Kumamoto J. Math.
Vol.**26** (2013), 9–22

On some generalized convergent lacunary sequence spaces defined by a Musielak-Orlicz function

Kuldip Raj

(Received August 30, 2012)
(Accepted January 25, 2013)

Abstract. In the present paper we introduce some generalized convergent lacunary sequence spaces defined by a Musielak-Orlicz function $\mathcal{M} = (M_i)$. We also make an effort to study some topological properties and prove some inclusion relations between these spaces.

1 Introduction and Preliminaries

The notion of difference sequence spaces was introduced by Kızmaz [6], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [3] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, n be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta_n^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $n = 1$, we get the spaces which were studied by Et and Çolak [3]. Taking $m = n = 1$, we get the spaces which were introduced and studied by Kızmaz [6].

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$.

Mathematical Subject Classification (2010): 40A05, 40C05, 46A45
Key words: Orlicz function, Musielak-Orlicz function, lacunary sequence, paranorm space

Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{(x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||(x_k)|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \le 1\Big\}.$$

Also, it was shown in [8] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \ge 1)$. An Orlicz function M satisfies the Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \le K(L)M(u)$ for all values of $u \ge 0$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \ge 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
A sequence $\mathcal{M} = (M_i)$ of Orlicz functions is called a Musielak-Orlicz function (see [11, 15]). A sequence $\mathcal{N} = (N_i)$ defined by

$$N_i(v) = \sup\{|v|u - M_i(u) : u \ge 0\},\ i = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{i=1}^{\infty}(M_i)(x_i), x = (x_i) \in t_{\mathcal{M}}.$$

We equip $t_{\mathcal{M}}$ with the Luxemburg norm

$$||x|| = \inf\Big\{i > 0 : I_{\mathcal{M}}\Big(\frac{x}{i}\Big) \le 1\Big\}$$

or equip it with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{i}\Big(1 + I_{\mathcal{M}}(ix)\Big) : i > 0\Big\}.$$

A Musielak-Orlicz function (M_i) is said to satisfy the Δ_2-condition if there exist constants $a, K > 0$ and a sequence $c = (c_i)_{i=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality

$$M_i(2u) \leq KM_i(u) + c_i$$

holds for all $i \in N$ and $u \in R_+$ whenever $M_i(u) \leq a$.
Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called a paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called a total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [20], Theorem 10.4.2, pp. 183). For more details about sequence spaces (see [12, 13, 18, 19]) and reference therein.
Let l_∞, c and c_0 denotes the sequence spaces of bounded, convergent and null sequences $x = (x_i)_{i=1}^{\infty}$ respectively. A linear functional $\mathcal{L}$ on ℓ_∞ is said to be a Banach limit (see [1]) if it has the properties:

1. $\mathcal{L}(x) \geq 0$ if $x \geq 0$(*i.e.* $x_n \geq 0$ for all n),
2. $\mathcal{L}(e) = 1$, where $e = (1, 1, \cdots)$,
3. $\mathcal{L}(Dx) = \mathcal{L}(x)$,

where the shift operator D is defined by $(Dx_n) = (x_{n+1})$.
Let $\mathfrak{B}$ be the set of all Banach limits on ℓ_∞. A sequence x is said to be almost convergent to a number L if $\mathcal{L}(x) = L$ for all $\mathcal{L} \in \mathfrak{B}$. Lorentz [5] has shown that x is almost convergent to L if and only if

$$t_{km} = t_{km}\Big(\{x_j\}_{j=1}^{\infty}\Big) = \frac{x_m + x_{m+1} + \cdots + x_{m+k}}{k+1} \to L \text{ as } k \to \infty, \text{ uniformly in } m.$$

Also a sequence $x = (x_i) \in l_\infty$ is said to be almost convergent if all Banach limits of $x = (x_i)$ coincide. In [7], it was shown that

$$c = \Big\{x = (x_i) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_{i+s} \text{ exists, uniformly in } s\Big\}.$$

Example : Consider $(\Omega, \mathcal{F}, P) = ([0, 1], \mathcal{B}, dx|[0, 1])$.

$$\text{If } 0 \leq a \leq b \leq 1 \text{ and } E = [a, b], \text{ then } P(E) = b - a.$$

Define a sequence of random variables $\{X_n\}$ as follows

$$X_n(\omega) = \omega^n, \ \forall \omega \in \Omega.$$

For a fixed sample point $\omega \in [0,1)$, the sequence of real numbers $\{X_n(\omega)\}$ has limit

$$\lim_{n\to\infty} X_n(\omega) = \lim_{n\to\infty} \omega^n = 0.$$

For $\omega = 1$, the sequence of real numbers $\{X_n(\omega)\}$ has limit

$$\lim_{n\to\infty} X_n(\omega) = \lim_{n\to\infty} \omega^n = \lim_{n\to\infty} 1^n = 1.$$

Therefore, the sequence of random variables $\{X_n\}$ does not converge pointwise to $X = 0$, because $\lim_{n\to\infty} X_n(\omega) \neq X(\omega)$ for $\omega = 1$. However, the set of sample points ω such that $\{X_n(\omega)\}$ does not converge to $X(\omega)$ is a zero-probability event

$$P\Big(\Big\{\omega \in \Omega : \{X_n(\omega)\} \text{ does not converge to } X(\omega)\Big\}\Big) = P(\{1\}) = 1 - 1 = 0.$$

Therefore, the sequence $\{X_n\}$ almost convergent to $X = 0$.
In ([9, 10]), Maddox defined a sequence $x = (x_i)$ to be strongly almost convergent to a number L if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |x_{i+s} - L| = 0, \text{ uniformly in } s.$$

By a lacunary sequence $\theta = (k_r)$, $r = 0, 1, 2, ...$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $h_r = (k_r - k_{r-1}) \to \infty (r \to \infty)$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman [4] as follows:

$$N_\theta = \Big\{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r}\sum_{i\in I_r} |x_i - L| = 0, \text{ for some } L\Big\}.$$

Let M be an Orlicz function. Güngör and Et [5] defined the following sequence spaces:

$$[\hat{c}, M](\Delta^m) = \Big\{x = (x_i) : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} M\Big(\frac{|\Delta^m x_{i+s} - L|}{\rho}\Big) = 0, \text{uniformly in } s, \text{ for some } \rho > 0 \text{ and } L > 0\Big\},$$

$$[\hat{c}, M]_0(\Delta^m) = \Big\{x = (x_i) : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} M\Big(\frac{|\Delta^m x_{i+s}|}{\rho}\Big) = 0, \quad \text{uniformly in } s, \text{ for some } \rho > 0 \Big\}$$

and

$$[c, M]_\infty(\Delta^m) = \Big\{x = (x_i) : \sup_{n,s} \frac{1}{n}\sum_{i=1}^{n} M\Big(\frac{|\Delta^m x_{i+s}|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$. In the second section of this paper we want to generalize the result of Güngör and Et [4].

Let $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function, $p = (p_i)$ be a bounded sequence of positive real numbers and $u = (u_i)$ be a sequence of strictly positive real numbers. Then we define the following sequence spaces:

$$[\hat{c}, \mathcal{M}, u, p, \Delta_n^m] = \Big\{x = (x_i) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s} - L|}{\rho}\Big)^{p_i} = 0, \text{ uniformly in } s \text{ for some } L \text{ and } \rho > 0\Big\},$$

$$[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0 = \Big\{x = (x_i) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho}\Big)^{p_i} = 0, \text{ uniformly in } s \text{ for some } \rho > 0\Big\}$$

and

$$[c, \mathcal{M}, u, p, \Delta_n^m]_\infty = \Big\{x = (x_i) : \sup_{n,s} \frac{1}{n} \sum_{i=1}^{n} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho}\Big)^{p_i} < \infty, \text{ for some } \rho > 0\Big\}.$$

Examples 1. Let $M_i(x) = x^2$, for all $i \in \mathbb{N}$. Let $p_i = 2$, $u_i = i$, $m = 1$, $n = 0$ and $s = 0$. Consider a sequence $x_k = k$, for all $k \in \mathbb{N}$. Then $x_k \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]$ but $x_k \notin [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0$.

2. Let $M_i(x) = x^4$, for all $i \in \mathbb{N}$. Let $p_i = \frac{1}{i}$, $u_i = 4$, $m = 2$, $n = 2$ and $s = 0$. Consider a sequence $x_k = k + 1$, for all $k \in \mathbb{N}$. Then $x_k \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0$ but $x_k \notin [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]$.

3. Let $M_i(x) = x^3$, for all $i \in \mathbb{N}$. Let $p_i = i$, $u_i = 9$, $m = 3$, $n = 2$ and $s = 0$. Consider a sequence $x_k = k^3$, for all $k \in \mathbb{N}$. Then $x_k \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty$ but $x_k \notin [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0$.

The subject studied in this paper is important because we have generated some new sequence spaces and studied some algebraic and interesting topological properties of these spaces. The applications of sequence spaces has been found in quantum mechanics and matrix transformations. It is related to the other fields of mathematics because topologist may study different type of topology on these spaces. Also if one can from operator theory may study the properties of operators like as boundedness, compactness, Frdholmness etc. on these spaces. So that it is related to other types of mathematics.

Relation between the sequence spaces and operator theory: Bhardwaj and Singh in [2] introduced the sequence space $W_0(A, f)$ and studied some topological and algebraic properties of this sequence space. In [16] Raj, Komal and Khosla studied the properties of composition and weighted composition operators like boundedness, compactness, closed range, Fredholmness, invertibility etc. on this space whereas the properties of multiplication operators are studied by Raj and Khosla in [17]. Recently, Mursaleen and Mohiuddine [14] studied the properties of the matrix transformation on the sequence spaces $V_\sigma(\theta)$ and $V_\sigma^\infty(\theta)$. So that in this way one can define the relation between the sequence spaces and other mathematics.

2 Some strongly almost convergent sequence spaces

In this section of the paper we shall study some topological properties and inclusion relation between the spaces $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]$, $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0$ and $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty$.

Theorem 2.1 *Let $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function, $p = (p_i)$ be a bounded sequence of positive real numbers and $u = (u_i)$ be a sequence of strictly positive real numbers. Then the spaces $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]$, $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0$ and $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_i), y = (y_i) \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive integers ρ_1 and ρ_2 such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho_1}\Big)^{p_i} = 0, \text{ uniformly in } s$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} M_i\Big(\frac{|u_i \Delta_n^m y_{i+s}|}{\rho_2}\Big)^{p_i} = 0, \text{ uniformly in } s.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing convex function, and so by using inequality (1), we have

$$\begin{aligned}
&\frac{1}{n} \sum_{i=1}^{n} M_i\Big(\frac{|u_i \Delta_n^m (\alpha x_{i+s} + \beta y_{i+s})|}{\rho_3}\Big)^{p_i} \\
&\leq \frac{1}{n} \sum_{i=1}^{n} M_i\Big[\Big(\frac{|u_i \Delta_n^m (\alpha x_{i+s})|}{\rho_3}\Big) + \Big(\frac{|u_i \Delta_n^m (\beta y_{i+s})|}{\rho_3}\Big)\Big]^{p_i} \\
&\leq K \frac{1}{n} \sum_{i=1}^{n} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho_1}\Big)^{p_i} + K \frac{1}{n} \sum_{i=1}^{n} M_i\Big(\frac{|u_i \Delta_n^m y_{i+s}|}{\rho_2}\Big)^{p_i}. \\
&\to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

Therefore $\alpha x + \beta y \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0$. Hence $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0$ is a linear space. Similarly, we can prove that $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]$ and $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty$ are linear spaces.

Theorem 2.2 *Let $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function, $p = (p_i)$ be a bounded* *sequence of positive real numbers and* $u = (u_i)$ *be a sequence of strictly positive real numbers. Then the space* $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty$ *is a paranormed space with the paranorm defined by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \sup_{n,s}\Big(\frac{1}{n}\sum_{i=1}^{n} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}|}{\rho}\Big)^{p_i}\Big)^{\frac{1}{H}} \leq 1, r = 1, 2, ...\Big\},$$

where $H = \max(1, \sup_i p_i) < \infty$.

Proof. It is easy to prove so we omit the details.

Theorem 2.3 *Let* $\mathcal{M} = (M_i)$ *be a Musielak-Orlicz function. If* $\sup_i [M_i(x)]^{p_i} < \infty$ *for all fixed* $x > 0$*, then* $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0 \subset [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty$.

Proof. Let $x = (x_i) \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0$. There exists some positive ρ_1 such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\Big[M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}|}{\rho_1}\Big)\Big]^{p_i} = 0, \text{ uniformly in } s.$$

Define $\rho = 2\rho_1$. Since $\mathcal{M} = (M_i)$ is non-decreasing and convex, by using inequality (1), we have

$$\begin{aligned}
\sup_s \frac{1}{n}\sum_{i=1}^{n}\Big[M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}|}{\rho}\Big)\Big]^{p_i} &= \sup_s \frac{1}{n}\sum_{i=1}^{n}\Big[M_i\Big(\frac{\Delta_n^m x_{i+s} - L + L}{\rho}\Big)\Big]^{p_i} \\
&\leq K\sup_s \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{2^{p_i}}M_i\Big(\frac{|u_i\Delta_n^m x_{i+s} - L|}{\rho_1}\Big)\Big]^{p_i} \\
&+ K\sup_s \frac{1}{n}\sum_{i=1}^{n}\Big[\frac{1}{2^{p_k}}M_i\Big(\frac{|L|}{\rho_1}\Big)\Big]^{p_i} \\
&\leq K\sup_s \frac{1}{n}\sum_{i=1}^{n}\Big[M_i\Big(\frac{|u_i\Delta_n^m x_{i+s} - L|}{\rho_1}\Big)\Big]^{p_i} \\
&+ K\sup_s \frac{1}{n}\sum_{i=1}^{n}\Big[M_i\Big(\frac{|L|}{\rho_1}\Big)\Big]^{p_i} \\
&< \infty.
\end{aligned}$$

Hence $x = (x_i) \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty$. This completes the proof.

3 Generalized lacunary sequence spaces

In this section of the paper we introduce some generalized lacunary almost convergent sequence spaces by using a Musielak-Orlicz function. Let $\mathcal{M} = (M_i)$

be a Musielak-Orlicz function, $p = (p_i)$ be a bounded sequence of real numbers, $u = (u_i)$ be a sequence of strictly positive real numbers. We define the following sequence spaces in this paper:

$$[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta = \Big\{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s} - L|}{\rho}\Big)^{p_i} = 0, \text{ uniformly in } s,$$

$$\text{for some } L \text{ and } \rho > 0\Big\},$$

$$[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0^\theta = \Big\{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho}\Big)^{p_i} = 0, \text{ uniformly in } s$$

$$\text{for some } \rho > 0\Big\}$$

and

$$[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty^\theta = \Big\{x = (x_i) : \sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho}\Big)^{p_i} < \infty, \text{ for some } \rho > 0\Big\}.$$

Examples 1. Let $M_i(x) = x$, for all $i \in \mathbb{N}$. Let $\theta = (2^r)$, $p_i = i$, $u_i = 3$, $m = 1$, $n = 0$ and $s = 0$. Consider a sequence $x_k = k^2$, for all $k \in \mathbb{N}$. Then $x_k \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta$ but $x_k \notin [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0^\theta$.

2. Let $M_i(x) = x^2$, for all $i \in \mathbb{N}$. Let $\theta = (2^r)$, $p_i = \frac{1}{i}$, $u_i = i$, $m = 2$, $n = 2$ and $s = 0$. Consider a sequence $x_k = k$, for all $k \in \mathbb{N}$. Then $x_k \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0^\theta$ but $x_k \notin [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta$.

3. Let $M_i(x) = x^4$, for all $i \in \mathbb{N}$. Let $\theta = (2^r)$, $p_i = 4$, $u_i = 3$, $m = 3$, $n = 2$ and $s = 0$. Consider a sequence $x_k = k + 1$, for all $k \in \mathbb{N}$. Then $x_k \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty^\theta$ but $x_k \notin [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0^\theta$.

If $x = (x_i) \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta$, we say that $x = (x_i)$ is lacunary strongly almost generalized Δ_n^m-convergent to the number L with respect to the Musielak-Orlicz function (M_i). In this case we write $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta - \lim x = L$. If $M(x) = x$, then above spaces reduces to the following spaces :

$$[\hat{c}, u, p, \Delta_n^m]^\theta = \Big\{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big(\frac{|u_i \Delta_n^m x_{i+s} - L|}{\rho}\Big)^{p_i} = 0, \text{ uniformly in } s,$$

$$\text{for some } L \text{ and } \rho > 0\Big\},$$

$$[\hat{c}, u, p, \Delta_n^m]_0^\theta = \Big\{x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho}\Big)^{p_i} = 0, \text{ uniformly in } s$$

$$\text{for some } \rho > 0\Big\}$$

and

$$[\hat{c}, u, p, \Delta_n^m]_\infty^\theta = \Big\{x = (x_i) : \sup_{r,s} \frac{1}{h_r} \sum_{i \in I_r} \Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho}\Big)^{p_i} < \infty, \text{ for some } \rho > 0\Big\}.$$

One may note that the examples on page no.13 for strongly almost convergent sequence spaces whereas the examples on page no.16 for generalized lacunary almost convergent sequence spaces.

The purpose of this paper is to introduce new sequence spaces by using the concept of lacunary almost generalized Δ_n^m-convergence and a Musielak-Orlicz function. I also make an efforts to study some topological properties and interesting inclusion relations between these spaces. An attempt is also made to establish some relations between the spaces defined in section II and the spaces defined in this section. These spaces also generalizes the well known Orlicz sequence space l_M, strongly summable sequence spaces $[c, 1], [c, 1]_0$ and $[c, 1]_\infty$.

Theorem 3.1 *For any Musielak-Orlicz function $\mathcal{M} = (M_i)$ and any sequence $p = (p_i)$ of strictly positive real numbers, then $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta, [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0^\theta$ and $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty^\theta$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_i), y = (y_i) \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0^\theta$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive integers ρ_1 and ρ_2 such that

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho_1}\Big)^{p_i} = 0, \text{ uniformly in } s$$

and

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} M_i\Big(\frac{|u_i \Delta_n^m y_{i+s}|}{\rho_2}\Big)^{p_i} = 0, \text{ uniformly in } s.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is a non-decreasing convex function and so by using inequality (1), we have

$$\begin{aligned}
&\frac{1}{h_r} \sum_{i \in I_r} M_i\Big(\frac{|u_i \Delta_n^m (\alpha x_{i+s} + \beta y_{i+s}|}{\rho_3}\Big)^{p_i} \\
&\quad \leq \frac{1}{h_r} \sum_{i \in I_r} M_i\Big[\Big(\frac{|u_i \Delta_n^m (\alpha x_{i+s})|}{\rho_3}\Big) + \Big(\frac{|u_i \Delta_n^m (\beta y_{i+s})|}{\rho_3}\Big)\Big]^{p_i} \\
&\quad \leq K \frac{1}{h_r} \sum_{i \in I_r} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho_1}\Big)^{p_i} + K \frac{1}{h_r} \sum_{i \in I_r} M_i\Big(\frac{|u_i \Delta_n^m y_{i+s}|}{\rho_2}\Big)^{p_i}.
\end{aligned}$$

Therefore $\alpha x + \beta y \in [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0^\theta$. Hence $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_0^\theta$ is a linear space. Similarly,

we can prove that $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta$ and $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty^\theta$ are linear spaces.

Theorem 3.2 *For any Musielak-Orlicz function* $\mathcal{M} = (M_i)$, $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]_\infty^\theta$ *is a semi-normed linear space, semi-normed by*

$$h_{\Delta_n^m}(x) = \sum_{i=1}^m |x_i| + \inf\Big\{\rho > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho}\Big)^{p_i} \leq 1,\ r, s = 1, 2, \cdots\Big\}.$$

Proof. Clearly $h_{\Delta_n^m}(x) = h_{\Delta_n^m}(-x), x = \bar{0}$ implies $u_i\Delta_n^m x_{i+s} = 0$ for all $i, s \in \mathbb{N}$ and as

such $M_i(\bar{0}) = 0$, where $\bar{0} = (0, 0, \cdots)$. Therefore $h_{\Delta_n^m}(\bar{0}) = 0$. Next let ρ_1 and ρ_2 be such that

$$\sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho_1}\Big)^{p_i} \leq 1$$

and

$$\sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m y_{i+s}|}{\rho_2}\Big)^{p_i} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\begin{aligned}
&\sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m (x_{i+s} + y_{i+s})|}{\rho}\Big)^{p_i} \\
\leq\ & \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big) \sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho_1}\Big)^{p_i} + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big) \sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m y_{i+s}|}{\rho_2}\Big)^{p_i} \\
\leq\ & 1.
\end{aligned}$$

Since the $\rho' s$ are non-negative, so we have

$$\begin{aligned}
&h_{\Delta_n^m}(x+y) \\
=\ & \sum_{i=1}^m |x_i| + \inf\Big\{\rho > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m (x_{i+s} + y_{i+s})|}{\rho}\Big)^{p_i} \leq 1,\ r, s = 1, 2, \cdots\Big\} \\
\leq\ & \sum_{i=1}^m |x_i| + \inf\Big\{\rho_1 > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m x_{i+s}|}{\rho_1}\Big)^{p_i} \leq 1,\ r, s = 1, 2, \cdots\Big\} \\
+\ & \sum_{i=1}^m |y_i| + \inf\Big\{\rho_2 > 0 : \sup_{r,s} \frac{1}{h_r} \sum_{i\in I_r} M_i\Big(\frac{|u_i \Delta_n^m y_{i+s}|}{\rho_2}\Big)^{p_i} \leq 1,\ r, s = 1, 2, \cdots\Big\}.
\end{aligned}$$

So, $h_{\Delta_n^m}(x+y) \leq h_{\Delta_n^m}(x) + h_{\Delta_n^m}(y)$. Finally for $\lambda \in \mathbb{C}$, without loss of generality $\lambda \neq 0$, then

$$\begin{aligned}
&h_{\Delta_n^m}(\lambda x)\\
&= \sum_{i=1}^{m}|\lambda x_i| + \inf\Big\{\rho>0 : \sup_{r,s}\frac{1}{h_r}\sum_{i\in I_r} M_i\Big(\frac{|u_i\Delta_n^m \lambda x_{i+s}|}{\rho}\Big)^{p_i} \le 1,\ r,s=1,2,\cdots\Big\}\\
&= |\lambda|\sum_{i=1}^{m}|x_i| + \inf\Big\{|\lambda| r>0 : \sup_{r,s}\frac{1}{h_r}\sum_{i\in I_r} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}|}{r}\Big)^{p_i} \le 1,\ r,s=1,2,\cdots\Big\}\\
&= |\lambda| h_{\Delta_n^m}(\lambda x),
\end{aligned}$$

where $r = |\frac{\rho}{\lambda}|$. This completes the proof of the theorem.

Theorem 3.3 *If* $\theta=(k_r)$ *be a lacunary sequence with* $\liminf q_r > 1$*, then*

$$[\hat{c}, \mathcal{M}, u, p, \Delta_n^m] \subset [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta.$$

Proof. Let $\liminf q_r > 1$. Then there exists $\eta > o$ such that $q_r > 1+\eta$ and hence

$$\frac{h_r}{k_r} = 1 - \frac{k_{r-1}}{k_r} > 1 - \frac{1}{1+\eta} = \frac{\eta}{1+\eta}.$$

Therefore,

$$\begin{aligned}
\frac{1}{k_r}\sum_{i=1}^{k_r} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L|}{\rho}\Big)^{p_i} &\ge \frac{1}{k_r}\sum_{i\in I_r} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L|}{\rho}\Big)^{p_i}\\
&\ge \frac{\eta}{1+\eta}\frac{1}{h_r}\sum_{i\in I_r} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L|}{\rho}\Big)^{p_i}
\end{aligned}$$

and if $x=(x_i)\in[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]$, then it follows that $x=(x_i)\in[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta$.

Theorem 3.4 *If* $\theta=(k_r)$ *be a lacunary sequence with* $\limsup q_r < \infty$*, then*

$$[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta \subset [\hat{c}, \mathcal{M}, u, p, \Delta_n^m].$$

Proof. Let $x=(x_i)\in[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta$. Choose $\delta>0$ be arbitrarily, then there exists σ_0 such that for every $\sigma\ge\sigma_0$ and for all $s\in\mathbb{N}$

$$a_{\sigma s} = \frac{1}{h_\sigma}\sum_{i\in I_\sigma} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L|}{\rho}\Big)^{p_i} < \delta$$

That is, we can find some positive constant W, such that

$$a_{\sigma s} < W \tag{2}$$

for all σ and s. Given $\limsup q_r < \infty$ implies that there exists some positive number K such that

$$q_r < K \tag{3}$$

for all $r \geq 1$. Therefore, for $k_{r-1} < n \leq k_r$, from (2) and (3) we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L|}{\rho}\Big)^{p_i} &\leq \frac{1}{k_{r-1}}\sum_{i=1}^{k_r} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L|}{\rho}\Big)^{p_i} \\
&\leq \frac{1}{k_{r-1}}\sum_{\sigma=1}^{r}\sum_{i\in I_\sigma} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L|}{\rho}\Big)^{p_i} \\
&= \frac{1}{k_{r-1}}\Big[\sum_{\sigma=1}^{\sigma_0}+\sum_{\sigma=\sigma_0+1}^{r}\Big]\sum_{i\in I_\sigma} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L|}{\rho}\Big)^{p_i} \\
&\leq \frac{1}{k_{r-1}}\Big(\sup_{1\leq p\leq\sigma_0} a_{ps}\Big)k_{\sigma_0}+\delta\frac{k_r-k_{\sigma_0}}{k_{r-1}} \\
&\leq W\frac{k_{\sigma_0}}{k_{r-1}}+\delta K.
\end{aligned}$$

Since $k_{r-1}\to\infty$ as $r\to\infty$, we get $x=(x_i)\in[\hat{c},\mathcal{M},u,p,\Delta_n^m]$. This completes the proof of the theorem.

Theorem 3.5 *If* $\theta=(k_r)$ *be a lacunary sequence with* $1<\liminf q_r\leq\limsup q_r<\infty$, *then*

$$[\hat{c},\mathcal{M},u,p,\Delta_n^m]=[\hat{c},\mathcal{M},u,p,\Delta_n^m]^\theta$$

Proof. The proof of Theorem 3.5 follows from Theorems 3.3 and 3.4.

Theorem 3.6 *Let* $x=(x_i)\in[\hat{c},\mathcal{M},u,p,\Delta_n^m]^\theta\cap[\hat{c},\mathcal{M},u,p,\Delta_n^m]$. *Then*

$$[\hat{c},\mathcal{M},u,p,\Delta_n^m]^\theta-\lim x=[\hat{c},\mathcal{M},u,p,\Delta_n^m]-\lim x$$

and $[\hat{c},\mathcal{M},u,p,\Delta_n^m]^\theta-\lim x$ *is unique for any lacunary sequence* $\theta=(k_r)$.

Proof. Let $x=(x_i)\in[\hat{c},\mathcal{M},u,p,\Delta_n^m]^\theta\cap[\hat{c},\mathcal{M},u,p,\Delta_n^m]$ and $[\hat{c},\mathcal{M},u,p,\Delta_n^m]^\theta-\lim x=L_0,[\hat{c},\mathcal{M},u,p,\Delta_n^m]-\lim x=L$. We can see that

$$M_i\Big(\frac{|L-L_0|}{\rho}\Big)^{p_i} \leq \frac{1}{h_r}\sum_{i\in I_r} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L|}{\rho}\Big)^{p_i}+\frac{1}{h_r}\sum_{i\in I_r} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L_0|}{\rho}\Big)^{p_i}.$$

Taking limit as $r\to\infty$, we have

$$M_i\Big(\frac{|L-L_0|}{\rho}\Big)^{p_i} \leq \lim_r\sup_s\frac{1}{h_r}\sum_{i\in I_r} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L|}{\rho}\Big)^{p_i}.$$

Hence, there exist s and r_0 such that for $r>r_0$

$$\frac{1}{h_r}\sum_{i\in I_r} M_i\Big(\frac{|u_i\Delta_n^m x_{i+s}-L|}{\rho}\Big)^{p_i}>\frac{1}{2}M_i\Big(\frac{|L-L_0|}{\rho}\Big)^{p_i}.$$

Since $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m] - \lim x = L$, it follows that

$$0 \geq \limsup_r \Big(\frac{h_r}{k_r}\Big) M_i \Big(\frac{|L - L_0|}{\rho}\Big)^{p_i} \geq \liminf_r M_i \Big(\frac{|L - L_0|}{\rho}\Big)^{p_i} \geq 0$$

and so $\lim_r q_r = 1$. Therefore by Theorem 3.4, $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta \subset [\hat{c}, \mathcal{M}, u, p, \Delta_n^m]$ and $[\hat{c}, \mathcal{M}, u, p, \Delta_n^m]^\theta - \lim x = L_0, \quad [\hat{c}, \mathcal{M}, u, p, \Delta_n^m] - \lim x = L$. Further

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} M_i \Big(\frac{|u_i \Delta_n^m x_{i+s} - L|}{\rho}\Big)^{p_i} + \frac{1}{n}\sum_{i=1}^{n} M_i \Big(\frac{|u_i \Delta_n^m x_{i+s} - L_0|}{\rho}\Big)^{p_i} &\geq M_i \Big(\frac{|L - L_0|}{\rho}\Big)^{p_i} \\ &\geq 0 \end{aligned}$$

and taking the limit on both sides as $n \to \infty$, we have $M_i \Big(\frac{|L-L_0|}{\rho}\Big)^{p_i} = 0$, that is, $L = L_0$ for any sequence of Orlicz functions $\mathcal{M} = (M_i)$. This completes the proof of the theorem.

Acknowledgement : The author thanks the referee for his valuable suggestions that improved the presentation of the paper.

References

[1] S. Banach, *Theorie operations linearies*, Chelsea Publishing Co., New York, (1955).

[2] V. K. Bhardwaj and N. Singh, *On some sequence spaces defined by a modulus*, Indian J. Pure Appl. Math., **30** (1999), 809-817.

[3] M. Et and R. Çolak, *On generalized difference sequence spaces*, Soochow J. Math., **21** (1995), 377-386.

[4] A. R. Freedman, J. J. Sember and M. Raphael, *Some Cesaro-type summability spaces*, Proc. London Math. Soc., 37(3) (1978), 508-520.

[5] M. Güngör and M. Et, Δ^r*-strongly almost summable sequences defined by Orlicz functions*, Indian J. Pure Appl. Math., 34(8) (2003), 1141-1151.

[6] H. Kızmaz, *On certain sequence spaces*, Canad. Math-Bull., **24** (1981), pp. 169-176.

[7] G. G. Lorentz, *A contribution to the theory of divergent sequences* , Acta Mathematica, 80(1)(1948), 167-190.

[8] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math., **10** (1971), pp. 345-355.

[9] I. J. Maddox, *Spaces of strongly summable sequences* , Quart. J. Math., 18(1967), 345-355.

[10] I. J. Maddox, *A new type of convergence* , Math. Proc. Camb. Phil. Soc., 83(1978), 61-64.

[11] L. Maligranda, *Orlicz spaces and interpolation*, Seminars in Mathematics **5**, Polish Academy of Science, 1989.

[12] M. Mursaleen, *Matrix transformation between some new sequence spaces*, Houston J. Math., **9** (1983), 505-509.

[13] M. Mursaleen, *On some new invariant matrix methods of summability*, Quart. J. Math., Oxford **34** (1983), 77 - 86.

[14] M. Mursaleen and S. A. Mohiuddine, *Some matrix transformation of convex and paranormed sequence spaces into the space of invariant means*, Journal of Function Spaces and Applications, (2012), Article ID 612671, 10 pages.

[15] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, **1034** (1983).

[16] K. Raj, B. S. Komal and V. Khosla, *Composition and weighted composition operators acting on sequence spaces defined by modulus function*, Int. J. Math. Archive, Vol. **2**(2011), 217-221.

[17] K. Raj and V. Khosla, *On operators of multiplication acting on sequence spaces defined by modulus function*, Int. Math. Forum, Vol. **5**(2010), 3073-3077.

[18] K. Raj, A. K. Sharma and S. K. Sharma, *A Sequence space defined by Musielak-Orlicz functions*, Int. J. Pure Appl. Math., Vol. **67**(2011), 475-484.

[19] K. Raj and S. K. Sharma, *Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function*, Acta Univ. Sapientiae Math., **3** (2011), 97-109.

[20] A. Wilansky, *Summability through Functional Analysis*, North- Holland Math. Stud. 85(1984).

Kuldip Raj
School of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K, India
e-mail: kuldipraj68@gmail.com