Kumamoto J. Math.
Vol.**25** (2012), 17–51

# Six points/planes in the 3-space

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(Received August 9, 2011)
(Accepted December 30, 2011)

**Abstract.** There is only one arrangement of $n+3$ hyperplanes in general position in $n$-dimensional real projective space up to continuous move. For example, six planes in the space cut out two cubes, six tetrahedrons, twelve triangular prisms, and six gyozas (a gyoza is bounded by two pentagons, two triangles and two rectangles); we study how they are arranged. Passing to the dual situation, for given six points in the space, we consider the arrangement of the planes passing through three points out of the six; we study also this arrangement.

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**Mathematical Subject Classification (2010)**: 52C35
**Key words**: hyperplane arrangements

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# 1 Introduction

There is only one arrangement of $n+2$ or less hyperplanes in general position in $n$-dimensional real projective space $\mathbf{P}^n$ up to projective move. There is only one arrangement of $n+3$ or less hyperplanes in general position in $\mathbf{P}^n$ up to continuous move. If $n \geq 2$, arrangements of more than $n+3$ hyperplanes are not unique.

Since we consider only the hyperplanes in general position in this paper, we often omit '*in general position*'.

Five lines in the plane cut out five triangles, five rectangles and a pentagon; six planes in the space cut out two cubes, six tetrahedrons, twelve triangular prisms, and six gyozas (a gyoza is bounded by two pentagons, two triangles and two rectangles); we describe in detail how they are arranged.

Starting from six planes bounding a cube, we move the planes slightly to make the three pairs of parallel planes non-parallel, and study the happenings.

For given five points in the plane, we consider the arrangement of the lines passing through two points out of the three; the complete pentagon. For given six points in the space, we consider the arrangement of the planes passing through three points out of the six; we study this arrangement.

# 2 Some generalities

In this section only, we work in the $n$-dimensional real projective space $\mathbf{P}^n$.

## 2.1 Uniqueness of the arrangement of $n+3$ hyperplanes in $\mathbf{P}^n$

In this subsection we prove

**Theorem 1** *The set of arrangements of $n+3$ hyperplanes in general position (no $n+1$ hyperplane meet at a point) in $\mathbf{P}^n$ is connected.*

We prove the dual statement: The set of arrangements of $n+3$ points in $\mathbf{P}^n$ in general position (no $n+1$ points are collinear) is connected.

Any $n+2$ points in general position in $\mathbf{P}^n$ can be transformed projectively into the $n+2$ points:

$$1:0:\cdots:0,\quad 0:1:0:\cdots:0,\quad \ldots,\quad 0:\cdots:0:1,\quad 1:\cdots:1.$$

Projective transformations still operate on these points as permutations of $n+2$ points. $n$ points out of these $n+2$ points span hyperplanes defined by:

$$x_j=0,\quad x_j=x_k\quad (j,k=1,\ldots,n+1,\ j\neq k).$$

These hyperplanes divide the space $\mathbf{P}^n$ into simplices (if non-empty) defined by

$$x_{i_1}<x_{i_2}<\cdots<x_{i_{n+1}},\quad \{i_1,\ldots,i_{n+1}\}\subset\{0,1,\ldots,n+1\},$$

where $x_0=0, x_{n+1}=1$. The symmetric group on $n+2$ letters acts transitively on these simplices. This completes the proof.

## 2.2 A unique cubic curve passing through six points in the space

In this subsection we prove

**Theorem 2** *For any $n+3$ points in general position in $\mathbf{P}^n$, there is a unique (irreducible rational) curve of degree $n$ passing through these points.*

Without loss of generality, we consider $n+3$ points:

$$\begin{array}{rccccc}
 & x_1: & x_2 & :\cdots: & x_n: & x_{n+1} \\
 & & & & & \\
p_0= & 1: & 1 & :\cdots: & 1: & 1, \\
p_1= & 1: & 0 & :\cdots: & 0: & 0, \\
p_2= & 0: & 1 & :\cdots: & 0: & 0, \\
\vdots & & & & & \\
p_n= & 0: & 0 & :\cdots: & 1: & 0, \\
p_{n+1}= & 0: & 0 & :\cdots: & 0: & 1, \\
p_{n+2}= & a_1: & a_2 & :\cdots: & a_n: & a_{n+1},
\end{array}$$

where $0<a_1<\cdots<a_n<a_{n+1}$. We will find a curve

$$C: t\longmapsto x_1(t):\cdots:x_{n+1}(t),$$

such that

$$C(q_0) = p_0, \quad C(q_1) = p_1, \ \ldots, \ C(q_{n+1}) = p_{n+1}, \quad C(r) = p_{n+2}.$$

If we normalize as

$$q_0 = \infty, \quad q_1 = 0, \quad q_2 = 1,$$

then the above condition is equivalent to the system

$$\begin{array}{lll}
(x_1(r) =) & c(r-q_2)(r-q_3)(r-q_4)\cdots(r-q_{n+1}) & = a_1, \\
(x_2(r) =) & c(r-q_1)(r-q_3)(r-q_4)\cdots(r-q_{n+1}) & = a_2, \\
(x_3(r) =) & c(r-q_1)(r-q_2)(r-q_4)\cdots(r-q_{n+1}) & = a_3, \\
\vdots & & \\
(x_{n+1}(r) =) & c(r-q_1)(r-q_2)(r-q_4)\cdots(r-q_n) & = a_{n+1},
\end{array}$$

with $n+1$ unknowns $q_3, \ldots, q_{n+1}, r$ and $c$. From the first and the second equations, $r$ is solved, from the second and the third equation, $p_3$ is solved,..., and we obtain a unique set of solutions:

$$r = \frac{a_2}{a_2 - a_1}, \quad q_j = \frac{(a_j - a_1)a_2}{(a_2 - a_1)a_j} \quad (j = 3, \ldots n+1)$$

We do not care the value of $c$. Since

$$q_3 - 1 = \frac{(a_3 - a_2)a_1}{(a_2 - a_1)a_3}, \quad q_j - q_i = \frac{a_1 a_2}{a_2 - a_1} \cdot \frac{a_j - a_i}{a_j a_i}, \quad r - q_j = \frac{a_1 a_2}{a_2 - a_1} \cdot \frac{1}{a_4}$$

we have

$$q_1 = 0 < q_2 = 1 < q_3 < \cdots < q_{n+1} < r.$$

# 3 Chambers cut out by six planes in $\mathbf{P}^3$

It is known (cf. [CY]) and not difficult to show that any six planes cut out two cubes, six gyozas (defined in §3.2), twelve prisms and six tetrahedrons. We describe in this section how they are situated:

**Theorem 3** *1. The six gyozas form a solid torus.*

*2. The two cubes touch each other at two anti-podal points, forming homotopically a circle (cf. Figure 3).*

*3. The two cubes fattened by the twelve prisms form another solid torus (cf. Figure 6).*

*4. The two solid tori are glued along their rectangular faces; there are six tetrahedral openings (cf. Figure 7).*

Every statement, once stated, can be checked anyway; so we do not bother to give a proof. We make our best effort to explain and describe the happenings.

Thanks to the theorems in the previous section, any six planes in general position can be considered as the osculating planes of a cubic curve $\mathbf{P}^1 \to \mathbf{P}^3$ at six points (cf. [CY]) . Labeling the pre-images as $t_1 < t_2 < \cdots < t_6$, we label the corresponding six planes as $1, \ldots, 6$, respectively. Then every statement implies the other statements made by application of the shift $j \to j+1 \bmod 6$. Let $\mathbf{Z}_6$ be the cyclic group generated by this shift. Summing up, we have

**Proposition 1** *The chambers cut out by six planes* $1, \ldots, 6$ *are the* $\mathbf{Z}_6$*-orbits:*

| | | | |
|---|---|---|---|
| two cubes : | $12 \times 34 \times 56,$ | $23 \times 45 \times 61,$ | |
| twelve prisms : | $123 \times 45,\ 123 \times 56,$ | $234 \times 56,\ 234 \times 61,$ | $\ldots,$ |
| six tetrahedrons : | $1234,$ | $2345,$ | $\ldots,$ |
| six gyozas : | $1234\underline{56},$ | $2345\underline{61},$ | $\ldots.$ |

## 3.1 Restricted arrangements

The intersection line of the planes 5 and 6 is denoted by $\{56\} = \{65\}$, and the intersection point of the planes 4, 5 and 6 is denoted by $\{456\} = \{546\} = \cdots$. On plane 6, there are five lines

$$\{16\}, \quad \{26\}, \quad \ldots \quad , \{56\} \quad \subset \text{plane } 6$$

surrounding a pentagon $6 : 12345$ in this order as is shown in Figure 1. Note that around the pentagon, there are five triangles

$$123, \quad 234, \quad 345, \quad 451, \quad 512 \quad \subset \text{plane } 6,$$

and five rectangles

$$12 \times 34, \quad 23 \times 45, \quad 34 \times 51, \quad 45 \times 12, \quad 51 \times 23 \quad \subset \text{plane } 6$$

## 3.2 Six gyozas form a solid torus

The six planes carry six pentagons:

$$1 : 23456, \quad 2 : 34561, \quad \ldots \quad , 5 : 61234, \quad 6 : 12345.$$

There is a polytope bounded by the two pentagons in the planes 5 and 6, two triangles in the planes 1 and 4, and two rectangles in the planes 2 and 3, which will be denoted by $1234\underline{56}$; this is shown in Figure 2 left. Such a polytope is called a **gyoza**.[1]

Two gyozas $1234\underline{56}$ and $6123\underline{45}$ are adjacent through the pentagon $5 : 61234$. A gyoza is adjacent to a tetrahedron through a triangular face, to a prism through a rectangular face. The six gyozas form a solid torus, which are bounded by twelve triangles and twelve rectangles.

[1] Gyoza is a familiar Chinese dumpling.

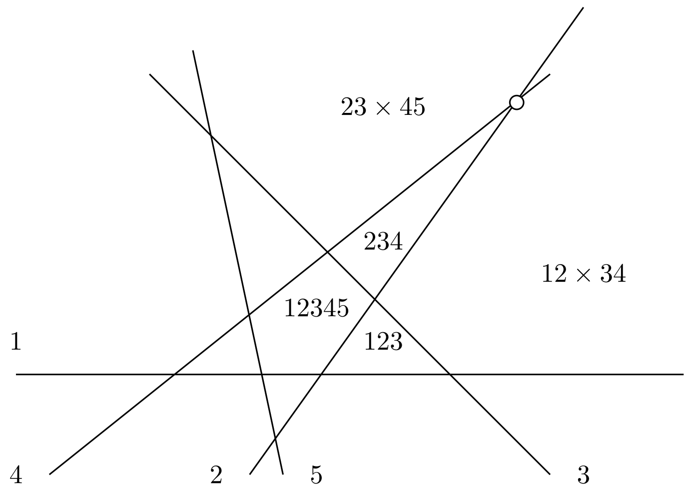


Figure 1: Five lines $1, 2, 3, 4, 5$ in the plane surrounding a pentagon 12345

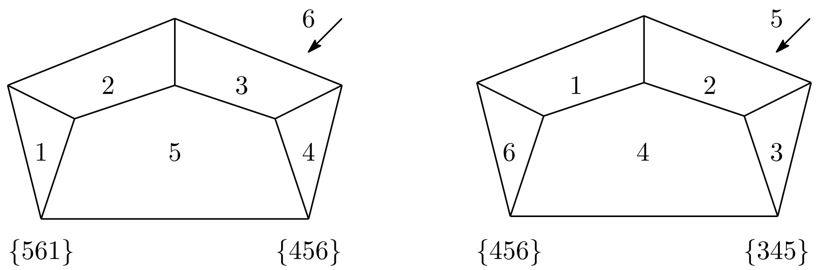


Figure 2: Gyozas $1234\underline{56}$ and $6123\underline{45}$

### 3.3 Six tetrahedrons

The six tetrahedrons are coded by serial four letters:

$$1234, \quad 2345, \quad \ldots, \quad 6123.$$

The tetrahedron 1234 is adjacent to the two gyozas: $5612\underline{34}$ and $3456\underline{12}$, through planes 2 and 3, respectively. Its face 1 is adjacent to the prism $61 \times 234$, and face 4 to the prism $34 \times 561$.

The other way round: gyoza $1234\underline{56}$ is adjacent to two tetrahedrons 5612 and 3456.

## 3.4 Two cubes

There are two cubes $12 \times 34 \times 56$ and $23 \times 45 \times 61$. They share two vertices $\{135\}$ and $\{246\}$ (see Figure 3); the union of the two cubes is homotopic to a circle.

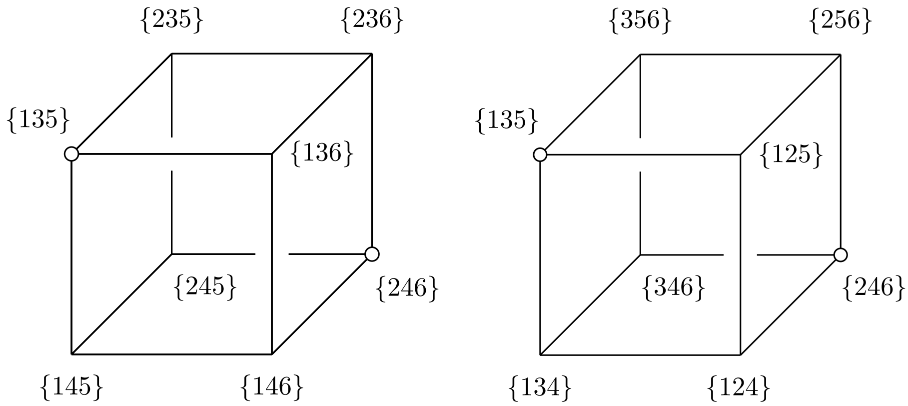


Figure 3: Two cubes$12 \times 34 \times 56, \quad 23 \times 45 \times 61$

On plane 6, there are two rectangles $6 : 12 \times 34$ and $6 : 23 \times 45$ (touching at the point $\{246\}$), which are faces of the two cubes (cf. Figure 1). The cubes are adjacent only to prisms not to any gyozas.

## 3.5 Twelve prisms

On each plane, there is a unique triangle with vertex $\{135\}$ or $\{246\}$; on plane 6, the triangle 234 with vertex $\{246\}$ (see Figures 1, 4). Among the five triangles on a plane, this special triangle is the intersection of two prisms. There are six such special triangles, of which both sides are twelve prisms.

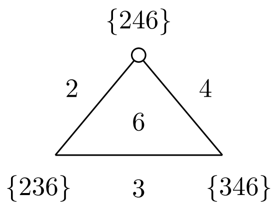


Figure 4: The special triangle $6 : 234$ on plane 6

Figure 5 describes the two prisms touching along the triangle $6 : 234$. Each prism is adjacent to a prism through a triangular face, to another prism through a rectangular face (marked $P$), to a tetrahedron through the other triangular face, to a cube through a rectangular face (marked $C$), and to a gyoza through a rectangular face (marked $G$).

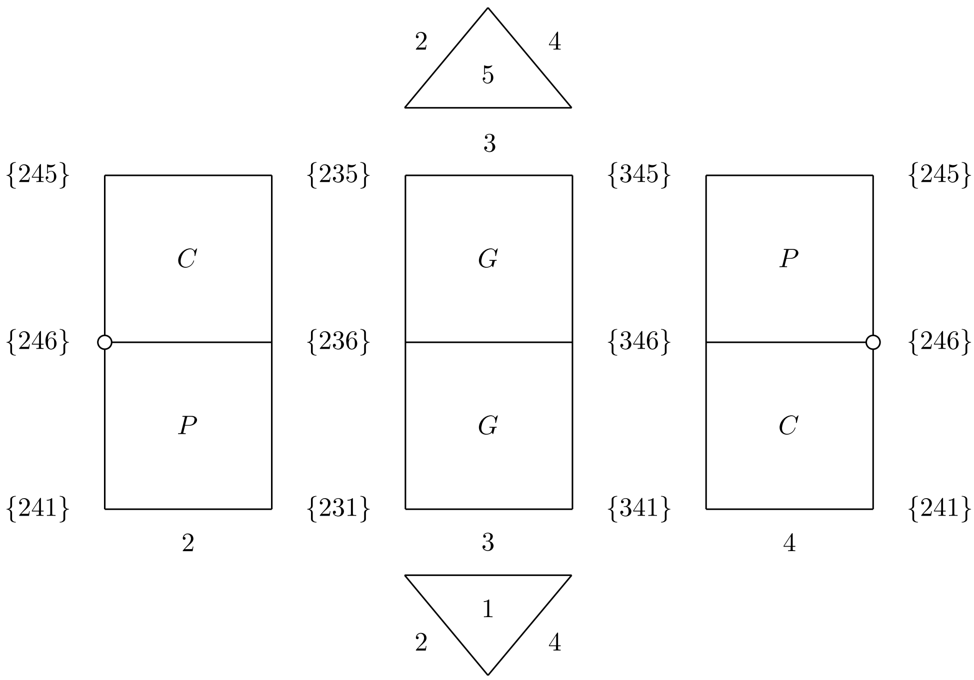


Figure 5: The two prisms $234 \times 56$ and $234 \times 61$ glued around the triangle $6 : 234$

## 3.6 Prisms around the cubes

Let us enrich the union of the two cubes touching at the two points by the twelve prisms to make it a solid torus. Figure 6 shows the two cubes and the six prisms

$$12 \times 456, \quad 23 \times 456; \qquad 34 \times 612, \quad 45 \times 612; \qquad 56 \times 234, \quad 61 \times 234$$

around the vertex $\{246\}$. In front we can see four rectangles and two triangles on plane 5.

This solid torus is bounded by twelve triangles and twelve rectangles. Through each adjacent pair of triangles is a tetrahedron. Let us regard the six tetrahedrons niches between this solid torus and the solid torus made by the six gyozas. Then the two solid tori are glued along the bubbled torus weaved by twelve rectangles and six tetrahedrons as in Figure 7. In the figure, $GP$ means that the rectangle is (part of) the intersection of a gyoza and a prism, $PT$ means that the triangle is

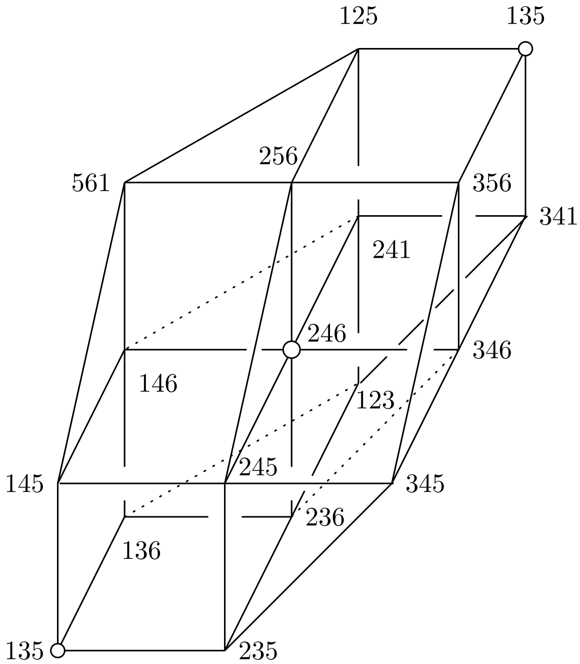


Figure 6: Six prisms around the two cubes

(part of) the intersection of a prism and a tetrahedron, and so on. You can see in the middle-top two adjacent rectangles marked −3−; these are the rectangles on the plane 3 shown in the middle of Figure 5. The two squares on the right suggest that the tetrahedron with vertices

$$\{123\}, \quad \{341\}, \quad \{124\}, \quad \{234\}$$

is surrounded by two prisms and two gyozas.

On the boundary of the solid torus shown in Figure 6, we can see a meridian

$$123 \longrightarrow 156 \longrightarrow 345 \longrightarrow 123$$

traveling diagonally the six rectangular faces. Tracing this curve on the boundary of the solid torus made by six gyozas, we find that this curve travels the longitude twice.

## 3.7 Both sides of a plane

Each plane is divided into eleven polygons. Each polygon is the intersection of two polyhedra. Figure 8 shows the kind of these polyhedra, where $T$ stands for a

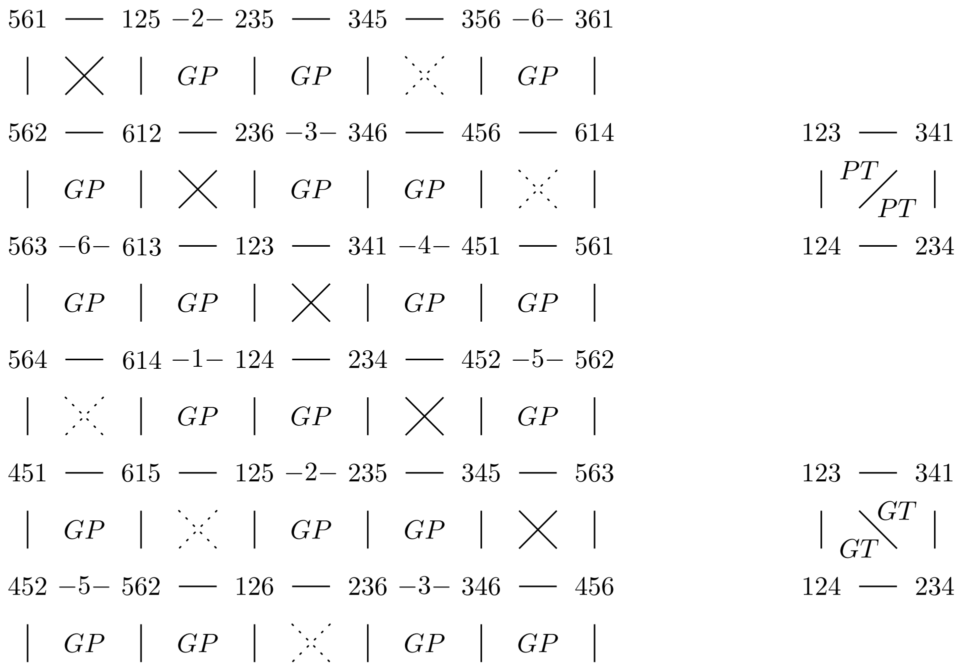


Figure 7: Bubbled torus weaved by twelve rectangles and six simplices

tetrahedron, $P$ for a prism, $C$ for a cube, and $G$ for a gyoza. The vertex marked by a circle is the vertex $\{135\}$ or $\{246\}$. The triangle with this vertex is the special triangle stated in Section 3.5.

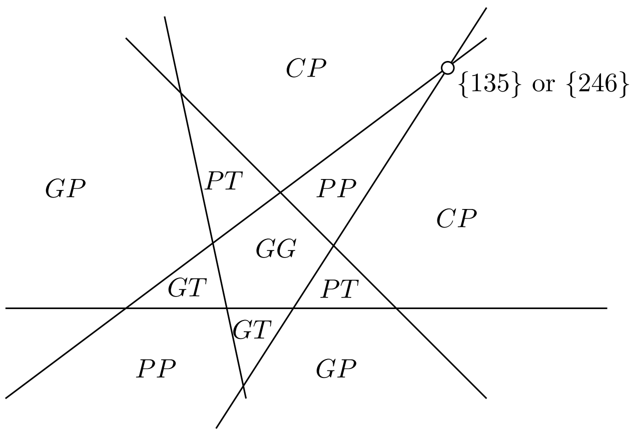


Figure 8: Eleven polygons on a plane bounding polyandry

## 3.8 Switch with respect to a simplex

Consider the arrangement of six planes $1, ..., 6$ in the space as in §3. We choose a simplex, say 3456, and move one of the four planes, say 3, bounding the simplex so that the simplex reduces to a point, and re-appears again. This move is called the **switch with respect to the simplex** 3456. We are interested in the change of the 26 chambers.

### 3.8.1 2D case

We start from the arrangement of five lines in the plane bounding the pentagon 12345 in this order. We study the switch with respect to the simplex 345: we move one of the three lines, say 3, bounding the simplex so that the simplex reduces to a point, and re-appears again (cf. Figure 9). Note that the new arrangement bound

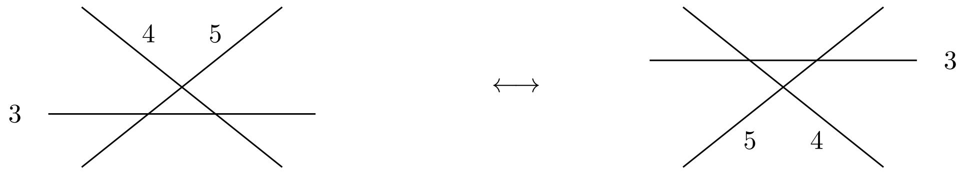


Figure 9: Switch with respect to the simplex 345

the pentagon 21345 in this order, which is obtained from 12345 by exchanging 1 and 2. By this switch the chambers change as follows:

- The chambers stable under the exchange $1 \leftrightarrow 2$ do not change; besides the simplex 345, they are the simplices $\mathbf{12}3$, $5\mathbf{12}$, and the rectangles $\mathbf{12} \times 34$, $\mathbf{12} \times 45$.
- Simplices vs rectangles:

$$\mathbf{2}34 \leftrightarrow 34 \times 5\mathbf{2}, \qquad \mathbf{51} \times 34 \leftrightarrow \mathbf{1}34,$$

$$45\mathbf{1} \leftrightarrow 45 \times \mathbf{1}3, \qquad \mathbf{2}3 \times 45 \leftrightarrow 45\mathbf{2}.$$

- A rectangle vs the pentagon:

$$\mathbf{2}3 \times 5\mathbf{1} \leftrightarrow \mathbf{21}345, \qquad \mathbf{12}345 \leftrightarrow \mathbf{1}3 \times 5\mathbf{2}.$$

### 3.8.2 3D case

- The chambers stable under the exchange $1 \leftrightarrow 2$ do not change; besides the simplex 3456, they are the simplices $\mathbf{12}34$, $6\mathbf{12}3$, the prisms

$$\mathbf{12} \times 345,\ \mathbf{12} \times 456,\ 34 \times 6\mathbf{12},\ 45 \times 6\mathbf{12},\ 45 \times \mathbf{12}3,\ 56 \times \mathbf{12}3,$$

the cube $\mathbf{12} \times 34 \times 56$, and the gyoza $3456\underline{\mathbf{12}}$.

- Simplices vs prisms:

$$\mathbf{2}345 \leftrightarrow \mathbf{2}6 \times 345, \quad \mathbf{2}3 \times 456 \leftrightarrow \mathbf{2}456,$$

$$456\mathbf{1} \leftrightarrow \mathbf{1}3 \times 456, \quad 6\mathbf{1} \times 345 \leftrightarrow \mathbf{1}345.$$

- Prisms vs gyozas:

$$\mathbf{2}3 \times 56\mathbf{1} \leftrightarrow \mathbf{21}34\underline{56}, \quad \mathbf{12}34\underline{56} \leftrightarrow \mathbf{1}3 \times \mathbf{2}56,$$

$$6\mathbf{1} \times \mathbf{2}34 \leftrightarrow 56\mathbf{21}\underline{34}, \quad 56\mathbf{12}\underline{34} \leftrightarrow \mathbf{2}6 \times \mathbf{1}34,$$

- Prisms vs prisms:

$$34 \times 56\mathbf{1} \leftrightarrow 56 \times \mathbf{1}34, \quad 56 \times \mathbf{2}34 \leftrightarrow 34 \times \mathbf{2}56.$$

- Gyozas vs gyozas:

$$\mathbf{2}345\underline{6\mathbf{1}} \leftrightarrow \mathbf{2}654\underline{3\mathbf{1}}, \quad 456\mathbf{1}\underline{\mathbf{2}3} \leftrightarrow \mathbf{1}345\underline{6\mathbf{2}}.$$

- A gyoza and a cube:

$$6\mathbf{12}3\underline{45} \leftrightarrow \mathbf{1}3 \times \mathbf{2}6 \times 45, \quad \mathbf{2}3 \times 45 \times 6\mathbf{1} \leftrightarrow 3\mathbf{12}6\underline{45}.$$

These changes are caused by the moves shown in Figure 10.

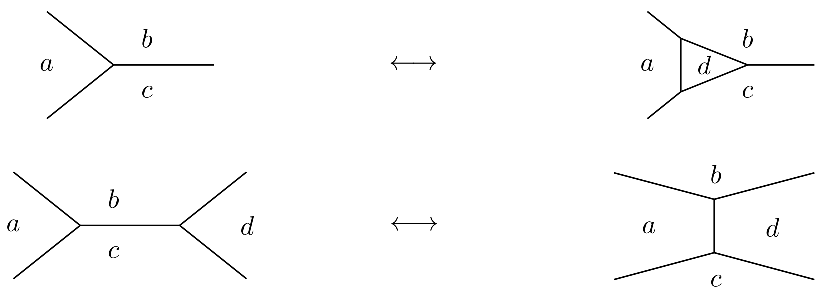


Figure 10: Switch with respect to the simplex 3456; $\{a, b, c, d\} = \{3, 4, 5, 6\}$

#### 3.8.3 $n$D case

Consider the arrangement of $n+3$ hyperplanes $1, ..., m = n+3$ in the projective space $\mathbf{P}^n$ as in [CY], where the chambers are labeled by $\pm$ sequences with length $m$. We perform the switch with respect to the simplex $3 \cdots m$.

- The chambers stable under the exchange $1 \leftrightarrow 2$ do not change.
- Any other chamber changes as follows: if $\epsilon_1 \cdots \epsilon_m$ denotes its label, then apply the exchange $1 \leftrightarrow 2$ to the chamber with label

$$-\epsilon_1 \ -\epsilon_2 \ \epsilon_3 \ \cdots \ \epsilon_m.$$

# 4 Displacement of six planes bounding a cube

The reader might think that to get six planes in general position, one has only to put a cube, bounded by six planes, in the space and move the planes slightly to make the three pairs of parallel planes non-parallel. It is not too easy to see how the chambers are deformed; in this section we describe this move.

## 4.1 Five lines on the plane

In this section we play with a baby model. We start from a square (two pairs of parallel lines: $L, R$; $U, D$) and the line $\infty$ at infinity (Step 0). We have two triple points $X$ and $Y$ at infinity. Figure 11 shows them and the intersection points.

Step 1: Move $U$ and $D$ to kill the triple point $X$, then the new triangle $ex_1x_2$ comes out (Figure 12).

Step 2: Move $L$ and $R$ to kill the triple point $Y$, then the new triangle $fy_1y_2$ comes out (Figure 13).

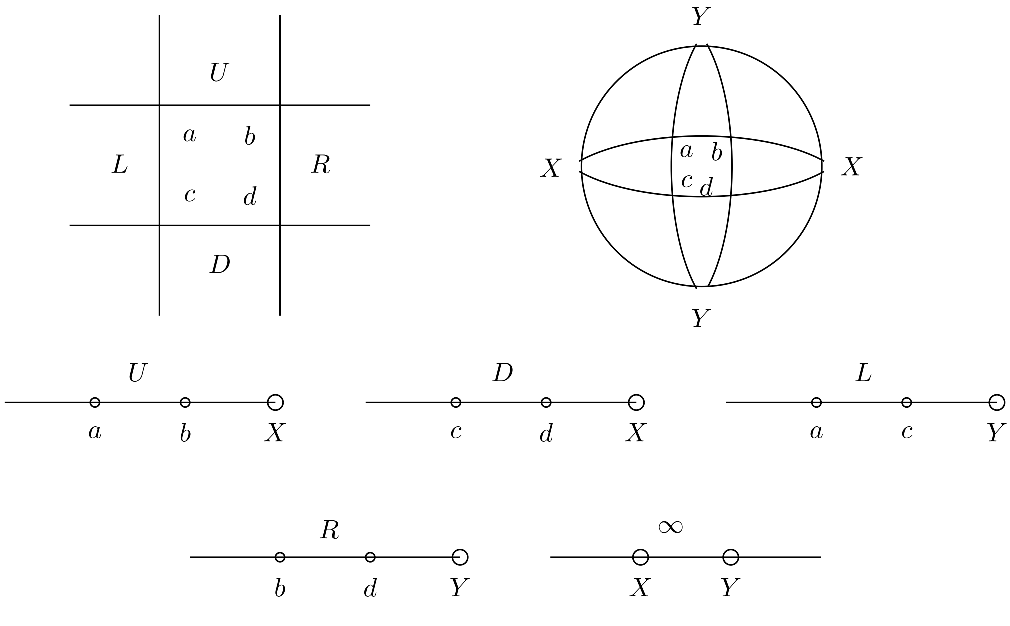


Figure 11: A square

During this process, some triangles change into squares, and a square changes into a pentagon:

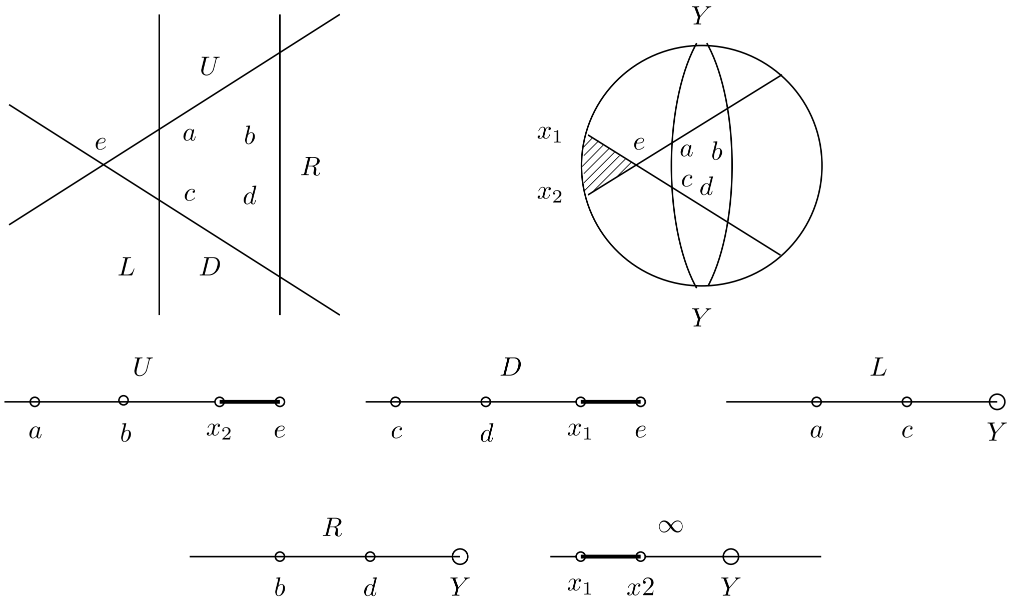


Figure 12: Move $U$ and $D$, the triangle $ex_1x_2$ comes out

$$
\begin{array}{lclcl}
\text{Step0} & & \text{Step1} & & \text{Step2} \\
 & & & & \\
S(abcd) & \to & S(abcd) & \to & S(abcd), \\
 & & & & \\
T(aXY) & \Rightarrow & S(aex_1Y) & \to & S(aex_1y_2), \\
T(bXY) & \to & T(bx_2Y) & \to & T(bx_2y_1), \\
T(cXY) & \Rightarrow & S(cex_2Y) & \Rightarrow & P(cex_2y_1f), \\
T(dXY) & \to & T(dx_1Y) & \Rightarrow & S(dx_1y_2f), \\
 & & & & \\
T(acX) & \to & T(ace) & \to & T(ace), \\
T(cdY) & \to & T(cdY) & \to & T(cdf), \\
T(dbX) & \Rightarrow & S(dbx_2x_1) & \to & S(dbx_2x_1), \\
T(baY) & \to & T(baY) & \Rightarrow & S(bay_2y_1,) \\
 & & & & \\
 & & T(ex_1x_2) & \to & T(ex_1x_2), \\
 & & & & \\
 & & & & T(fy_1y_2),
\end{array}
$$

where $T, S, P$ stand for triangle, square, and pentagon, respectively; for example, $S(abcd)$ is a square (rectangle) with vertices $a, , b, c, d$ in this order.

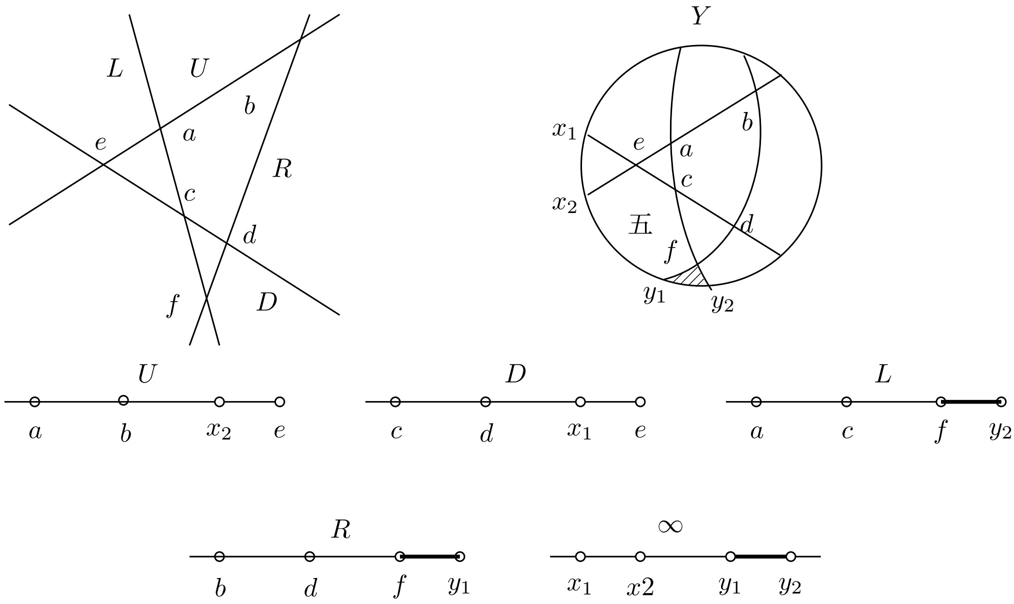


Figure 13: Move $L$ and $R$, the triangle $fy_1y_2$ comes out

## 4.2 Starting from a cube

In the space, put a cube $C$ bounded by pairwise parallel planes

$$U, \quad D; \qquad L, \quad R; \qquad F, \quad B;$$

they are initial letters of up, down, left, right, front and back, respectively. We name the eight vertices as

$$b = ULF, \quad a = ULB, \quad c = URF, \quad d = URB,$$

$$f = ULF, \quad e = ULB, \quad g = URF, \quad h = URB,$$

where $ULF = U \cap L \cap F$ (see Figure 14). The six planes divide the space into chambers; every chamber other than $C$ has a face or an edge or a vertex in common with $C$.

- a pyramid: a face in common with $C$; four faces meet at a point in infinity,
- a tetrahedron: an edge in common with $C$; two faces meet the plane at infinity along an edge,
- a tetrahedron: a vertex in common with $C$; it meets the plane at infinity along a triangle.

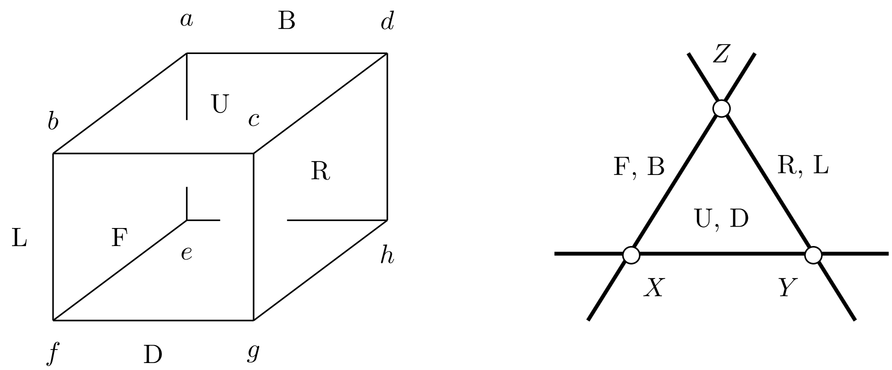


Figure 14: The cube and the plane at infinity

The sphere at infinity (the celestial sphere) is triangulated to be an octahedron which is dual to the cube $C$. Intersections of the six planes and the plane at infinity in $\mathbf{P}^3$ is shown in Figure 14. Note that the two tetrahedrons with a pair of antipodal vertices in common with $C$ are glued along the triangle at infinity, and form a *double tetrahedron*. Thus the number of chambers in $\mathbf{P}^3$ cut out by the six planes is

$$1 + 6 + 12 + 8/2 = 23.$$

Figure 15 shows the restricted arrangements on the six planes. Each plane is compactified by adding circles; identification of antipodal points gives projective planes. We displace the six planes in three steps $1, 2, 3$, killing the quadruple points $Z, X, Y$, respectively (see Figure 16). At each step, a new tetrahedron appears, and the process will end up with $13 + 3 = 26$ chambers, as expected. We study

- restricted arrangements,
- the intersection of the six planes and the plane at infinity,
- kind of chambers cut out by the six planes.

The change of the intersection of the six planes and the plane at infinity is described in Figure 16. Here the points

$$w, \quad \zeta_1, \quad \zeta_1', \quad \zeta_2, \quad \zeta_2', \quad v$$

are intersections of two planes and the plane at infinity, not of three planes, i.e. not the vertices of the arrangement.

In the course of the deformation process, we keep the two planes $U$ and $D$ being parallel, because any two planes intersect anyway, and we can assume without loss of generality that the intersection line is at infinity.

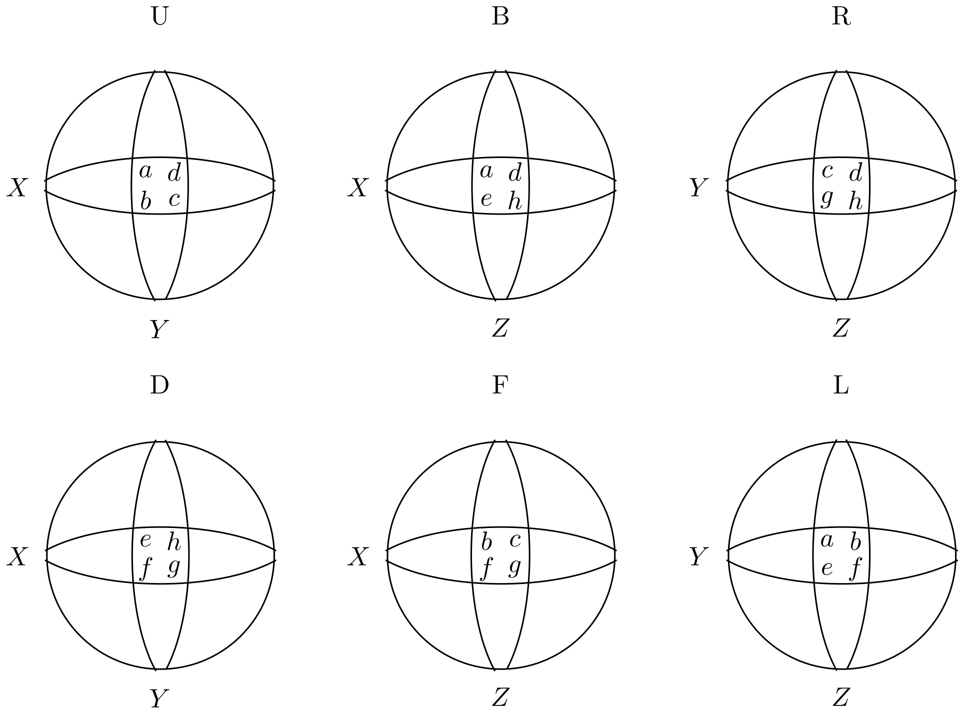


Figure 15: Restricted arrangements 0

## 4.3 Step 1

We let $B$ lean forward, and $F$ lean backward; the two planes $B$ and $F$ intersect above the cube $C$. See Figure 17. The quadruple point $Z$ is resolved, and the 23 chambers change their shape as is stated in detail later. The new tetrahedron $ijz_1z_2$ appears.

## 4.4 Step 2

We move $F$ and $B$ to the left as in Figure 18. The quadruple point $X$ is resolved, and the new tetrahedron $klx_1x_2$ appears.

## 4.5 Step 3

Finally, we move $L$ and $R$ to intersect in front of $C$ as in Figure 19. The quadruple point $Y$ is resolved, and the new tetrahedron $opy_1y_2$ appears.

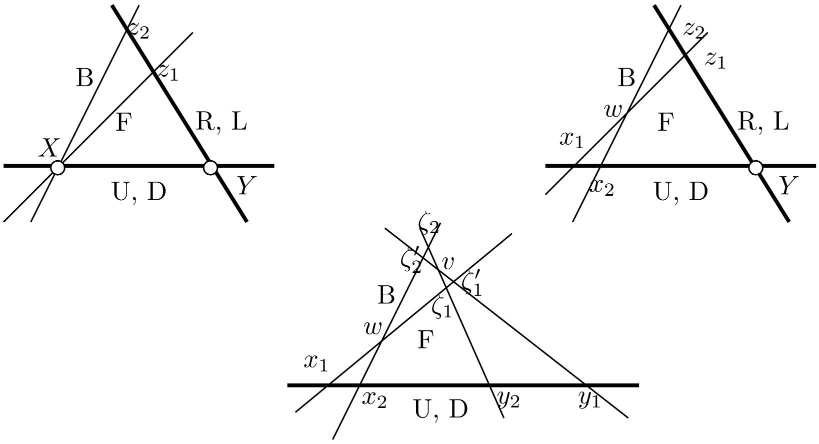


Figure 16: The six lines at infinity, in steps $1, 2, 3$

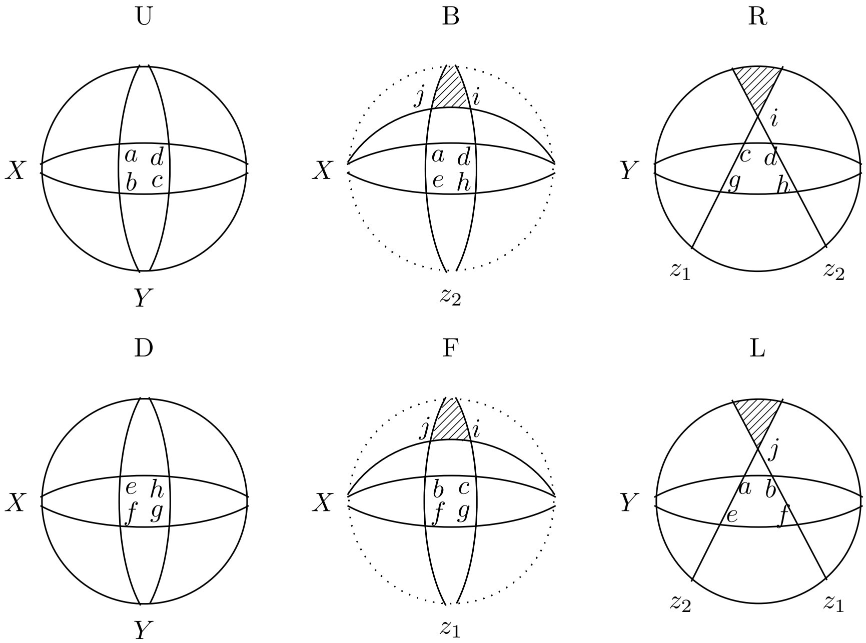


Figure 17: Restricted arrangements 1

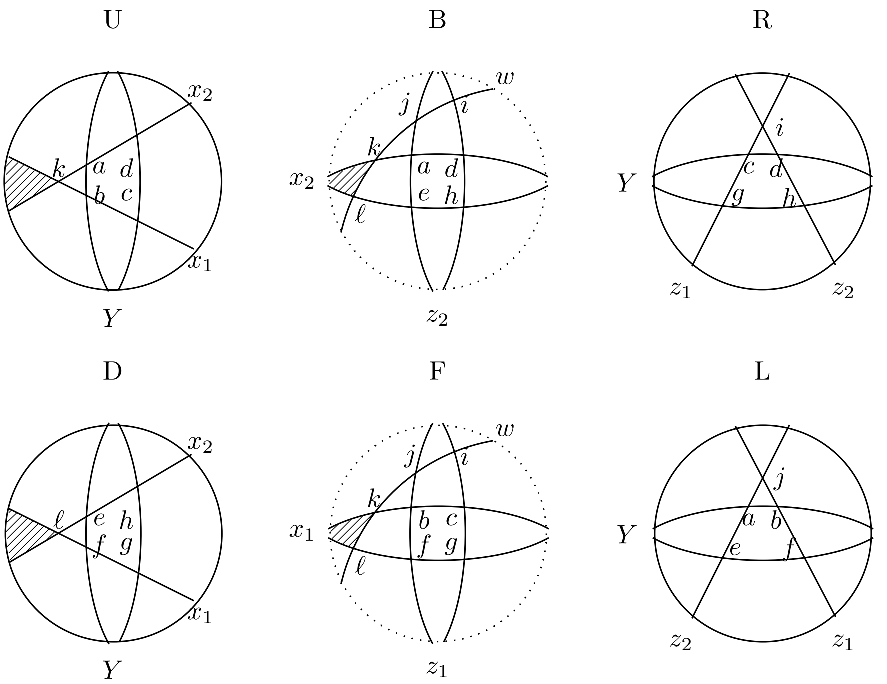


Figure 18: Restricted arrangements 2

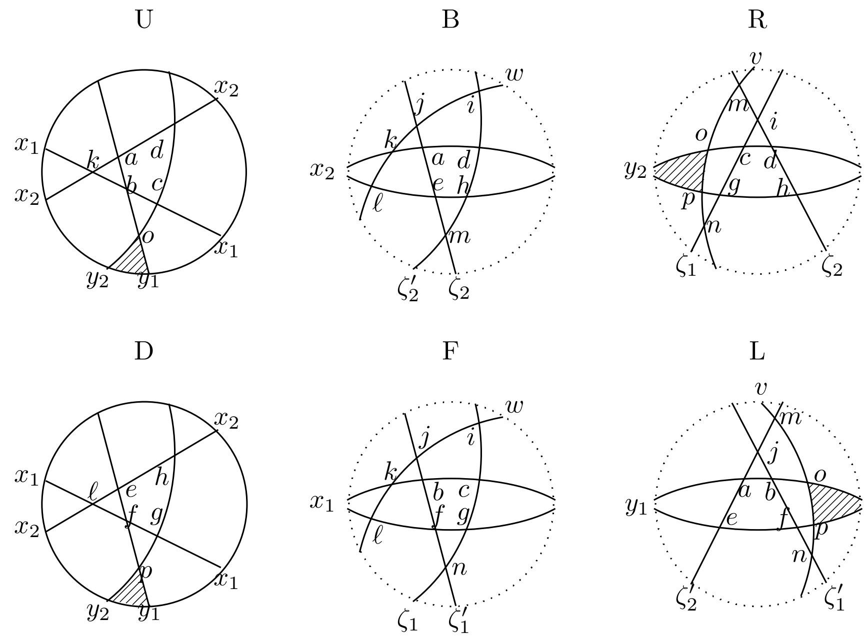


Figure 19: Restricted arrangements 3

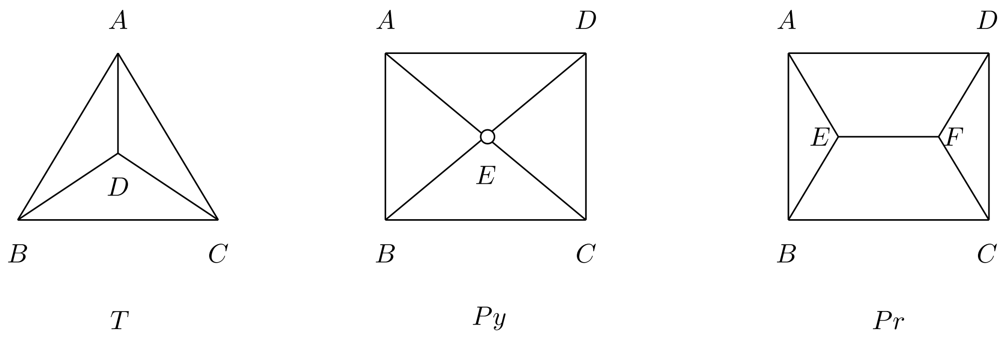


Figure 20: Polyhedra 1

## 4.6 Polyhedra and their possible deformation

During the deformation from Step 0 to Step 3, the following polyhedra appear:

T: tetrahedron, Py: Pyramid, Pr: Prism,

DT: double tetrahedron, C', N, G', C: cube, G: gyoza,

which are shown in Figures 20, 21. They will be denoted respectively by

$$T(A,B,C,D), \quad Py\begin{pmatrix} A & - & D \\ | & \mathbf{E} & | \\ B & - & C \end{pmatrix}, \quad Pr\begin{pmatrix} A & D \\ E & F \\ B & C \end{pmatrix},$$

$$DT\begin{pmatrix} & A & \\ \mathbf{B} & \mathbf{C} & \mathbf{E} \\ & D & \end{pmatrix}, \quad N\begin{pmatrix} \mathbf{A} & - & F \\ B & \backslash & E \\ C & - & \mathbf{D} \end{pmatrix},$$

$$C'\begin{pmatrix} A & B & & F \\ G & C & \mathbf{D} & E \end{pmatrix}, \quad G'\begin{pmatrix} B & A & E \\ | & \mathbf{F} & G \\ C & - & D \end{pmatrix},$$

$$C\begin{pmatrix} A & B & H & F \\ G & C & D & E \end{pmatrix}, \quad G\begin{pmatrix} B & A & E \\ F & G & H \\ C & - & D \end{pmatrix},$$

where quadruple points are printed in boldface. In the course of killing the quadruple points, the polyhedra change their shape as is shown in Figures 22 and 23.

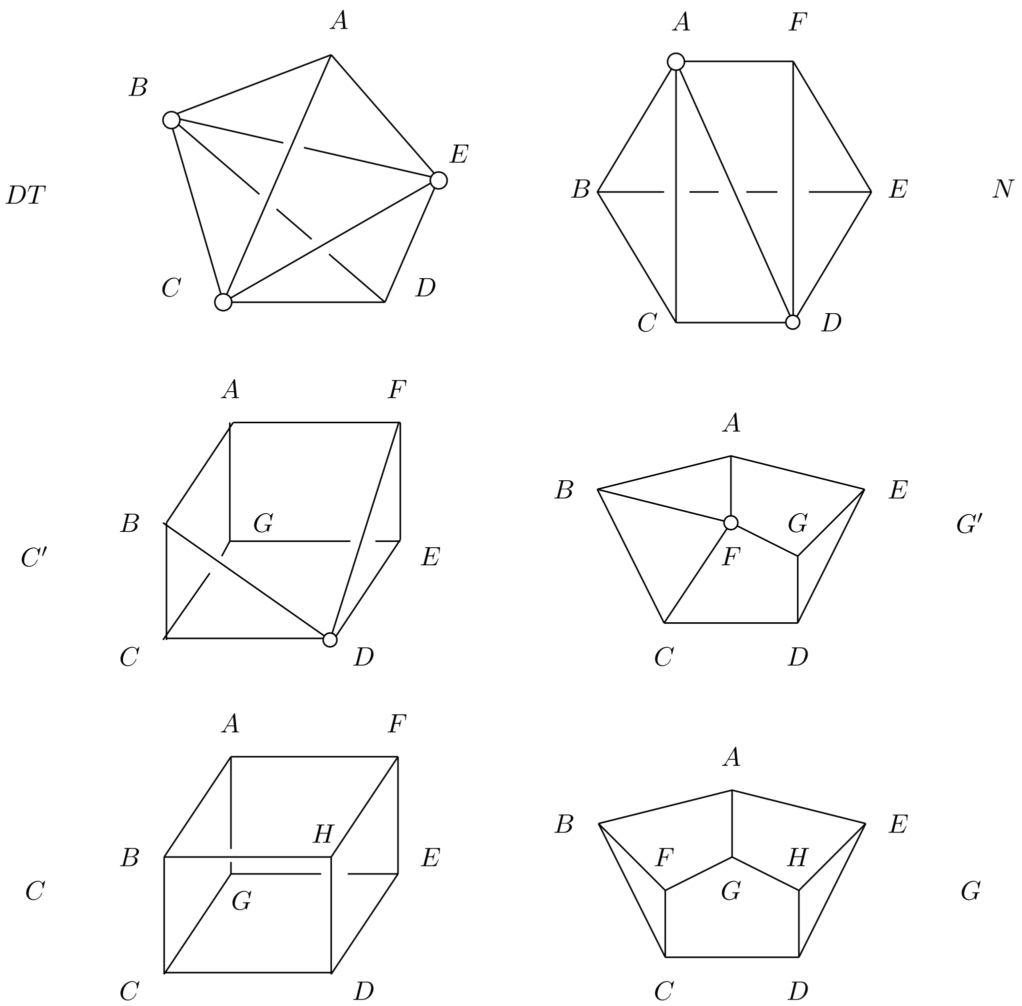


Figure 21: Polyhedra II

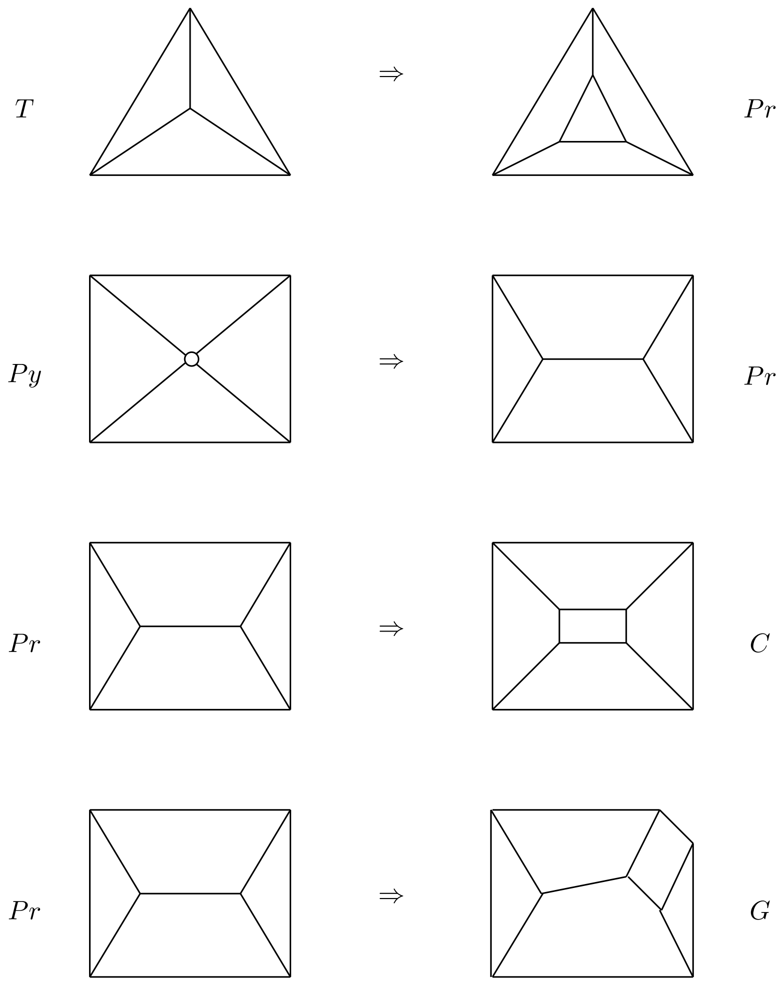


Figure 22: Change of shapes I

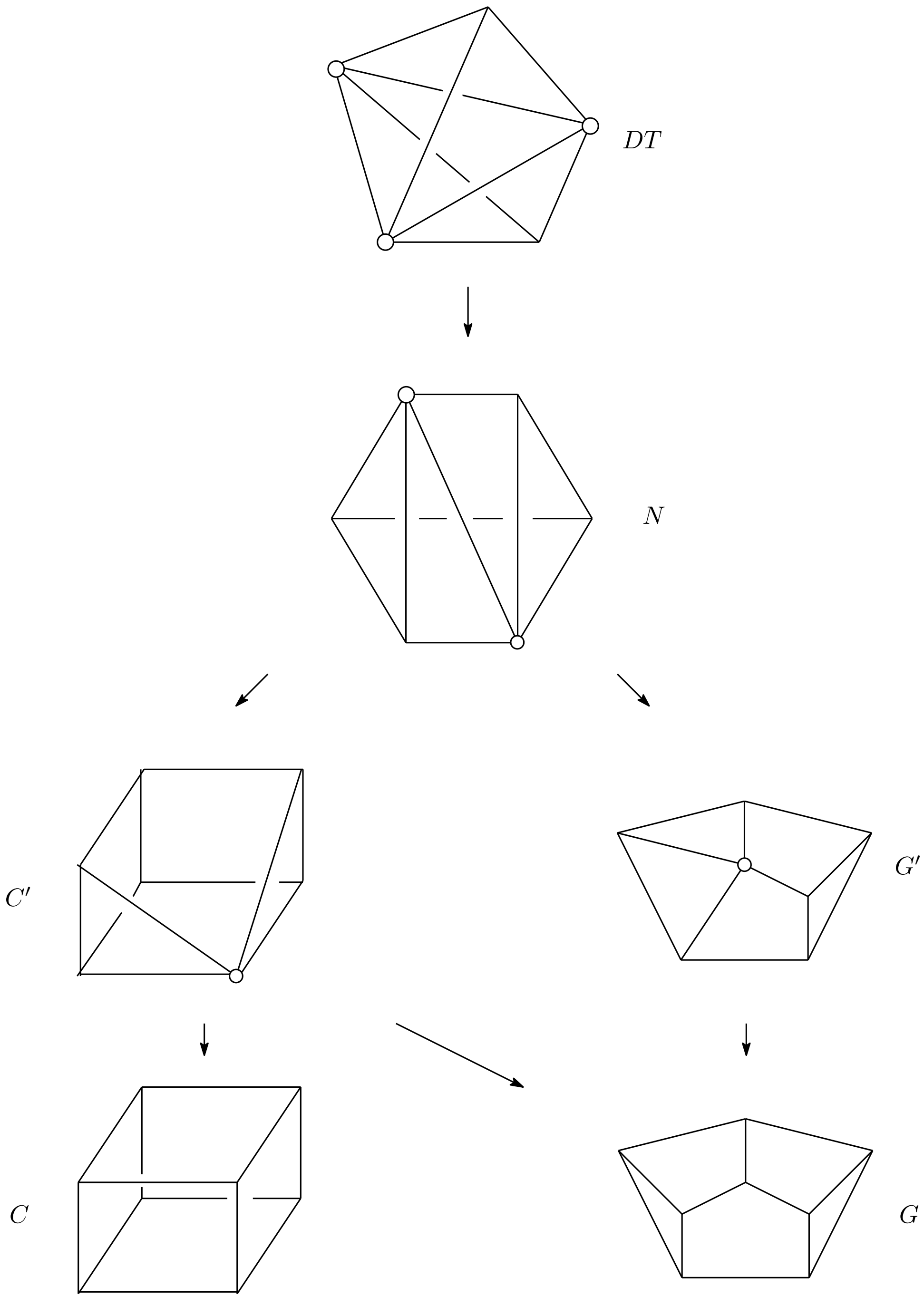


Figure 23: Change of shapes II

## 4.7 Deformation

We describe the deformation of the 22 chambers according to the change:

$$\text{Step } 0 \longrightarrow \text{Step } 1 \longrightarrow \text{Step } 2 \longrightarrow \text{Step } 3.$$

The head of each formula of deformation is the intersection of the chamber and the cube $C$.

We explain, for instance, the first one '**Face** $abcd$' and the last one '**Vertices** $d$ **and** $f$':

The pyramid with rectangular bottom $abcd$ and with vertex $Z$ changes to a prism by the first move resolving the quadruple point $Z$ at infinity. This prism does not change by the second and the third move.

The tetrahedron $T(d, X, Y, Z)$ with vertex at $d$ and with bottom triangle $XYZ$ (in the plane at infinity) is glued along this triangle with the tetrahedron $T(f, Z, Y, X)$ to form a double tetrahedron $DT$ with three quadruple points $\{X, Y, Z\}$. By the first move, the quadruple point $Z$ is resolved and it changes to a polyhedron $N$ with two quadruple points $\{X, Y\}$. By the second move, $X$ is resolved and $N$ changes to a polyhedron $C'$ with a quadruple point $Y$. By the third move $Y$ is resolved and $C'$ changes to a gyoza $G$.

**Face** $abcd$

$$Py\begin{pmatrix} a & - & d \\ | & \mathbf{Z} & | \\ b & - & c \end{pmatrix} \to Pr\begin{pmatrix} a & d \\ j & i \\ b & c \end{pmatrix} \to \text{ibid} \to \text{ibid}$$

**Face** $efgh$

$$Py\begin{pmatrix} f & - & e \\ | & \mathbf{Z} & | \\ g & - & h \end{pmatrix} \to Pr\begin{pmatrix} f & e \\ z_1 & z_2 \\ g & h \end{pmatrix} \to \text{ibid} \to Pr\begin{pmatrix} f & e \\ n & m \\ g & h \end{pmatrix}$$

**Face** $aefb$

$$Py\begin{pmatrix} e & - & a \\ | & \mathbf{X} & | \\ f & - & b \end{pmatrix} \to \text{ibid} \to Pr\begin{pmatrix} e & a \\ \ell & k \\ f & b \end{pmatrix} \to \text{ibid}$$

**Face** $cghd$

$$Py\begin{pmatrix} c & - & d \\ | & \mathbf{X} & | \\ g & - & h \end{pmatrix} \to \text{ibid} \to Pr\begin{pmatrix} c & d \\ x_1 & x_2 \\ g & h \end{pmatrix} \to \text{ibid}$$

**Face** $bfgc$

$$Py\begin{pmatrix} f & - & b \\ | & \mathbf{Y} & | \\ g & - & c \end{pmatrix} \to \text{ibid} \to \text{ibid} \to Pr\begin{pmatrix} f & b \\ p & o \\ g & c \end{pmatrix}$$

**Face** $aehd$

$$Py\begin{pmatrix} a & - & d \\ | & \mathbf{Y} & | \\ e & - & h \end{pmatrix} \to \text{ibid} \to \text{ibid} \to Pr\begin{pmatrix} a & d \\ y_1 & y_2 \\ e & h \end{pmatrix}$$

**Edge** $ab$

$$T(a,b,X,Z) \to T(a,b,X,j) \to T(a,b,k,j) \to \text{ibid}$$

**Edge** $cd$

$$T(c,X,d,Z) \to T(c,X,d,i) \to Pr\begin{pmatrix} x_1 & c \\ w & i \\ x_2 & d \end{pmatrix} \to \text{ibid}$$

**Edge** $bc$

$$T(b,c,Y,Z) \to Pr\begin{pmatrix} b & j \\ Y & z_2 \\ c & i \end{pmatrix} \to \text{ibid} \to Pr\begin{pmatrix} b & j \\ o & m \\ c & i \end{pmatrix}$$

**Edge** $ad$

$$T(a,d,Y,Z) \to Pr\begin{pmatrix} a & j \\ Y & z_1 \\ d & i \end{pmatrix} \to \text{ibid} \to G\begin{pmatrix} j & a & y_1 \\ i & d & y_2 \\ n & - & p \end{pmatrix}$$

**Edge** $ef$

$$T(e,f,X,Z) \to Pr\begin{pmatrix} e & z_2 \\ X & i \\ f & z_1 \end{pmatrix} \to \text{ibid} \to Pr\begin{pmatrix} e & m \\ \ell & i \\ f & n \end{pmatrix}$$

**Edge** $gh$

$$T(g,h,X,Z) \to Pr\begin{pmatrix} g & z_1 \\ X & j \\ h & z_2 \end{pmatrix} \to G\begin{pmatrix} x_1 & g & z_1 \\ x_2 & h & z_2 \\ k & - & j \end{pmatrix} \to G\begin{pmatrix} x_1 & g & n \\ x_2 & h & m \\ k & - & j \end{pmatrix}$$

**Edge** $fg$

$$T(f,g,Y,Z) \to T(f,g,Y,z_1) \to \text{ibid} \to T(f,g,p,n)$$

**Edge** $eh$

$$T(e,h,Y,Z) \to T(e,h,Y,z_2) \to \text{ibid} \to Pr\begin{pmatrix} e & y_1 \\ m & o \\ h & y_2 \end{pmatrix}$$

**Edge** $bf$

$$T(b,f,X,Y) \to \text{ibid} \to Pr\begin{pmatrix} b & k \\ Y & x_2 \\ f & \ell \end{pmatrix} \to G\begin{pmatrix} k & b & o \\ \ell & f & p \\ x_2 & - & y_2 \end{pmatrix}$$

**Edge** $ae$

$$T(a,e,X,Y) \to \text{ibid} \to Pr\begin{pmatrix} a & k \\ Y & x_1 \\ e & \ell \end{pmatrix} \to Pr\begin{pmatrix} a & k \\ y_1 & x_1 \\ e & \ell \end{pmatrix}$$

**Edge** $cg$

$$T(c,g,X,Y)\to \text{ibid}\to T(c,g,x_1,Y)\to Pr\begin{pmatrix} c & o\\ x_1 & y_1\\ g & p\end{pmatrix}$$

**Edge** $dh$

$$T(d,h,X,Y)\to \text{ibid}\to T(d,h,x_2,Y)\to T(d,h,x_2,y_2)$$

**Vertices** $b$ **and** $h$

$$T(b,X,Y,Z)+T(h,Z,X,Y)=DT\begin{pmatrix} & b & \\ \mathbf{X} & \mathbf{Y} & \mathbf{Z}\\ & h & \end{pmatrix}\to N\begin{pmatrix}\mathbf{X} & - & h\\ j & \backslash & z_2\\ b & - & \mathbf{Y}\end{pmatrix}$$

$$\to G'\begin{pmatrix} h & z_2 & j\\ | & \mathbf{Y} & b\\ x_2 & - & k\end{pmatrix}\to G\begin{pmatrix} h & m & j\\ y_2 & o & b\\ x_2 & - & k\end{pmatrix}$$

**Vertices** $a$ **and** $g$

$$T(a,X,Y,Z)+T(g,Z,X,Y)=DT\begin{pmatrix} & a & \\ \mathbf{X} & \mathbf{Y} & \mathbf{Z}\\ & g & \end{pmatrix}\to N\begin{pmatrix}\mathbf{X} & - & g\\ j & \backslash & z_1\\ a & - & \mathbf{Y}\end{pmatrix}$$

$$\to G'\begin{pmatrix} g & x_1 & k\\ | & \mathbf{Y} & a\\ z_1 & - & j\end{pmatrix}\to G\begin{pmatrix} g & x_1 & k\\ p & y_1 & a\\ n & - & j\end{pmatrix}$$

**Vertices** $c$ **and** $e$

$$T(c,X,Y,Z)+T(e,Z,X,Y)=DT\begin{pmatrix} & c & \\ \mathbf{X} & \mathbf{Y} & \mathbf{Z}\\ & e & \end{pmatrix}\to N\begin{pmatrix}\mathbf{X} & - & e\\ i & \backslash & z_2\\ c & - & \mathbf{Y}\end{pmatrix}$$

$$\to C'\begin{pmatrix} i & c & & z_2\\ \ell & x_1 & \mathbf{Y} & e\end{pmatrix}\to C\begin{pmatrix} c & x_1 & \ell & i\\ o & y_1 & e & m\end{pmatrix}$$

**Vertices** $d$ **and** $f$

$$T(d,X,Y,Z)+T(f,Z,Y,X)=DT\begin{pmatrix} & d & \\ \mathbf{X} & \mathbf{Y} & \mathbf{Z}\\ & f & \end{pmatrix}\to N\begin{pmatrix}\mathbf{X} & - & f\\ i & \backslash & z_1\\ d & - & \mathbf{Y}\end{pmatrix}$$

$$\to C'\begin{pmatrix} i & d & & z_1\\ \ell & x_2 & \mathbf{Y} & f\end{pmatrix}\to G\begin{pmatrix} d & i & n\\ x_2 & \ell & f\\ y_2 & - & p\end{pmatrix}$$

# 5 Six points in the space

If we pass to the dual situation, six planes (in general position) become six lines (in general position). In this section we study the planes spanned by three points out of the six; there are twenty of them.

## 5.1 Five points in the plane

Before going further, we would like to see the situation of five points on the plane $\mathbf{P}^2$. Any four points in general position are projectively moved to

$$p_0 = 1:1:1, \quad p_1 = 1:0:0, \quad p_2 = 0:1:0, \quad p_3 = 0:0:1.$$

The lines joining two of them are given by

$$x_j = 0, \quad x_i - x_j = 0, \quad 1 \le i, j \le 3, \quad i \ne j.$$

They divide the plane $\mathbf{P}^2$ into $(5-1)!/2 = 12$ triangles coded by 5-juzus[2]; permutations of five letters, say, $\{0,1,2,3,4\}$ with the identifications such as

$$01234 = 12340 = \cdots = 40123 = 43210 = \cdots.$$

Indeed if we put the fifth point $p_4$ in one of these triangles then there is a unique conic passing these five points (Theorem 2 in §2.2) according to the order of the corresponding 5-juzu. For example, the triangle $0 \le x_1 \le x_2 \le x_3 = 1$ corresponds to the juzu 01234. If we join the points $p_0$ and $p_1$, $p_1$ and $p_2$,..., $p_4$ and $p_0$, then these five lines are in general position, and cut out a unique pentagon, five triangles and five rectangles. If we join two points out of the five points (there are five lines anew) then regardless of the position of $p_4$ in the triangle, the intersection pattern of these ten lines is the same (see Figure 24). From this figure, we can see that there are four kinds of arrangements of six points (lines) in general position (see [Yo]). Note that the new five lines cut a rectangle into three pieces (a rectangle and two triangles), and the pentagon into eleven pieces (a pentagon and ten triangles), while they do not cut a triangle. Important remark: more than two lines meet only at $p_0, \cdots, p_4$.

## 5.2 Six points in the space; Results

We work on the 3-space $\mathbf{P}^3$. We are interested in the intersection pattern of the planes passing through three points out of the six points (there are fourteen planes anew). Not like the five points on the plane stated above, four such planes happen to meet besides the given six points.

**Proposition 2** *Let $p_0, \ldots, p_5$ be six points in general position in $\mathbf{P}^3$, and $H_{ijk}$ the plane passing through $p_i, p_j$ and $p_k$. Four such planes never meet at a point besides the given six points unless the four planes are $\{H_{012}, H_{234}, H_{035}, H_{145}\}$, up to re-numbering.*

Any five points in general position are projectively moved to

$$p_0 = 1:1:1:1, \quad p_1 = 1:0:0:0, \quad \ldots \quad p_4 = 0:0:0:1.$$

The planes passing through three of them are given by

$$x_j = 0, \quad x_i - x_j = 0, \quad 1 \le i, j \le 4, \quad i \ne j.$$

[2] Juzu is a Buddhism version of Christian rosary; a mathematics terminology in Japanese.

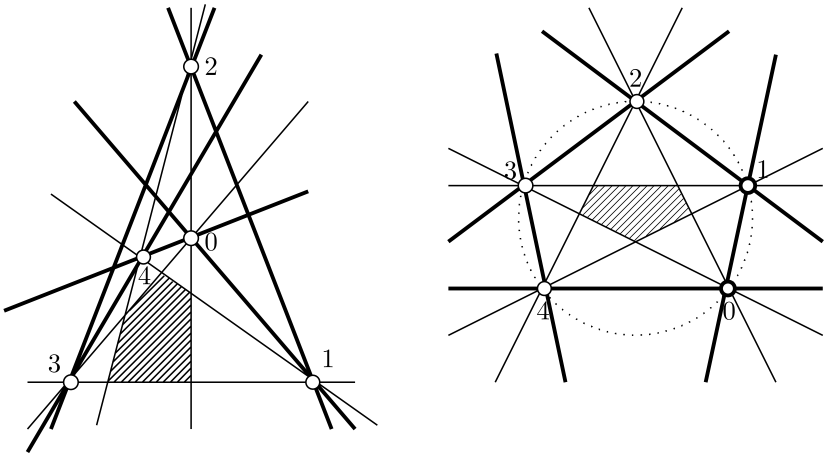


Figure 24: Right and left are combinatorially the same

They divide the space $\mathbf{P}^3$ into $(6-1)!/2 = 60$ tetrahedrons coded by 6-juzus. For example, the tetrahedron defined by

$$0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 = 1$$

corresponds to the 6-juzu 01234. If we put the sixth point $p_5$ in one of these tetrahedron then there is a unique cubic curve passing through these six points (Theorem 2 in §2.2) according to the order of the corresponding 6-juzu. Note that the six planes passing through $\{p_0, p_1, p_2\}$, $\{p_1, p_2, p_3\}$,..., $\{p_5, p_0, p_1\}$ are in general position.

**Theorem 4** *Let the five points $p_0, \dots, p_4$ be above, and $p_5 = a_1 : a_2 : a_3 : a_4$ in the simplex*

$$T : 0 < a_1 < a_2 < a_3 < a_4 = 1.$$

*Four planes in $\{H_{ijk} \mid 0 \leq i < j < k \leq 5\}$ meet at a point besides $p_0, \dots, p_5$ only in the following cases:*

1. $H_{012} \cap H_{234} \cap H_{045} \cap H_{135} : f_1 := a_2a_3 - a_1a_2 - a_2a_4 + a_1a_4 = 0,$
2. $H_{012} \cap H_{244} \cap H_{034} \cap H_{135} : f_2 := a_2a_3 - a_1a_4 = 0,$
3. $H_{014} \cap H_{025} \cap H_{135} \cap H_{234} : f_3 := a_4a_3 + a_1a_2 - a_1a_4 - a_2a_4 = 0,$
4. $H_{023} \cap H_{045} \cap H_{124} \cap H_{135} : f_4 := (a_2 - a_4)a_3 - (a_1 - a_4)a_2 = 0,$

*The four surfaces $S_j : f_j = 0$ meet at a point*

$$a_1 : a_2 : a_3 : a_4 = \frac{1}{3} : \frac{1}{2} : \frac{2}{3} : 1,$$

*and they divide the simplex $T$ into twelve chambers.*

The above point $p_5$ can be characterized as

**Proposition 3** *There is a unique cubic curve passing through $p_0, \dots, p_5$ in this order. These form a regular hexagon if and only if $p_5$ is the point stated in the previous theorem.*

We prove the above statements in the following subsections.

## 5.3 The twenty planes passing three points out of the six

Let the sixth point $p_5$ be coordinatized by $a_1 : \cdots : a_4$, and $H_{ijk}$ denote the plane passing through the points $p_i, p_j$ and $p_k$. The twenty planes $H_{ijk}$ are defined as follows

$$
\begin{array}{lcccc}
H_{012}: & 0 & 0 & 1 & -1 \\
H_{013}: & 0 & 1 & 0 & -1 \\
H_{014}: & 0 & 1 & -1 & 0 \\
H_{023}: & 1 & 0 & 0 & -1 \\
H_{024}: & 1 & 0 & -1 & 0 \\
H_{034}: & 1 & -1 & 0 & 0 \\
H_{123}: & 0 & 0 & 0 & 1 \\
H_{124}: & 0 & 0 & 1 & 0 \\
H_{134}: & 0 & 1 & 0 & 0 \\
H_{234}: & 1 & 0 & 0 & 0 \\
H_{015}: & 0 & a_3-a_4 & a_4-a_2 & a_2-a_3 \\
H_{025}: & a_3-a_4 & 0 & a_4-a_1 & a_1-a_3 \\
H_{035}: & a_2-a_4 & a_4-a_1 & 0 & a_1-a_2 \\
H_{045}: & a_3-a_2 & a_1-a_3 & a_2-a_1 & 0 \\
H_{125}: & 0 & 0 & -a_4 & a_3 \\
H_{135}: & 0 & -a_4 & 0 & a_2 \\
H_{145}: & 0 & a_3 & -a_2 & 0 \\
H_{235}: & -a_4 & 0 & 0 & a_1 \\
H_{245}: & a_3 & 0 & -a_1 & 0 \\
H_{345}: & a_2 & -a_1 & 0 & 0,
\end{array}
$$

where for example $H_{012}$ is defined by $x_3 - x_4 = 0$.

## 5.4 Four planes meeting at a point

By computing $4\times 4$-minors of the above table, we find that four planes among the twenty can meet at a point besides $p_0, \dots, p_5$, not like the ten lines in the plane shown in Figure 24, and that this can happen if the set of four planes belongs to the $S_6$-orbit (consists of thirty sets) of

$$\{H_{012}, \quad H_{234}, \quad H_{035}, \quad H_{145}\}.$$

Let $d(012, 234, 035, 145)$ denote the determinant of the equations of the above four planes modulo multiplicative constant and the factors $a_i - a_j (i \neq j)$. They are given as

$$\begin{array}{lll}
d(012,234,035,145): & (a_1-a_2)a_3-(a_4-a_1)a_2, & \\
d(012,234,045,135): & a_2a_3-a_1a_2-a_2a_4+a_1a_4 & =f_1, \\
d(012,235,035,145): & a_1a_3-a_2a_4, & \\
d(012,235,045,134): & a_1a_3-a_1a_2+a_2a_4-a_1a_4, & \\
d(012,244,034,135): & a_2a_3-a_1a_4 & =f_2, \\
d(012,235,035,134): & (a_2-a_1)a_3-a_1(a_2-a_4), & \\
d(013,024,145,235): & a_4a_3-a_1a_2, & \\
d(013,025,145,234): & (a_1+a_2-a_4)a_3-a_1a_2, & \\
d(013,045,124,235): & (a_1-a_4)a_3-a_1(a_2-a_4), & \\
d(013,045,125,234): & (-a_2+a_1+a_4)a_3-a_1a_4, & \\
d(014,025,135,234): & a_4a_3+a_1a_2-a_1a_4-a_2a_4 & =f_3, \\
d(014,035,125,234): & (a_1-a_4)a_3-(a_1-a_2)a_4, & \\
d(023,045,124,135): & (a_2-a_4)a_3-(a_1-a_4)a_2 & =f_4, \\
d(023,045,125,134): & (a_2-a_1+a_4)a_3-a_2a_4, & \\
d(024,045,125,134): & (a_4-a_2)a_3-(a_1-a_2)a_4; &
\end{array}$$

there are fifteen of them. The remaining fifteen sets are the complementary ones, which give the same equation, for example $d(345,015,124,023)$ coincides with $d(012,234,035,145)$.

## 5.5 The sixth point in a tetrahedron

Without loss of generality, we assume that the sixth point $p_5 = a_1 : a_2 : a_3 : a_4$ $(a_4 = 1)$ is in the tetrahedron $T$ defined by

$$0 < a_1 < a_2 < a_3 < a_4 = 1.$$

Then only four (out of fifteen) surfaces defined by $d(*,*,*)$ pass through $T$; they are $f_1, \dots, f_4$ shown above. The (non-singular) surfaces $S_1 : f_1 = 0, \dots, S_4 : f_4 = 0$ in $T$ meet at a point

$$a_1 : a_2 : a_3 : a_4 = \frac{1}{3} : \frac{1}{2} : \frac{2}{3} : 1,$$

and cut $T$ into twelve pieces. Indeed if we set

$$a_1 = y, \quad a_2 = x, \quad a_3 = c,$$

then the surfaces $S_1, \dots, S_4$ are given by

$$y = \frac{x(c-1)}{x-1}, \quad y = cx, \quad y = \frac{x-c}{1-c}, \quad y = x + 1 - \frac{x}{c}, \quad (0 < y < x < c < 1)$$

respectively; we cut by level planes $c$ =constant and easily draw sections of $T$(Figure 25).

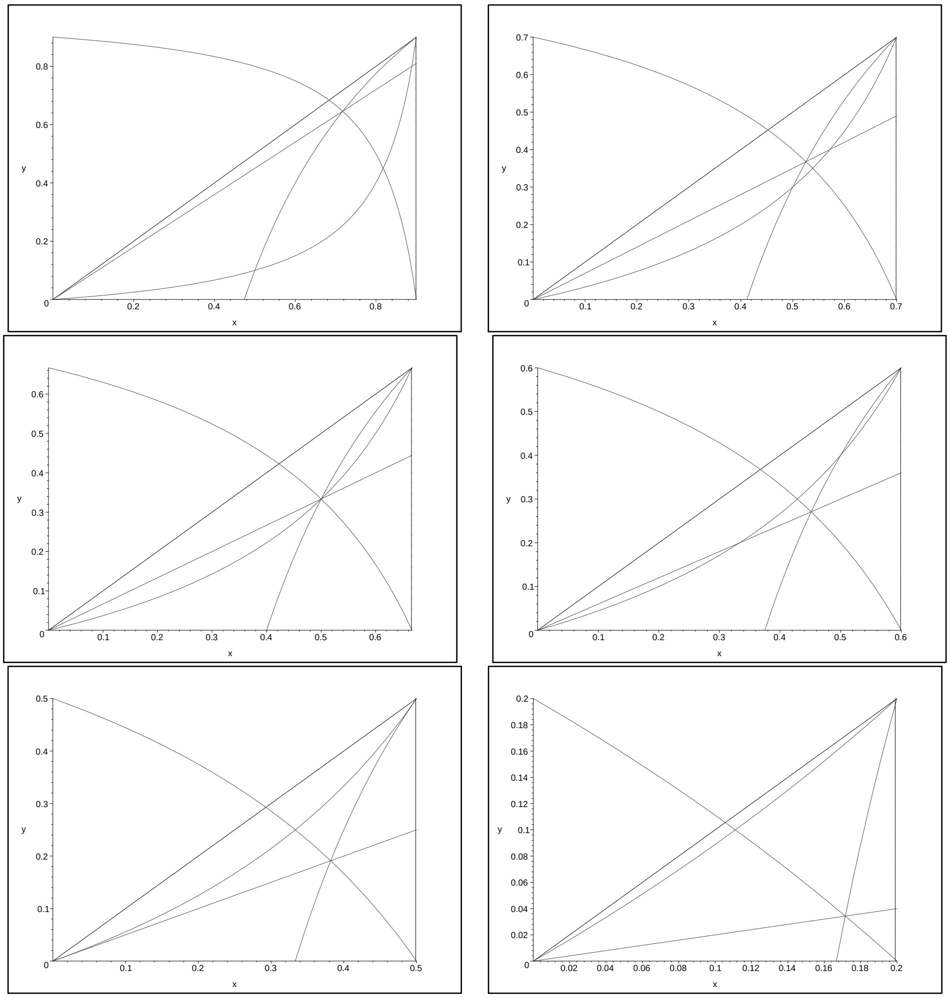

Figure 25: Sections of the simplex $T : 0 < y < x < c$, for $c = 0.8, 0.7, 2/3, 0.6, 0.5, 0.2$

## 5.6 Regular $m$-gon

Set $\zeta = \exp 2\pi i/m$. On the complex $z$-plane, we consider a regular $m$-gon with vertices

$$z = 1, \quad \zeta, \quad \cdots, \zeta^{m-1};$$

this is invariant under the rotation $r : z \mapsto \zeta z$. The transformation

$$x = \frac{z-1}{z-\zeta^{m-1}} \cdot \frac{\zeta - \zeta^{m-1}}{\zeta - 1}$$

takes the unit circle in the $z$-plane into the real $x$-line, and send the vertices as

$$x(1) = 0, \quad x(\zeta) = 1, \quad \dots, \quad x(\zeta^{m-2}) = |1+\zeta|^2, \quad x(\zeta^{m-1}) = \infty.$$

Since the inverse map is given by

$$z = \frac{(1-\zeta^{m-1})x - \zeta + \zeta^{m-1}}{(\zeta - 1)x - \zeta + \zeta^{m-1}},$$

the transformation $r$ in $x$-variable is given by

$$R(x) = \frac{|\zeta+1|^2}{|\zeta+1|^2 - x}.$$

By using this transformation, we can easily show that when $m = 6$, the image of the vertices are

$$x = \quad 0, \quad 1, \quad \frac{3}{2}, \quad 2, \quad 3, \quad \infty.$$

## 5.7 The cubic curve passing through the six points

In this section we find the cubic curve

$$C : t \mapsto C(t) = x_1(t) : x_2(t) : x_3(t) : x_4(t) \in \mathbf{P}^3$$

passing through the six points $p_0, \dots, p_4$ and

$$p_5 = a_1 : \cdots : a_4 = \frac{1}{3} : \frac{1}{2} : \frac{2}{3} : 1,$$

the intersection points of the four surfaces $S_1, \dots, S_4$. Since $0 < a_1 < \cdots < a_4$, the six points will be arranged on $C$ in this order. Let us normalize as

$$C(\infty) = p_0, \quad C(0) = p_1, \quad C(1) = p_2$$

and put

$$C(p) = p_3, \quad C(q) = p_4, \quad C(r) = p_5.$$

Our task is to solve the system:

$$\begin{array}{lcl} (x_1(r) =) & c(r-1)(r-p)(r-q) & = a_1, \\ (x_2(r) =) & cr(r-p)(r-q) & = a_2, \\ (x_3(r) =) & cr(r-1)(r-q) & = a_3, \\ (x_4(r) =) & cr(r-1)(r-p) & = a_4, \end{array}$$

with unknown $p, q, r, c$ $(1 < p < q < r)$. This can be solved as

$$p = \frac{3}{2}, \quad q = 2, \quad r = 3, \qquad c = 9.$$

This shows that the surfaces $S_1, \dots, S_4$ meet at a point if and only if the six points $p_0, \dots, p_5$ form a regular hexagon on the cubic curve; this sounds quite natural.

## 5.8 The twenty planes when the six points form a regular hexagon

When the six points $p_0, \ldots, p_5$ form a regular hexagon in this order as above, we show in Figure 26 the plane $H_{012}$ with the intersection lines with the other nineteen planes.

- five thick lines are intersections with the five planes $H_{123}, H_{234}, H_{345}, H_{450}, H_{501}$.,
- three bullets stand for $p_0, p_1$ and $p_2$,
- the line joining $p_0$ and $p_1$ is the intersection of three planes; same for $\{p_1, p_2\}$ and for $\{p_2, p_0\}$,
- if we move $p_5$ slightly out of the surfaces $S_1, \ldots, S_4$, then all the multiple points other than $p_0, p_1$ and $p_2$ disappear.

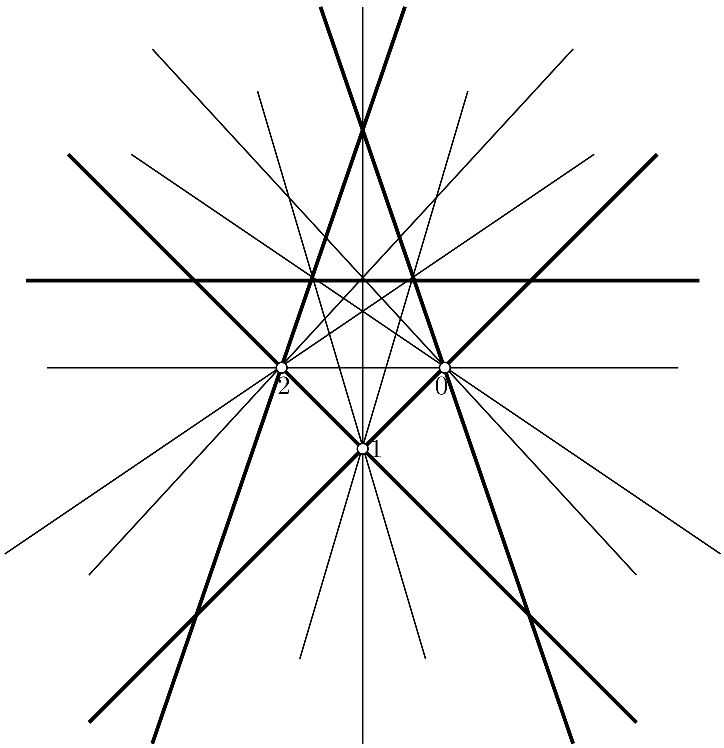


Figure 26: Intersection with other 19 planes

# References

[CY] K. Cho and M. Yoshida, Veronese arrangements of hyperplanes in real projective spaces, to appear in Internat J of Math.

[Ya] K. Yada, Arrangement of six planes in the space, Master thesis Kyushu University, 2011.

[Yo] M. Yoshida, *Hypergeometric Functions, My Love: Modular Interpretations of Configurations Spaces,* Vieweg Verlag, Wiesbaden, 1997.

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