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On the radical of linear forms over the ring of arithmetical functions

Alexandru Zaharescu and Mohammad Zaki

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Abstract. In this paper we study the radical of a linear expression of the form $c_1x_1 + c_2x_2 + \ldots + c_mx_m$, with $c_1, \ldots, c_m, x_1, \ldots, x_m$ relatively prime elements in the ring $A_r(K)$ of arithmetical functions in r variables over a field K of characteristic zero.

1 Introduction

Given an integer $r \geq 1$ and a field K of characteristic zero, the set of arithmetical functions in r variables over K is given by $A_r = A_r(K) = \{f : \mathbb{N}^r \to K\}$, with multiplication defined by the convolution

$$(f * g)(n_1, ..., n_r) = \sum_{d_1|n_1} ... \sum_{d_r|n_r} f(d_1, ..., d_r)g(\frac{n_1}{d_1}, ..., \frac{n_r}{d_r}), \tag{1}$$

for any $f, g \in A_r$. Here K has a natural embedding in A_r, and A_r with addition and convolution defined as above becomes a K-algebra. For some work on rings of arithmetical functions the reader is referred to [11], [4], [5], [6], [10], [7], [8], [2], [9], [1], [12], and [3]. In particular, by generalizing a theorem of Cashwell and Everett in [12] it is shown that the ring A_r is factorial. This opens up the possibility of studying various Diophantine equations over the ring A_r. With this in mind, an anlog of the well-known ABC conjecture of Masser and Oesterlé was investigated in [13]. In order to state the main result of [13], let us first recall the construction from [1] of a class of absolute values on A_r, which generalize the one discovered by Schwab and Silberberg [9]. Let $\underline{t} = (t_1, \ldots, t_r) \in \mathbb{R}^r$ with $t_1, \ldots, t_r$ linearly independent over $\mathbb{Q}$, and $t_i > 0, (i = 1, 2, \ldots, r)$. For each $n \in \mathbb{N}$ denote by $\Omega(n)$ the total number of prime factors of n counting multiplicities, and define $\Omega_r : \mathbb{N}^r \to \mathbb{N}^r$ by

$$\Omega_r(n_1, \ldots, n_r) = (\Omega(n_1), \ldots, \Omega(n_r)).$$

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For any $f \in A_r$, f not identically zero, let $supp(f) = \{\underline{n} \in \mathbb{N}^r | f(\underline{n}) \neq 0\}$, and define

$$V_{\underline{t}}(f) = \min_{\underline{n} \in supp(f)} \underline{t} \cdot \Omega_r(\underline{n}).$$

We also put $V_{\underline{t}}(0) = \infty$. It is shown in [1] that for any $f, g \in A_r$,

$$V_{\underline{t}}(f+g) \geq \min(\{V_{\underline{t}}(f), V_{\underline{t}}(g)\}),$$

and

$$V_{\underline{t}}(f * g) = V_{\underline{t}}(f) + V_{\underline{t}}(g).$$

Next, using the valuation $V_{\underline{t}}$ one defines a nonarchimedean absolute value $|.|_{\underline{t}}$ by

$$|x|_{\underline{t}} = \rho^{V_{\underline{t}}(x)} \ \text{ if } x \neq 0, \text{ and } |x|_{\underline{t}} = 0 \ \text{ if } x = 0,$$

where ρ is a fixed real number in $(0,1)$.

Since A_r is a unique factorization domain (see [12]), every $f \in A_r$, can be written as $f = up_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where $p_1, \ldots, p_m$ are irreducible elements of A_r, u is a unit in A_r, and the factorization is unique up to the order of factors and multiplication of $p_1, \ldots, p_m$ by units in A_r. The radical of f is defined by $\mathrm{rad}(f) = p_1 p_2 \cdots p_m$, which is well-defined up to multiplication by a unit. Moreover, for any $\underline{t}$ as above, the absolute value $|\mathrm{rad}(f)|_{\underline{t}}$ is well-defined, since by the construction of $|.|_{\underline{t}}$ the absolute value of any unit of A_r equals 1.

In [13] it is proved that for any nonzero relatively prime elements f, g, and h of A_r satisfying $|f|_{\underline{t}} < |g|_{\underline{t}}$ and $f + g = h$, and any $\underline{t}$ as above,

$$|\mathrm{rad}(fgh)|_{\underline{t}} \leq \max\{|f|_{\underline{t}}, |g|_{\underline{t}}, |h|_{\underline{t}}\}. \tag{2}$$

In the present paper we employ Wronskians with entries in A_r, defined with respect to certain derivations which were introduced in [1], in combination with the method from [13], to obtain the following generalization of the above result.

Theorem 1 *Let r be a positive integer and K a field of characteristic zero. Let $\underline{t} = (t_1, \ldots, t_r) \in \mathbb{R}^r$ with $t_1, \ldots, t_r$ linearly independent over $\mathbb{Q}$, and $t_i > 0$, $i = 1, \ldots, r$. Let $c_1, \ldots, c_m$ be relatively prime non-zero elements of $A_r(K)$ and consider the linear form $L(X_1, \ldots, X_m) := c_1 X_1 + \cdots + c_m X_m$. Let $x_1, \ldots, x_m$ be relatively prime elements of $A_r(K)$ that are also relatively prime to the product $c_1 \cdots c_m$, and assume that the values $|c_1 x_1|_{\underline{t}}, \ldots, |c_m x_m|_{\underline{t}}$ are pairwise distinct. Then one has*

$$|rad(L(x_1, \ldots, x_m))|_{\underline{t}}^{m-1} \leq \frac{\max\{|c_1 x_1|_{\underline{t}}, \ldots, |c_m x_m|_{\underline{t}}\}}{\left| rad\left(\prod_{j=1}^{m} c_j\right) rad\left(\prod_{j=1}^{m} x_j\right)\right|_{\underline{t}}^{m-1}}. \tag{3}$$

2 Preliminaries

An arithmetical function $f \in A_r$ is called completely additive provided

$$f(n_1m_1, ..., n_rm_r) = f(n_1, ..., n_r) + f(m_1, ..., m_r),$$

for any $n_1, ..., n_r, m_1, ..., m_r \in \mathbb{N}$.

Let $\psi \in A_r$ be a completely additive function, and define the map $D_\psi : A_r \to A_r$ by

$$D_\psi(f)(n_1, \ldots, n_r) = f(n_1, \ldots, n_r)\psi(n_1, \ldots, n_r),$$

for all $n_1, \ldots, n_r \in \mathbb{N}$. Then we have that (see [1]) for all $f, g \in A_r$ and $c \in K$,
(a) $D_\psi(f + g) = D_\psi(f) + D_\psi(g)$,
(b) $D_\psi(fg) = fD_\psi(g) + gD_\psi(f)$,
(c) $D_\psi(cf) = cD_\psi(f)$.
Thus, D_ψ is a derivation on A_r over K.

Let $\psi \in A_r$ be a completely additive function. Let f be a nonzero element of A_r, and $f = up_1^{e_1} \cdots p_m^{e_m}$ be its prime factorization. Then,

$$\begin{aligned} D_\psi(f) = {} & ue_1p_1^{e_1-1}D_\psi(p_1)p_2^{e_2} \cdots p_m^{e_m} + up_1^{e_1}e_2p_2^{e_2-1}D_\psi(p_2)p_3^{e_3} \cdots p_m^{e_m} \\ & + \ldots + up_1^{e_1} \cdots p_{m-1}^{e_{m-1}}e_mp_m^{e_m-1}D_\psi(p_m) + D_\psi(u)p_1^{e_1} \cdots p_m^{e_m}, \end{aligned}$$

where

$$D_\psi(u)(1, \ldots, 1) = D_\psi(p_1)(1, \ldots, 1) = \cdots = D_\psi(p_m)(1, \ldots, 1) = 0.$$

So we may write $D_\psi(f)$ as $D_\psi(f) = p_1^{e_1-1}p_2^{e_2-1} \cdots p_m^{e_m-1}f_\psi$, for some $f_\psi \in A_r$ with $f_\psi(1, \ldots, 1) = 0$, and consequently $|f_\psi|_{\underline{t}} < 1$. Since $\frac{f}{\mathrm{rad}(f)} = p_1^{e_1-1}p_2^{e_2-1} \cdots p_m^{e_m-1}$, it is a divisor of $D_\psi(f)$, and so we have $|D_\psi(f)|_{\underline{t}}|\mathrm{rad}(f)|_{\underline{t}} < |f|_{\underline{t}}$. Also, since the greatest common divisor $(f, D_\psi(f))$ is a multiple of $\frac{f}{\mathrm{rad}(f)}$,

$$|(f, D_\psi(f))|_{\underline{t}} \leq \left| \frac{f}{\mathrm{rad}(f)} \right|_{\underline{t}}.$$

We will assume that ψ has the property that $\psi(n_1, \ldots, n_r) \neq 0$ for all r-tuples $(n_1, \ldots n_r) \neq (1, \ldots, 1)$. Over a field K of characteristic zero one can easily construct such functions ψ. For example, define $\psi(n_1, \ldots, n_r) = \Omega(n_1) + \cdots + \Omega(n_r)$ for all positive integers $n_1, \ldots, n_r$, and then use the canonical embedding of $\mathbb{Z}$ in K in order to send the values of ψ in K. If ψ has the above property, then any element f of A_r satisfying $|f|_{\underline{t}} < 1$, and $D_\psi(f)$ will have the same support. Therefore we have $|D_\psi(f)|_{\underline{t}} = |f|_{\underline{t}}$.

3 Proof of Theorem 1

Let $c_1, \ldots, c_m, x_1, \ldots, x_m \in A_r$ satisfy the hypothesis of Theorem 1. Let $\psi \in A_r$ be a completely additive function such that $\psi(n_1, \ldots, n_r) \neq 0$ for all r-tuples $(n_1, \ldots n_r) \neq (1, \ldots, 1)$. Since $L(x_1, \ldots, x_m) = c_1x_1 + \ldots + c_mx_m$ and the absolute values $|c_1x_1|_{\underline{t}}, \ldots, |c_mx_m|_{\underline{t}}$, are assumed to be distinct, it follows that $|L(x_1,\ldots,x_m)|_{\underline{t}} = \max\{|c_1x_1|_{\underline{t}},\ldots,|c_mx_m|_{\underline{t}}\}$. For $f \in A_r(K)$, define $D^0_\psi f = f$, and inductively $D^n_\psi f = D_\psi(D^{n-1}_\psi f)$ for any positive integer n. To simplify our notation, in what follows we denote $L(x_1,\ldots,x_m)$ by L. By applying the operator D_ψ repeatedly to the equality $c_1x_1 + \ldots + c_mx_m = L$, we know that

$$\begin{aligned}
&D_\psi c_1x_1 + \ldots + D_\psi c_mx_m = D_\psi L,\\
&D^2_\psi c_1x_1 + \ldots + D^2_\psi c_mx_m = D^2_\psi L,\\
&\ldots\\
&D^{m-1}_\psi c_1x_1 + \ldots + D^{m-1}_\psi c_mx_m = D^{m-1}_\psi L.
\end{aligned}$$

Next, we consider the Wronskian

$$W_\psi(c_1x_1,\ldots,c_mx_m) = \begin{vmatrix} c_1x_1 & c_2x_2 & \ldots & c_mx_m \\ D_\psi c_1x_1 & D_\psi c_2x_2 & \ldots & D_\psi c_mx_m \\ \vdots & \vdots & \vdots & \vdots \\ D^{m-1}_\psi c_1x_1 & D^{m-1}_\psi c_2x_2 & \ldots & D^{m-1}_\psi c_mx_m \end{vmatrix}$$

From the above equations, it follows that

$$W_\psi(c_1x_1,\ldots,c_mx_m) = W_\psi(c_1x_1,\ldots,c_{m-1}x_{m-1},L).$$

Our next goal is to show that

$$|W_\psi(c_1x_1,\ldots,c_mx_m)|_{\underline{t}} = \left(\prod_{j=1}^m |c_j|_{\underline{t}}\right)\left(\prod_{j=1}^m |x_j|_{\underline{t}}\right). \tag{4}$$

To proceed, let us denote $y_1 = c_1x_1, \ldots, y_m = c_mx_m$. We also denote by $\mathbb{B}_H$ the set $\Omega_r[supp(H)]$ for any $H \in A_r$. Let $\underline{n_1} = (n_{11},\ldots,n_{1r}) \in supp(y_1)$, $\underline{n_2} = (n_{21},\ldots,n_{2r}) \in supp(y_2), \ldots,$ $\underline{n_m} = (n_{m1},\ldots,n_{mr}) \in supp(y_m)$. Suppose that $\underline{l_1} = (l_{11},\ldots,l_{1r}) \in \mathbb{B}_{y_1}$, $\underline{l_2} = (l_{21},\ldots,l_{2r}) \in \mathbb{B}_{y_2},\ldots,$ $\underline{l_m} = (l_{m1},\ldots,l_{mr}) \in \mathbb{B}_{y_m}$ satisfy the equations $\Omega_r(\underline{n_1}) = \underline{l_1}, \ldots, \Omega_r(\underline{n_m}) = \underline{l_m}$ respectively. Also assume that $\underline{n_1},\ldots,\underline{n_m}$ are chosen such that

$$\begin{aligned}
&V_{\underline{t}}(y_1) = t_1l_{11} + \ldots + t_rl_{1r} = t_1\Omega(n_{11}) + \ldots + t_r\Omega(n_{1r}),\\
&\vdots\\
&V_{\underline{t}}(y_m) = t_1l_{m1} + \ldots + t_rl_{mr} = t_1\Omega(n_{m1}) + \ldots + t_r\Omega(n_{mr}).
\end{aligned}$$

Let us define for each $1 \le i \le m$, the set

$$\mathfrak{C}_{y_i} = \{\underline{a} \in \mathbb{N}^r : y_i(\underline{a}) \neq 0 \ \ and \ \ \Omega_r(\underline{a}) = \underline{l_i}\}$$

Also, to make a choice, let us assume that for each $1 \le i \le m$, $\underline{n_i}$ was chosen so that it is the smallest element of $\mathfrak{C}_{y_i}$ with respect to the lexicographical ordering. We have that

$$V_{\underline{t}}(y_1) + \ldots + V_{\underline{t}}(y_m) = \underline{t}\cdot\underline{l_1} + \ldots + \underline{t}\cdot\underline{l_m} = \underline{t}\cdot\Omega_r(\underline{u}),$$

where $\underline{u} = (n_{11} \cdots n_{m1}, \ldots, n_{1r} \cdots n_{mr})$

By the hypothesis from the statement of the theorem and the definition of $y_1, \ldots, y_m$ we know that $|y_1|_{\underline{t}}, \ldots, |y_m|_{\underline{t}}$ are pairwise distinct. Therefore the r-tuples $\underline{n_1}, \ldots, \underline{n_m}$ are distinct. Let $\underline{d_1} = (d_{11}, \ldots, d_{1r})$, $\underline{d_2} = (d_{21}, \ldots, d_{2r}), \ldots,$ $\underline{d_m} = (d_{m1}, \ldots, d_{mr})$ be tuples in $\mathbb{N}^r$, and define

$$D(\underline{d_1}, \ldots, \underline{d_m}) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \psi(\underline{d_1}) & \psi(\underline{d_2}) & \ldots & \psi(\underline{d_m}) \\ \vdots & \vdots & \vdots & \vdots \\ \psi^{m-1}(\underline{d_1}) & \psi^{m-1}(\underline{d_2}) & \ldots & \psi^{m-1}(\underline{d_m}) \end{vmatrix}.$$

Here $D(\underline{d_1}, \ldots, \underline{d_m})$ is a Vandermonde determinant, and so

$$D(\underline{d_1}, \ldots, \underline{d_m}) = \prod_{1 \le i < j \le m} \Big(\psi(\underline{d_j}) - \psi(\underline{d_i}) \Big).$$

To simplify our notation, we will write $f^{(n)} = D_\psi^{(n)}(f)$ for any f in A_r and any positive integer n. Consider $w = y_1 y_2^{(1)} y_3^{(2)} \cdots y_m^{(m-1)}$. We have

$$w(\underline{u}) = (y_1 y_2^{(1)} y_3^{(2)} \cdots y_m^{(m-1)})(\underline{u}) = \sum_{d_{11} \cdots d_{m1} = n_{11} \cdots n_{m1}} \cdots \sum_{d_{1r} \cdots d_{mr} = n_{1r} \cdots n_{mr}}$$

$$y_1(d_{11}, \ldots, d_{1r}) \cdots y_m(d_{m1}, \ldots, d_{mr}) \psi(d_{21}, \ldots, d_{2r}) \cdots \psi^{(m-1)}(d_{m1}, \ldots, d_{mr}).$$

We now expand the determinant $W_\psi(y_1, \ldots, y_m)(\underline{u})$ as a sum of terms of the form $\pm w(\underline{u})$ with w similar to the one above, and apply to each of them the above relation. It follows that

$$W_\psi(y_1, \ldots, y_m)(\underline{u}) = \sum_{d_{11} \cdots d_{m1} = n_{11} \cdots n_{m1}} \cdots \sum_{d_{1r} \cdots d_{mr} = n_{1r} \cdots n_{mr}}$$

$$y_1(d_{11}, \ldots, d_{1r}) \cdots y_m(d_{m1}, \ldots, d_{mr}) D(\underline{d_1}, \ldots, \underline{d_m}).$$

We claim that here each term $y_1(d_{11}, \ldots, d_{1r}) \cdots y_m(d_{m1}, \ldots, d_{mr}) D(\underline{d_1}, \ldots, \underline{d_m})$ is zero with the possible exception of the term $(\underline{d_1}, \ldots, \underline{d_m}) = (\underline{n_1}, \ldots, \underline{n_m})$. Indeed, if $\underline{d_1}$ is such that the dot product $\underline{t} \cdot \Omega_r(\underline{d_1})$ is strictly smaller than $\underline{t} \cdot \Omega_r(\underline{n_1})$, then by the definition of $\underline{n_1}$ we must have $y_1(\underline{d_1}) = 0$. Similarly for the other $\underline{d_j}$'s. Also, if $\underline{t} \cdot \Omega_r(\underline{d_i}) > \underline{t} \cdot \Omega_r(\underline{n_i})$ for some i, then there will be a j for which $\underline{t} \cdot \Omega_r(\underline{d_j}) < \underline{t} \cdot \Omega_r(\underline{n_j})$, and then we will have $y_j(\underline{d_j}) = 0$ for that j. Thus the only terms that may survive in the sum are those for which we simultaneously have $\underline{t} \cdot \Omega_r(\underline{d_i}) = \underline{t} \cdot \Omega_r(\underline{n_i})$ for each i. This means $\Omega_r(\underline{d_i}) = \Omega_r(\underline{n_i})$ for each i, since the components of $\underline{t}$ are linearly independent over the rationals. Next, if there is an i for which $\underline{d_i}$ is strictly larger than $\underline{n_i}$ in the lexicographical order, then there will be a j for which $\underline{d_j}$ is strictly smaller than $\underline{n_j}$ in the lexicographical order, and this contradicts our choice of $\underline{n_j}$. Also, no $\underline{d_i}$ can be strictly smaller than $\underline{n_i}$ in the lexicographical order, by our choice of $\underline{n_i}$. This forces each $\underline{d_i}$ to coincide with $\underline{n_i}$, proving our claim. In conclusion

$$W_\psi(y_1,\ldots,y_m)(\underline{u}) = y_1(\underline{n_1})\cdots y_m(\underline{n_m})D(\underline{n_1},\ldots,\underline{n_m}). \tag{5}$$

The above considerations hold for any completely additive function ψ. In what follows we need ψ to satisfy the additional property that $W_\psi(y_1,\ldots,y_m)$ does not vanish at the point $\underline{u}$ above. In view of (5), and taking into account that $y_1(\underline{n_1}),\ldots,y_m(\underline{n_m})$ are non-zero by our choice of $\underline{n_1},\ldots,\underline{n_m}$, the above condition on ψ reduces to the requirement of having $D(\underline{n_1},\ldots,\underline{n_m})$ non-zero. This being a Vandermonde determinant, the above condition asks for ψ to take distinct values at the points $\underline{n_1},\ldots,\underline{n_m}$.

It is easy to construct such a ψ, but our original construction was not particularly nice or illuminating. Below we present a more conceptual argument for the existence of such a ψ, which was kindly provided to us by the referee.

Let $\mathcal{V}$ be the set of all completely additive functions in A_r. Then $\mathcal{V}$ is a K-vector subspace of A_r. Since the multiplicative monoid $\mathbb{N}$ is a free monoid with the free basis given by all prime numbers, $\mathcal{V}$ is isomorphic to the K-dual of $\left(K^{\bigoplus\mathbb{N}}\right)^{\bigoplus r}$, the r-ple direct sum of the countable dimensional K-vector space considered with the basis indexed by the set of all prime numbers. Eventhough $\mathcal{V}$ is infinite dimensional, one should only look at a finite dimensional portion of it. Indeed, as one readily sees in the above calculation, the condition for non-vanishing of $D(\underline{n_1},\ldots,\underline{n_m})$ for the fixed $\underline{n}$ only involves the finitely many primes that divide the entries of $\underline{n}$. Since the condition $D(\underline{n_1},\ldots,\underline{n_m})=0$ gives rise to a single polynomial equation among the values of ψ for finitely many tuples of the form $(1,\ldots,1,p,1,\ldots,1)$ where p is a prime number, and since such values can be chosen freely in the infinite field K, the desired choice of ψ is always possible, as K^n is Zariski dense in $A^n_K =$ Spec $K[X_1,\ldots,X_n]$ since K is infinite.

With ψ as above, we have that

$$\begin{aligned}
|y_1|_{\underline{t}}\cdots|y_m|_{\underline{t}} &= \rho^{V_{\underline{t}}(y_1)+\ldots+V_{\underline{t}}(y_m)}\\
&= \rho^{\underline{t}\cdot\Omega_r(n_{11}\cdots n_{m1},\ldots,n_{1r}n_{mr})}\\
&\le \rho^{V_{\underline{t}}(W_\psi(y_1,\ldots,y_m))}\\
&= |W_\psi(y_1,\ldots,y_m)|_{\underline{t}}\\
&\le |y_1|_{\underline{t}}\cdots|y_m|_{\underline{t}}.
\end{aligned}$$

Hence, $|W_\psi(y_1,\ldots,y_m)|_{\underline{t}} = |y_1|_{\underline{t}}\cdots|y_m|_{\underline{t}}$, which completes the proof of (4).

Next, let us remark that the greatest common divisor $(c_ix_i, D_\psi c_ix_i,\ldots,D_\psi^{m-1}c_ix_i)$ divides $W_\psi(c_1x_1,\ldots,c_mx_m)$, and c_ix_i divides $(c_ix_i, D_\psi c_ix_i,\ldots,D_\psi^{m-1}c_ix_i)\mathrm{rad}(c_ix_i)^{m-1}$ for all $1\le i\le m$. Also, $(L, D_\psi L,\ldots,D_\psi^{m-1}L)$ divides $W_\psi(c_1x_1,\ldots,c_mx_m)$, and L divides $(L, D_\psi L,\ldots,D_\psi^{m-1}L)\mathrm{rad}(L)^{m-1}$. Thus $c_1x_1c_2x_2\cdots c_mx_mL$ divides

$$\mathrm{rad}(c_1x_1c_2x_2\cdots c_mx_mL)^{m-1}(L, D_\psi L,\ldots,D_\psi^{m-1}L)\prod_{i=1}^{m}(c_ix_i, D_\psi c_ix_i,\ldots,D_\psi^{m-1}c_ix_i). \tag{6}$$

Since c_1x_1, c_2x_2,…,c_mx_m, and L are coprime, so are the greatest common divisors which appear as the last $m+1$ factors in (6). Each of these factors is a divisor of $W_\psi(c_1x_1,\ldots,c_mx_m)$, therefore their product divides $W_\psi(c_1x_1,\ldots,c_mx_m)$, and we find that $c_1x_1c_2x_2\cdots c_mx_mL$ divides $\mathrm{rad}(c_1x_1c_2x_2\cdots c_mx_mL)^{m-1}W_\psi(c_1x_1,\ldots,c_mx_m)$. Thus,

$$|\mathrm{rad}(c_1x_1c_2x_2\cdots c_mx_mL)|_{\underline{t}}^{m-1} \le \frac{|L|_{\underline{t}}\prod_{i=1}^{m}|c_ix_i|_{\underline{t}}}{|W_\psi(c_1x_1,\ldots,c_mx_m)|_{\underline{t}}} = |L|_{\underline{t}},$$

which completes the proof of Theorem 1.

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Alexandru Zaharescu
Institute of Mathematics of the Romanian Academy
P.O.Box 1-764
Bucharest 014700, Romania
and Department of Mathematics
University of Illinois at Urbana-Champaign
1409 W. Green Street
Urbana, IL, 61801, USA
e-mail: zaharesc@math.uiuc.edu

Mohammad Zaki
Department of Mathematics and Statistics
Ohio Northern University
525 S Main Street
Ada OH 45810, USA
e-mail: m-zaki@onu.edu