a^{-1}} \left[\frac{1}{1-a^{-N}} + \frac{a^{-N}}{1-a^{-N}} \right]$$

and therefore

$$g(0) = \frac{1}{a-a^{-1}} \frac{a^{N/2}+a^{-N/2}}{a^{N/2}-a^{-N/2}}.$$

Moreover, putting $a = e^{2\alpha}$ $(\alpha > 0)$, we have

$$g(0) = \frac{1}{2\sinh(2\alpha)\tanh(N\alpha)} =$$
$$\frac{\sinh(N\alpha)\cosh(N\alpha)}{2\sinh(2\alpha)\sinh^2(N\alpha)} = \frac{1}{2}\frac{\sinh(2N\alpha)}{\sinh(2\alpha)}\frac{1}{\cosh(2N\alpha)-1}.$$

Here, using the relation $\cosh(x) = \cos(\sqrt{-1}x)$, $\sinh(x) = -\sqrt{-1}\sin(\sqrt{-1}x)$, we have

$$\cosh(Nx) = T_N(\cosh(x)), \qquad \frac{\sinh(Nx)}{\sinh(x)} = U_N(\cosh(x)).$$

From this relation, $g(0)$ is rewritten as

$$g(0) = \frac{1}{2}\frac{U_N(\cosh(2\alpha))}{T_N(\cosh(2\alpha))-1} = \frac{1}{2}\frac{U_N\left(\dfrac{q+2}{2}\right)}{T_N\left(\dfrac{q+2}{2}\right)-1},$$

where we have used the relation

$$\cosh(2\alpha) = \frac{e^{2\alpha}+e^{-2\alpha}}{2} = \frac{q+2}{2}.$$

This shows (1.2). (1.3) follows from the following properties of Chebyshev polynomials.

$$U_N(x) = \begin{vmatrix} 2x & -1 & & \\ -1 & 2x & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2x \end{vmatrix}_{(N-1)\times(N-1)},$$

$$2(T_N(x)-1) = \begin{vmatrix} 2x & -1 & & -1 \\ -1 & 2x & \ddots & \\ & \ddots & \ddots & -1 \\ -1 & & -1 & 2x \end{vmatrix}_{N\times N}.$$

The first formula is easy to prove. The scond formula is a direct consequence from the first formula and the following relations:

$$U_{N+1}(x) - U_{N-1}(x) = 2T_N(x), \qquad U_{N+1}(x) - 2xU_N(x) + U_{N-1}(x) = 0.$$

This proves (1.2) and (1.3) in Threorem 1.1. ■

5 Discrete Sobolev functional

In this section, we assume $q > 0$, that is the case of (I). Sobolev functional $S(\boldsymbol{u})$ defined by

$$S(\boldsymbol{u}) = \left(\max_{0 \le j \le N-1} |u(j)| \right)^2 / \, \|\boldsymbol{u}\|_H^2 \qquad (\boldsymbol{u} \in H = \mathbf{C}^N, \quad \boldsymbol{u} \neq \mathbf{0})$$

satisfies the following theorem:

Theorem 5.1.
(1) *For arbitrarily fixed j $(0 \le j \le N-1)$, we have*

$$\sup_{\boldsymbol{u} \in H, \ \boldsymbol{u} \neq \mathbf{0}} S(\boldsymbol{u}) = S(\boldsymbol{G}\boldsymbol{\delta}_j) = C_0.$$

(2) $$\inf_{\boldsymbol{u} \in H, \ \boldsymbol{u} \neq \mathbf{0}} S(\boldsymbol{u}) = S(c\boldsymbol{\varphi}_{n+\varepsilon}) = \frac{1}{N} \frac{1}{\widehat{a}(n+\varepsilon) + q},$$

$$\boldsymbol{\varphi}_{n+\varepsilon} = \begin{cases} \dfrac{1}{\sqrt{N}} \, {}^t(1, \omega^n, \omega^{2n}, \cdots, \omega^{(N-1)n}) & (\varepsilon = 0), \\ \dfrac{1}{\sqrt{N}} \, {}^t(1, -1, 1, -1, \cdots, 1, -1) & (\varepsilon = 1), \end{cases}$$

where c is an arbitrary complex number.

It is interesting to note that (2) in the above theorem is peculiar to discrete case. In the continuous limit $(N \to \infty)$, we only have a trivial inequality $\left(\max_{0 \le j \le N-1} |u(j)| \right)^2 \ge 0$.

Proof of Theorem 5.1 (1) is equivalent to Theorem 1.1. Thus we treat the case (2). Recalling that $\widehat{a}(k)$ takes its maximum at $k = n + \varepsilon$ (see (2.5)), we have

$$\|\boldsymbol{u}\|_H^2 = ((\boldsymbol{A} + q\boldsymbol{I})\boldsymbol{u}, \boldsymbol{u}) = \left(\sum_{k=0}^{N-1} (\widehat{a}(k) + q) \, \boldsymbol{E}_k \boldsymbol{u}, \ \sum_{l=0}^{N-1} \boldsymbol{E}_l \boldsymbol{u} \right) =$$
$$\sum_{k=0}^{N-1} (\widehat{a}(k) + q) \, \|\boldsymbol{E}_k \boldsymbol{u}\|^2 \le (\widehat{a}(n+\varepsilon) + q) \sum_{k=0}^{N-1} \|\boldsymbol{E}_k \boldsymbol{u}\|^2 = (\widehat{a}(n+\varepsilon) + q) \, \|\boldsymbol{u}\|^2. \tag{5.1}$$

The equality holds if $\boldsymbol{E}_k \boldsymbol{u} = \mathbf{0}$ $(k \neq n + \varepsilon)$. Hence, in that case, we have $\boldsymbol{u} = (\boldsymbol{E}_0 + \cdots + \boldsymbol{E}_{N-1})\boldsymbol{u} = \boldsymbol{E}_{n+\varepsilon}\boldsymbol{u}$. On the other hand, we have the following trivial inequality:

$$\|\boldsymbol{u}\|^2 = \sum_{j=0}^{N-1} |u(j)|^2 \le N \left(\max_{0 \le j \le N-1} |u(j)| \right)^2 \tag{5.2}$$

in which the equality holds for $|u(0)| = |u(1)| = \cdots = |u(N-1)|$. Combining (5.1) and (5.2), we have

$$\|\boldsymbol{u}\|_H^2 \leq (\widehat{a}(n+\varepsilon)+q)\, N \left(\max_{0 \leq j \leq N-1} |u(j)| \right)^2 . \tag{5.3}$$

Since $\boldsymbol{E}_{n+\varepsilon}\boldsymbol{\varphi}_{n+\varepsilon} = (\boldsymbol{\varphi}_{n+\varepsilon}\boldsymbol{\varphi}^*_{n+\varepsilon})\boldsymbol{\varphi}_{n+\varepsilon} = \boldsymbol{\varphi}_{n+\varepsilon}$ and $|\boldsymbol{\varphi}_{n+\varepsilon}(0)| = |\boldsymbol{\varphi}_{n+\varepsilon}(1)| = \cdots = |\boldsymbol{\varphi}_{n+\varepsilon}(N-1)|$, the equality holds for (5.3) when $\boldsymbol{u} = c\,\boldsymbol{\varphi}_{n+\varepsilon}$. Thus we have (2). ■

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