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# Two combinatorial structures in the binary Golay code of length 24

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Dedicated to Emeritus Professor Hiroyoshi Yamaki at Kumamoto University who is one of the author's best friends

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**Abstract.** In the present paper we show that the design equations for the binary Golay code $\mathcal{G}_{24}$ of length 24 will serve to establish that the set of codewords of weight 8 in the code $\mathcal{G}_{24}$ forms an association scheme, and we derive that a distance regular graph structure from the obtained association scheme. The derived distance regular graph is turned out to be the one already shown by Brouwer-Cohen-Neumaier. They call it as Witt graph. We would like to point out that the results of our present research do not use finite group theory or sporadic geometry.

## 1 Introduction

It is known that the set of codewords of weight 8 (resp. 12, 16) forms a 5-design (a special case of a theorem of Assmus-Mattson [1] ). By the works of Mendelsohn and Wilson the conditions for design are reformulated as the relations among the cardinalities of the intersections of certain subsets of a finite set $X$ (c.f. [11],[12],[13],[15],[16]).

Later H.Koch[1] [7],[8],[9] obtained the formulas[2] that are essentially equivalent to the formulas formulated in the line of Mendelsohn and Wilson by using the modular form theory and the lattice theory. Koch obtained one further formula which is not obtainable from the Assmus-Mattson Theorem. The design equations for binary linear doubly self-dual codes originally serves for establishing 5-designs or 3-designs or etc.

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[1]Koch attributes the design equations formulated by him to B. Venkov, but Venkov did not publish any of the formula in the form of research papers.

[2]One should remark that Koch's formulas are valid only for binary linear extremal codes. For the designs coming from other types of combinatorial structures one can not apply Koch's formulas directly.

In the present paper we show that the design equations for the binary Golay code $\mathcal{G}_{24}$ of length 24 will serve to establish that the set of codewords of weight 8 in the code $\mathcal{G}_{24}$ forms an association scheme, and we derive that a distance regular graph structure from the obtained association scheme. The derived distance regular graph is turned out to be the one already shown by Brouwer-Cohen-Neumaier [3], Chapter 11, Section 4. They call it as Witt graph. We would like to point out that the results of our present research do not use finite group theory or sporadic geometry.

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# 2 Some Basic Definitions

Let $\mathbb{F}_2 = GF(2)$ be the field of 2 elements. Let $V = \mathbb{F}_2^n$ be the vector space of dimension $n$ over $\mathbb{F}_2$ . A linear $[n, k]$ code $\mathbf{C}$ is a vector subspace of $V$ of dimension $k$. An element $\mathbf{x}$ in $\mathbf{C}$ is called a codeword of $\mathbf{C}$. Usually an $[n, k]$ code is defined by giving $k$ linearly independent vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ of length $n$. The matrix formed by

$$\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \cdots \\ \mathbf{u}_k \end{pmatrix}$$

is called the generator matrix of the code. For instance the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

is a generator matrix of the Hamming $[8, 4]$ code $H_8$.
In $V$ , the inner product, which is denoted by $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y}$ in $V$, is defined as usual. The dual code $\mathbf{C}^\perp$ of $\mathbf{C}$ is defined by

$$\mathbf{C}^\perp = \{\mathbf{u} \in V \mid \mathbf{u} \cdot \mathbf{v} = \mathbf{0}, \quad {}^\forall\mathbf{v} \in \mathbf{C}\}.$$

The code $\mathbf{C}$ is called self-orthogonal if it satisfies $\mathbf{C} \subseteq \mathbf{C}^\perp$, and the code $\mathbf{C}$ is called self-dual if it satisfies $\mathbf{C} = \mathbf{C}^\perp$.

Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ be two vectors in $V$, then the Hamming distance $d(\mathbf{x}, \mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$ is defined to be the number of $i's$ such that $x_i \neq y_i$ for $1 \leq i \leq n$. The Hamming weight $wt(\mathbf{x})$ of $\mathbf{x}$ is the number of non-zero coordinates $x_i$ of $\mathbf{x}$. The intersersection $\mathbf{x} * \mathbf{y}$ of $\mathbf{x}$ and $\mathbf{y}$ is defined to be the number of $i's$ such that $x_i = y_i = 1$ for $1 \leq i \leq n$. There is a relation which connects the weight with the intersection:

$$wt(\mathbf{x} + \mathbf{y}) = wt(\mathbf{x}) + wt(\mathbf{y}) - 2\mathbf{x} * \mathbf{y}. \tag{2.1}$$

It is also well-known that

$$d(\mathbf{x}, \mathbf{y}) = wt(\mathbf{x} + \mathbf{y}). \tag{2.2}$$

The minimum weight $d = d(\mathbf{C})$ of a code $\mathbf{C}$ is defined by

$$\min_{0 \neq \mathbf{u} \in \mathbf{C}} wt(\mathbf{u}).$$

A homogeneous weight enumerator of a code $\mathbf{C}$ is a polynomial in two independent variables $x, y$ defined by

$$W_{\mathbf{C}}(x, y) = \sum_{\mathbf{u} \in \mathbf{C}} x^{wt(\mathbf{u})} y^{(n - wt(\mathbf{u}))}.$$

The weight enumerator $W_{\mathbf{C}}(x, y)$ of a code $\mathbf{C}$ carries important informations on the code $\mathbf{C}$. It is well known that the weight enumerator of the Hamming code $H_8$ is

$$W_{H_8}(x, y) = x^8 + 14x^4y^4 + y^8,$$

and $d(H_8) = 4$.

In a self-dual binary code each codeword has even weight. If each codeword has weight that is divisible by 4 then the code is called doubly even. Doubly even self-dual codes exist only when the length $n$ of the code is a multiple of 8. The Hamming code $H_8$ is a doubly even self-dual code of length 8. Another famous doubly even self-dual code is the binary Golay code $\mathcal{G}_{24}$ of length 24. Here we give a generator matrix of $\mathcal{G}_{24}$:

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&1&0&0&0&1&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&1&0&0&0&1&1\\
0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1&1&1&0&0&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1&1&1&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1&1&1&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1&1&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&1&0&1&1&0&1&1&1\\
0&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&1&1&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&1&1&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&1&1&0&0&0&1&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0
\end{pmatrix}.$$

The weight enumerator $W_{\mathcal{G}_{24}}(x, y)$ of the Golay code $\mathcal{G}_{24}$ is known to be

$$W_{\mathcal{G}_{24}}(x, y) = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}.$$

## 3 Design Equations for binary Golay code $\mathcal{G}_{24}$

In a sequence of papers [7],[8],[9] H. Koch develops a method to obtain the relations between intersection of codewords of weight 8 in doubly even self-dual extremal codes of various lengths. In the present paper we only need the equations for the set of codewords of weight 8.

**Proposition 3.1** *Let $\mathcal{G}_{24}$ be the binary Golay code and $\mathbf{C}_8$ be the set of the codewords of weight 8 in the Golay code $\mathcal{G}_{24}$, then we have*

$$\sum_{\xi \in \mathbf{C}_8} (\xi * \mathbf{a}) = 253(\mathbf{a} * \mathbf{a}), \tag{3.1}$$

$$\sum_{\xi \in \mathbf{C}_8} (\xi * \mathbf{a})^2 = 77(\mathbf{a} * \mathbf{a})^2 + 176(\mathbf{a} * \mathbf{a}), \tag{3.2}$$

$$\sum_{\xi \in \mathbf{C}_8} (\xi * \mathbf{a})^3 = 21(\mathbf{a} * \mathbf{a})^3 + 168(\mathbf{a} * \mathbf{a})^2 + 64(\mathbf{a} * \mathbf{a}), \tag{3.3}$$

$$\sum_{\xi \in \mathbf{C}_8} (\xi * \mathbf{a})^4 = 5(\mathbf{a} * \mathbf{a})^4 + 96(\mathbf{a} * \mathbf{a})^3 + 216(\mathbf{a} * \mathbf{a})^2 - 64(\mathbf{a} * \mathbf{a}), \tag{3.4}$$

$$\sum_{\xi \in \mathbf{C}_8} (\xi * \mathbf{a})^5 = (\mathbf{a} * \mathbf{a})^5 + 40(\mathbf{a} * \mathbf{a})^4 + 260(\mathbf{a} * \mathbf{a})^3 + 80(\mathbf{a} * \mathbf{a})^2 - 128(\mathbf{a} * \mathbf{a}), \tag{3.5}$$

$$\begin{array}{c} \sum_{\xi \in \mathbf{C}_8} (\xi * \mathbf{a})^7 - \left[14 + \frac{7 \cdot (\mathbf{a} * \mathbf{a})}{6}\right] \sum_{\xi \in \mathbf{C}_8} (\xi * \mathbf{a})^6 \\ = \frac{1}{6} \left\{ -(\mathbf{a} * \mathbf{a})^7 - 84(\mathbf{a} * \mathbf{a})^6 - 1820(\mathbf{a} * \mathbf{a})^5 \right. \\ \left. -15120(\mathbf{a} * \mathbf{a})^4 - 33152(\mathbf{a} * \mathbf{a})^3 + 28672(\mathbf{a} * \mathbf{a})^2 \right\}. \end{array} \tag{3.6}$$

In the above formulas (3.1)$\sim$(3.5) the right hand sides are polynomials in $\mathbf{a} * \mathbf{a}$, and we use $f_t(\mathbf{a} * \mathbf{a})$ $(1 \leq t \leq 5)$ to denote those polynomials in natural order for later use.

# 4 An Association Scheme in $\mathcal{G}_{24}$

## 4.1 A Brief Definition of Association Scheme

As to the precise definitions for the theory of association scheme one may refer [2],[5] or [6].

Here we give a brief definition of the association scheme. An association scheme with $d$ classes is a pair $(X, \mathcal{R})$, where $X$ is a finite set with at least two elements and $\mathcal{R}$ is a partition of $X \times X$ with the following properties:
(i) $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$,
(ii) $R_0 = \{\langle x, x\rangle | x \in X\}$,
(iii) if $R_i$ is a member of $\mathcal{R}$ then the set $R_i^T = \{\langle y, x\rangle | \langle x, y\rangle \in R_i\}$ is also a member of $\mathcal{R}$ for $i = 0, 1, \cdots, d$,
(iv) for any pair $\langle x, y\rangle \in R_k$ the number $p_{i,j}^k$ of $z \in X$ such that both the conditions $\langle x, z\rangle \in R_i$ and $\langle y, z\rangle \in R_j$ hold does not depend on the choice of $\langle x, y\rangle \in R_k$. The numbers $p_{i,j}^k$ are called the intersection numbers of this association scheme

$(X, \mathcal{R})$.
The number $n_i = p^0_{i,i}$ is called the valency of $R_i$, and it holds that

$$|X| = \sum_{i=0}^{d} n_i.$$

## 4.2 Preliminaries

Before discussing the association scheme in $\mathcal{G}_{24}$ we prove a simple lemma:

**Lemma 4.1** *Let* $\mathbf{C}_k$ $(k = 8, 12, 16)$ *be the set of codewords of weight* $k$ *in* $\mathcal{G}_{24}$*, then the followings hold,*
*(i) if* $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$ *then the value of* $\mathbf{u} * \mathbf{v}$ *is one of 0,2,4 and 8,*
*(ii) if* $\mathbf{u} \in \mathbf{C}_8$ *and* $\mathbf{v} \in \mathbf{C}_{12}$ *then the value of* $\mathbf{u} * \mathbf{v}$ *is one of 2,4 and 6,*
*(iii) if* $\mathbf{u} \in \mathbf{C}_8$ *and* $\mathbf{v} \in \mathbf{C}_{16}$ *then the value of* $\mathbf{u} * \mathbf{v}$ *is one of 0,4,6 and 8.*

*Proof.* Proof of (i). If $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$, then from the relation (2.1) we have

$$wt(\mathbf{u} + \mathbf{v}) = 16 - 2 \cdot \mathbf{u} * \mathbf{v}.$$

$\mathbf{u} + \mathbf{v}$ is a codeword of $\mathcal{G}_{24}$ and its weight may be one of 0,8,12 and 16. From each possible weight the value of $\mathbf{u} * \mathbf{v}$ is obtained.
The cases (ii) and (iii) are similarly showed.

After Lemma 4.1 we use $I_*$ to denote the set $\{0, 2, 4, 8\}$. The following proposition is easy to prove and we present it without giving the proof.

**Proposition 4.2** *Let* $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$ *then the followings hold*

$$\begin{aligned}
\mathbf{u} * \mathbf{v} = 8 &\quad \Leftrightarrow \quad d(\mathbf{u}, \mathbf{v}) = 0, \\
\mathbf{u} * \mathbf{v} = 0 &\quad \Leftrightarrow \quad d(\mathbf{u}, \mathbf{v}) = 16, \\
\mathbf{u} * \mathbf{v} = 2 &\quad \Leftrightarrow \quad d(\mathbf{u}, \mathbf{v}) = 12, \\
\mathbf{u} * \mathbf{v} = 4 &\quad \Leftrightarrow \quad d(\mathbf{u}, \mathbf{v}) = 8.
\end{aligned}$$

We will use $\#U$ to denote the cardinality of a finite set $U$.

**Lemma 4.3** *Let* $\mathbf{v}$ *be a fixed codeword of weight 8 in* $\mathcal{G}_{24}$*, then we have*

$$\mathbf{u} * \mathbf{v} = \begin{cases} 8 & \textit{for one } \mathbf{u} \in \mathbf{C}_8, \\ 0 & \textit{for } 30\ \mathbf{u} \in \mathbf{C}_8, \\ 2 & \textit{for } 448\ \mathbf{u} \in \mathbf{C}_8, \\ 4 & \textit{for } 280\ \mathbf{u} \in \mathbf{C}_8. \end{cases}$$

*Proof.* Viewing Lemma 4.1 we may put $\lambda_i = \#\{\mathbf{u} \in \mathbf{C}_8 \mid \mathbf{u} * \mathbf{v} = i\}$ for $i = 8, 0, 2, 4$. It is obvious that $\lambda_8 = 1$. By the equation (3.1) we get a linear equation:

$$2\lambda_2 + 4\lambda_4 + 8\lambda_8 = 253 \times 8.$$

By the equation (3.2) we get another linear equation:

$$2^2\lambda_2 + 4^2\lambda_4 + 8^2\lambda_8 = 77 \times 8^2 + 176 \times 8.$$

Both equations are enough to solve $\lambda_4$ and $\lambda_2$, and the solutions are

$$\lambda_2 = 448, \lambda_4 = 280.$$

To obtain $\lambda_0$ one remarks that

$$\lambda_8 + \lambda_0 + \lambda_2 + \lambda_4 = |C_8| = 759,$$

from which $\lambda_0 = 30$ follows.

Here we consider the set $\mathbf{C}_8$. We define relations $\mathcal{P} = \{P_8, P_0, P_2, P_4\}$ between the elements of $\mathbf{C}_8$. First a pair $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$ belong to $P_j,\ j = 0, 2, 4, 8$ if and only if $\mathbf{u} * \mathbf{v} = j$. The condition $\mathbf{u} * \mathbf{v} = 8$ implies that $\mathbf{u} = \mathbf{v}$, hence $P_8 = \{\langle \mathbf{u}, \mathbf{u}\rangle | \mathbf{u} \in \mathbf{C}_8\}$ holds.
For $i, j, k \in I_*$ we define $\Lambda_{i,j}(k)$ and $\lambda_{i,j}(k)$ by

$$\begin{aligned} \Lambda_{i,j}(k) &= \{\mathbf{w} \in \mathbf{C}_8 \mid \mathbf{u} * \mathbf{w} = i, \mathbf{v} * \mathbf{w} = j\}, \\ \lambda_{i,j}(k) &= \#\Lambda_{i,j}(k), \end{aligned}$$

where $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$ and they satisfy $\mathbf{u} * \mathbf{v} = k$. One may easily see that the equation

$$\lambda_{i,j}(k) = \lambda_{j,i}(k) \tag{4.1}$$

holds for $i, j \in I_*$, since we can make a one to one correspondence between the two sets $\Lambda_{i,j}(k)$ and $\Lambda_{j,i}(k)$. We prove

**Lemma 4.4** *Let $k$ be an element of $I_*$. For any pair of elements $i, j \in \mathbf{C}_8$ satisfying $\mathbf{u} * \mathbf{v} = k$ it holds that*

$$\begin{aligned} \sum_{j \in I_*} \lambda_{0,j}(k) &= 30, \\ \sum_{j \in I_*} \lambda_{2,j}(k) &= 448, \\ \sum_{j \in I_*} \lambda_{4,j}(k) &= 280, \\ \sum_{j \in I_*} \lambda_{8,j}(k) &= 1. \end{aligned}$$

*Proof.* We see that for a fixed codeword $\mathbf{u} \in \mathbf{C}_8$ the sum $\sum_{j \in I_*} \lambda_{0,j}(k)$ counts all $\mathbf{w} \in \mathbf{C}_8$ satisfying $\mathbf{u} * \mathbf{w} = 0$. By Lemma 4.3 the sum equals 30. Other cases are proved in a similar way.

## 4.3 The Numbers $\lambda_{i,j}(0),\ i,j \in I_*$

First we consider the numbers $\lambda_{i,j}(0),\ i,j \in I_*$.
$\lambda_{0,2}(0) = 0$. The number counts $\mathbf{w}$ satisfying $\mathbf{u} * \mathbf{w} = 0$ and $\mathbf{v} * \mathbf{w} = 2$. But we see that $wt(\mathbf{u} + \mathbf{v} + \mathbf{w}) = 20$ which is impossible in the Golay code $\mathcal{G}_{24}$.
$\lambda_{0,0}(0) = 1$. Because the number counts $\mathbf{w}$ satisfying $\mathbf{u} * \mathbf{w} = 0$ and $\mathbf{v} * \mathbf{w} = 0$. Since $wt(\mathbf{u}+\mathbf{v}) = 16$ and $wt(\mathbf{u}+\mathbf{v}+\mathbf{w}) = 24$. This implies that $\mathbf{w} = \mathbf{u}+\mathbf{v}+\mathbf{1}$, and therefore $\lambda_{0,0}(0) = 1$. We also see that $\lambda_{0,8}(0) = 1$. By Lemma $\sum_{j \in I_*} \lambda_{0,j}(0) = 30$ from which $\lambda_{0,4}(0) = 28$ follows. It is obvious that $\lambda_{2,8}(0) = \lambda_{4,8}(0) = \lambda_{8,8}(0) = 0$.

**Lemma 4.5** *We let* $\mathbf{a} = \mathbf{u} + \mathbf{v}$ *with* $\mathbf{u} * \mathbf{v} = 0$. *Then it holds that*

$$\mathbf{w} * \mathbf{a} = \begin{cases} 2 & \textit{if}\ \mathbf{w} \in \Lambda_{0,2}(0) \cup \Lambda_{2,0}(0), \\ 4 & \textit{if}\ \mathbf{w} \in \Lambda_{0,4}(0) \cup \Lambda_{4,0}(0), \\ 4 & \textit{if}\ \mathbf{w} \in \Lambda_{2,2}(0), \\ 6 & \textit{if}\ \mathbf{w} \in \Lambda_{2,4}(0) \cup \Lambda_{4,2}(0), \\ 8 & \textit{if}\ \mathbf{w} \in \Lambda_{4,4}(0), \\ 8 & \textit{if}\ \mathbf{w} \in \Lambda_{0,8}(0) \cup \Lambda_{8,0}(0). \end{cases}$$

To determine the numbers $\lambda_{2,2}(0), \lambda_{2,4}(0), \lambda_{4,4}(0)$ we use the formulas (3.1)$\sim$(3.5). Take $\mathbf{a} = \mathbf{u} + \mathbf{v}$ and the formula (3.1) with minding of Lemma 4.5 reads

$$\begin{aligned}
&\sum_{\mathbf{w} \in \mathbf{C}_8} (\mathbf{w} * \mathbf{a}) \\
&= \sum_{\mathbf{w} \in \Lambda_{0,2}(0) \cup \Lambda_{2,0}(0)} (\mathbf{w} * \mathbf{a}) + \sum_{\mathbf{w} \in \Lambda_{0,4}(0) \cup \Lambda_{4,0}(0)} (\mathbf{w} * \mathbf{a}) \\
&\quad + \sum_{\mathbf{w} \in \Lambda_{2,2}(0)} (\mathbf{w} * \mathbf{a}) + \sum_{\mathbf{w} \in \Lambda_{2,4}(0) \cup \Lambda_{4,2}(0)} (\mathbf{w} * \mathbf{a}) \\
&\quad + \sum_{\mathbf{w} \in \Lambda_{4,4}(0)} (\mathbf{w} * \mathbf{a}) + \sum_{\mathbf{w} \in \Lambda_{0,8}(0) \cup \Lambda_{8,0}(0)} (\mathbf{w} * \mathbf{a}) \\
&= 2 \cdot 2 \cdot \lambda_{0,2}(0) + 2 \cdot 4 \cdot \lambda_{0,4}(0) + 2 \cdot 8 \cdot \lambda_{0,8}(0) \\
&\quad + 4 \cdot \lambda_{2,2}(0) + 2 \cdot 6 \cdot \lambda_{2,4}(0) + 8 \cdot \lambda_{4,4}(0) \\
&= 2 \cdot 4 \cdot 28 + 2 \cdot 8 + 4 \cdot \lambda_{2,2}(0) + 2 \cdot 6 \cdot \lambda_{2,4}(0) + 8 \cdot \lambda_{4,4}(0) \\
&= 253 \cdot 16.
\end{aligned}$$

The formula (3.2) implies

$$\begin{aligned}
&\sum_{\mathbf{w} \in \mathbf{C}_8} (\mathbf{w} * \mathbf{a})^2 \\
&= 2 \cdot 2^2 \cdot \lambda_{0,2}(0) + 2 \cdot 4^2 \cdot \lambda_{0,4}(0) + 2 \cdot 8^2 \cdot \lambda_{0,8}(0) \\
&\quad + 4^2 \cdot \lambda_{2,2}(0) + 2 \cdot 6^2 \cdot \lambda_{2,4}(0) + 8^2 \cdot \lambda_{4,4}(0) \\
&= 2 \cdot 16 \cdot 28 + 2 \cdot 64 + 16 \cdot \lambda_{2,2}(0) + 2 \cdot 36 \cdot \lambda_{2,4}(0) + 64 \cdot \lambda_{4,4}(0) \\
&= 77 \cdot 16^2 + 176 \cdot 16.
\end{aligned}$$

The formula (3.3) implies

$$
\begin{aligned}
&\sum_{\mathbf{w}\in\mathbf{C}_8} (\mathbf{w}*\mathbf{a})^3 \\
= &\ 2\cdot 2^3\cdot\lambda_{0,2}(0) + 2\cdot 4^3\cdot\lambda_{0,4}(0) + 2\cdot 8^3\cdot\lambda_{0,8}(0) \\
&+4^3\cdot\lambda_{2,2}(0) + 2\cdot 6^3\cdot\lambda_{2,4}(0) + 8^3\cdot\lambda_{4,4}(0) \\
= &\ 2\cdot 64\cdot 28 + 2\cdot 512 + 64\cdot\lambda_{2,2}(0) + 2\cdot 216\cdot\lambda_{2,4}(0) + 512\cdot\lambda_{4,4}(0) \\
= &\ 21\cdot 16^3 + 168\cdot 16^2 + 64\cdot 16.
\end{aligned}
$$

The above three linear equations are enough to determine the numbers $\lambda_{2,2}(0), \lambda_{2,4}(0), \lambda_{4,4}(0)$. Actually we have

$$\lambda_{2,2}(0) = 224, \lambda_{2,4}(0) = 224, \lambda_{4,4}(0) = 28.$$

## 4.4 The Numbers $\lambda_{i,j}(2),\ i, j \in I_*$

Next we treat the numbers $\lambda_{i,j}(2),\ i, j \in I_*$.
We find that this case needs certain subtle analysis in determining the numbers $\lambda_{2,2}(2), \lambda_{2,4}(2), \lambda_{4,4}(2)$. Generally for temporally fixed $\mathbf{u}, \mathbf{v}$ we put

$$
\begin{aligned}
\mathbf{u} &= (u_1, \cdots, u_{24}) \in \mathbf{C}_8, \\
\mathbf{v} &= (v_1, \cdots, v_{24}) \in \mathbf{C}_8, \\
\mathbf{w} &= (w_1, \cdots, w_{24}) \in \mathbf{C}_8, \\
\mathbf{u}*\mathbf{v}*\mathbf{w} &= \#\{h | u_h = v_h = v_h = 1, 1 \le h \le 24\}, \\
\nu_1(\mathbf{w}) &= \#\{h \mid u_h = w_h = 1, v_h = 0, 1 \le h \le 24\}, \\
\nu_2(\mathbf{w}) &= \#\{h \mid u_h = 0, w_h = v_h = 1, 1 \le h \le 24\}, \\
\Lambda^{(s)}_{i,j}(2) &= \{\mathbf{w} \in \mathbf{C}_8 \mid \mathbf{u}*\mathbf{w} = i, \mathbf{v}*\mathbf{w} = j, \mathbf{u}*\mathbf{v} = 2, \mathbf{u}*\mathbf{v}*\mathbf{w} = s\}, \\
\lambda^{(s)}_{i,j}(2) &= \#\Lambda^{(s)}_{i,j}(2),
\end{aligned}
$$

where $0 \le s \le \mathbf{u}*\mathbf{v} = 2$. By the definition we have $\lambda^{(0)}_{i,j}(2) + \lambda^{(1)}_{i,j}(2) + \lambda^{(2)}_{i,j}(2) = \lambda_{i,j}(2)$. It is easy to observe that $\lambda^{(2)}_{0,2}(2) = \lambda^{(1)}_{0,2}(2) = 0$ and $\lambda^{(0)}_{0,2}(2) = \lambda_{0,2}(2)$, and $\lambda^{(2)}_{0,4}(2) = \lambda^{(1)}_{0,4}(2) = 0$, $\lambda^{(0)}_{0,4}(2) = \lambda_{0,4}(2)$. We treat the rather easier cases:

**Lemma 4.6** *Let the notations be as above, then it holds that*
*(i)* $\lambda_{0,0}(2) = 0, \lambda^{(2)}_{2,2}(2) = 0, \lambda^{(2)}_{2,8}(2) = 1, \lambda^{(0)}_{4,4}(2) = 0,$
*(ii)* $\lambda^{(0)}_{2,4}(2) = \lambda^{(2)}_{4,4}(2),$
*(iii)* $\lambda^{(2)}_{2,4}(2) = \lambda_{0,4}(2).$

*Proof.* Proof of (i). Suppose there is $\mathbf{w} \in \mathbf{C}_8$ satisfying $\mathbf{w}*\mathbf{u} = 0, \mathbf{w}*\mathbf{v} = 0$ whereas $\mathbf{u}*\mathbf{v} = 2$. Then it follows from these conditions that $wt(\mathbf{u}+\mathbf{v}+\mathbf{w}) = 20,$

which is impossible in $\mathcal{G}_{24}$. $\lambda^{(2)}_{2,2}(2)$ is the number of $\mathbf{w} \in \mathbf{C}_8$ satisfying $\mathbf{u} * \mathbf{w} = 2, \mathbf{v} * \mathbf{w} = 2, \mathbf{u} * \mathbf{v} = 2, \mathbf{u} * \mathbf{v} * \mathbf{w} = 2$. Then $wt(\mathbf{u} + \mathbf{v} + \mathbf{w}) = 20$, which is impossible in the Golay code. $\lambda^{(0)}_{4,4}(2)$ is the number of $\mathbf{w} \in \mathbf{C}_8$ satisfying $\mathbf{u} * \mathbf{w} = 4, \mathbf{v} * \mathbf{4} = 2, \mathbf{u} * \mathbf{v} = 2, \mathbf{u} * \mathbf{v} * \mathbf{w} = 0$. The conditions imply that $wt(\mathbf{u} + \mathbf{v} + \mathbf{w}) = 4$, which is impossible in the Golay code.

Proof of (ii). Let $\mathbf{w} \in \Lambda^{(2)}_{4,4}(2)$, then by definition $\mathbf{w}$ satisfies $\mathbf{u} * \mathbf{w} = 4, \mathbf{v} * \mathbf{w} = 4, \mathbf{u} * \mathbf{v} = 2, \mathbf{u} * \mathbf{v} * \mathbf{w} = 2$. Then we verify that $\mathbf{w} + \mathbf{v} \in \Lambda^{(0)}_{2,4}(2)$. Conversely when $\mathbf{w} \in \Lambda^{(0)}_{2,4}(2)$, then it is verified that $\mathbf{w} + \mathbf{v} \in \Lambda^{(2)}_{4,4}(2)$. Both mappings are injective. Thus we have $\lambda^{(0)}_{2,4}(2) = \lambda^{(2)}_{4,4}(2)$.

Proof (iii). When $\mathbf{w} \in \Lambda^{(2)}_{2,4}(2)$ then it holds that $\mathbf{u} * \mathbf{w} = 2, \mathbf{v} * \mathbf{w} = 4, \mathbf{u} * \mathbf{v} = 2, \mathbf{u} * \mathbf{v} * \mathbf{w} = 2$. We see that $\mathbf{w} + \mathbf{v} \in \Lambda_{0,4}(2)$. Conversely when $\mathbf{w} \in \Lambda_{0,4}(2)$, then we see that $\mathbf{w} + \mathbf{v} \in \Lambda^{(2)}_{2,4}(2)$, and $\lambda^{(2)}_{2,4}(2) = \lambda_{0,4}(2)$ holds.

Let $\mathbf{u} = (u_1, \cdots, u_{24}), \mathbf{v} = (v_1, \cdots, v_{24}) \in \mathbf{C}_8$ satisfying $\mathbf{u} * \mathbf{v} = 2$. We consider a vector $\mathbf{u} \cup \mathbf{v} = (t_1, t_2, \cdots, t_{24})$ defined by $t_i = 1$ if $u_i = 1$ or $v_i = 1$, and $t_i = 0$ if $u_i = v_i = 0$ for $1 \leq i \leq 24$. Note that $wt(\mathbf{u} \cup \mathbf{v}) = 14$.

**Lemma 4.7** *We let $\mathbf{a} = \mathbf{u} \cup \mathbf{v}$ with $\mathbf{u} * \mathbf{v} = 2$. Then it holds that*

$$\mathbf{w}*\mathbf{a} = \begin{cases} 4 & \text{if } \mathbf{w} \in \Lambda^{(0)}_{2,2}(2), \\ 3 & \text{if } \mathbf{w} \in \Lambda^{(1)}_{2,2}(2), \\ 2 & \text{if } \mathbf{w} \in \Lambda^{(2)}_{2,2}(2), \end{cases} \quad \mathbf{w}*\mathbf{a} = \begin{cases} 6 & \text{if } \mathbf{w} \in \Lambda^{(0)}_{2,4}(2), \\ 5 & \text{if } \mathbf{w} \in \Lambda^{(1)}_{2,4}(2), \\ 4 & \text{if } \mathbf{w} \in \Lambda^{(2)}_{2,4}(2), \end{cases} \quad \mathbf{w}*\mathbf{a} = \begin{cases} 8 & \text{if } \mathbf{w} \in \Lambda^{(0)}_{4,4}(2), \\ 7 & \text{if } \mathbf{w} \in \Lambda^{(1)}_{4,4}(2), \\ 6 & \text{if } \mathbf{w} \in \Lambda^{(2)}_{4,4}(2). \end{cases}.$$

*Proof.* Let $\mathbf{w} \in \Lambda^{(s)}_{i,j}(2)$ $s = 0, 1, 2$. Then using $\nu_1, \nu_2$ introduced above we have $\mathbf{w} * \mathbf{a} = \nu_1 + s + \nu_2 = i - s + s + j - s = i + j - s$. All the nine equalities in the lemma follow from this.

**Proposition 4.8** *Let $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$ satisfying $\mathbf{u} * \mathbf{v} = 2$. Let $\lambda^{(s)}_{i,j}(2)$, where $i, j \in I_*, 0 \leq s \leq$ defined above. Then we have*

$$\begin{aligned} &\lambda_{0,0}(2) = 0, \lambda_{0,2}(2) = 15, \lambda_{0,4}(2) = 15, \lambda_{0,8}(2) = 0, \\ &\lambda_{2,0}(2) = 15, \lambda^{(0)}_{2,2}(2) = 180, \lambda^{(1)}_{2,2}(2) = 72, \lambda^{(2)}_{2,2}(2) = 0, \\ &\lambda_{4,0}(2) = 15, \lambda^{(0)}_{2,4}(2) = 45, \lambda^{(1)}_{2,4}(2) = 120, \lambda^{(2)}_{2,4}(2) = 15, \\ &\lambda_{8,0}(2) = 0, \lambda^{(0)}_{4,4}(2) = 0, \lambda^{(1)}_{4,4}(2) = 40, \lambda^{(2)}_{4,4}(2) = 45. \end{aligned}$$

*Proof.* We use the vector $\mathbf{a} = \mathbf{u} \cup \mathbf{v}$. This time $wt(\mathbf{a}) = 14$. To determine $\lambda^{(s)}_{i,j}(2)$'s we utilize the formulas (3.1)$\sim$(3.5). In viewing Lemma 4.7 the formula

(3.1) reads

$$
\begin{aligned}
&\sum_{\mathbf{w}\in \mathbf{C}_8} (\mathbf{w} * \mathbf{a}) \\
= \quad & \sum_{\mathbf{w}\in \Lambda_{0,2}(2)\cup\Lambda_{2,0}(2)} (\mathbf{w} * \mathbf{a}) + \sum_{\mathbf{w}\in \Lambda_{0,4}(2)\cup\Lambda_{4,0}(2)} (\mathbf{w} * \mathbf{a}) \\
& + \sum_{\mathbf{w}\in \Lambda_{2,2}^{(0)}(2)} (\mathbf{w} * \mathbf{a}) + \sum_{\mathbf{w}\in \Lambda_{2,2}^{(1)}(2)} (\mathbf{w} * \mathbf{a}) + \sum_{\mathbf{w}\in \Lambda_{2,2}^{(2)}(2)} (\mathbf{w} * \mathbf{a}) \\
& + \sum_{\mathbf{w}\in \Lambda_{2,4}^{(0)}(2)\cup\Lambda_{4,2}^{(0)}(2)} (\mathbf{w} * \mathbf{a}) + \sum_{\mathbf{w}\in \Lambda_{2,4}^{(1)}(2)\cup\Lambda_{4,2}^{(1)}(2)} (\mathbf{w} * \mathbf{a}) + \sum_{\mathbf{w}\in \Lambda_{2,4}^{(2)}(2)\cup\Lambda_{4,2}^{(2)}(2)} (\mathbf{w} * \mathbf{a}) \\
& + \sum_{\mathbf{w}\in \Lambda_{4,4}^{(0)}(2)} (\mathbf{w} * \mathbf{a}) + \sum_{\mathbf{w}\in \Lambda_{4,4}^{(1)}(2)} (\mathbf{w} * \mathbf{a}) + \sum_{\mathbf{w}\in \Lambda_{4,4}^{(2)}(2)} (\mathbf{w} * \mathbf{a}) \\
& + \sum_{\mathbf{w}\in \Lambda_{2,8}(2)\cup\Lambda_{8,2}(2)} (\mathbf{w} * \mathbf{a}) \\
= \quad & 2\cdot 2\cdot \lambda_{0,2}(2) + 2\cdot 4\cdot \lambda_{0,4}(2) \\
& +4\cdot \lambda_{2,2}^{(0)}(2) + 3\cdot \lambda_{2,2}^{(1)}(2) + 2\cdot \lambda_{2,2}^{(2)}(2) \\
& +2\cdot 6\cdot \lambda_{2,4}^{(0)}(2) + 2\cdot 5\cdot \lambda_{2,4}^{(1)}(2) + 2\cdot 4\cdot \lambda_{2,4}^{(2)}(2) + 8\cdot \lambda_{4,4}^{(0)}(2) \\
& +7\cdot \lambda_{4,4}^{(1)}(2) + 6\cdot \lambda_{4,4}^{(2)}(2) + \\
& +2\cdot 8\cdot \lambda_{2,8}(2) \\
= \quad & 253\cdot 14.
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\mathbf{w}\in \mathbf{C}_8} (\mathbf{w} * \mathbf{a})^t \\
= \quad & 2\cdot 2^t\cdot \lambda_{0,2}(2) + 2\cdot 4^t\cdot \lambda_{0,4}(2) \\
& +4^t\cdot \lambda_{2,2}^{(0)}(2) + 3^t\cdot \lambda_{2,2}^{(1)}(2) + 2^t\cdot \lambda_{2,2}^{(2)}(2) \\
& +2\cdot 6^t\cdot \lambda_{2,4}^{(0)}(2) + 2\cdot 5^t\cdot \lambda_{2,4}^{(1)}(2) + 2\cdot 4^t\cdot \lambda_{2,4}^{(2)}(2) + 8^t\cdot \lambda_{4,4}^{(0)}(2) \\
& +7^t\cdot \lambda_{4,4}^{(1)}(2) + 6^t\cdot \lambda_{4,4}^{(2)}(2) + \\
& +2\cdot 8^t\cdot \lambda_{2,8}(2) \\
= \quad & f_t(\mathbf{a} * \mathbf{a}),
\end{aligned}
$$

where $f_t(\mathbf{a} * \mathbf{a})$ $(2 \le t \le 5)$ is the right hand side of the formulas (3.2)$\sim$(3.5). The above five linear conditions on $\lambda_{0,2}(2), \lambda_{0,4}(2), \cdots, \lambda_{2,8}(2)$ together with Lemma 4.6 are enough to determine the values of $\lambda_{0,2}(2), \lambda_{0,4}(2), \cdots, \lambda_{2,8}(2)$ explicitly.

## 4.5 The Numbers $\lambda_{i,j}(4),\ i,j \in I_*$

Finally we treat the numbers $\lambda_{i,j}(4),\ i,j \in I_*$.

$$\begin{aligned}
\Lambda_{i,j}^{(s)}(4) &= \{\mathbf{w} \in \mathbf{C}_8 \mid \mathbf{u} * \mathbf{w} = i, \mathbf{v} * \mathbf{w} = j, \mathbf{u} * \mathbf{v} = 4, \mathbf{u} * \mathbf{v} * \mathbf{w} = s\}, \\
\lambda_{i,j}^{(s)}(4) &= \#\Lambda_{i,j}^{(s)}(4),
\end{aligned}$$

where $0 \leq s \leq \mathbf{u} * \mathbf{v} = 4$ and $\mathbf{u} * \mathbf{v} * \mathbf{w}$ is defined already.

**Lemma 4.9** *Let the notations be as above, then it holds that*
*(i)* $\lambda_{2,2}^{(3)}(4) = \lambda_{2,2}^{(4)}(4) = 0, \lambda_{2,4}^{(0)}(4) = \lambda_{2,4}^{(3)}(4) = \lambda_{2,4}^{(4)}(4) = 0,$
*(ii)* $\lambda_{4,4}^{(0)}(4) = 1,\ \lambda_{4,4}^{(1)}(4) = 0,$
*(iii)* $\lambda_{4,8}^{(0)}(4) = \lambda_{4,8}^{(1)}(4) = \lambda_{4,8}^{(2)}(4) = \lambda_{4,8}^{(3)}(4) = 0, \lambda_{4,8}^{(4)}(4) = 1.$

*Proof.* Proof of (i). Since $s = \mathbf{u}*\mathbf{v}*\mathbf{w}$ can not be greater than $\mathbf{u}*\mathbf{v}, \mathbf{u}*\mathbf{w}, \mathbf{v}*\mathbf{w}$ we have evidently $\lambda_{2,2}^{(3)}(4) = 0$ and $\lambda_{2,2}^{(4)}(4) = 0$. By the same reason we have $\lambda_{2,4}^{(3)}(4) = \lambda_{2,4}^{(4)}(4) = 0$. As to $\lambda_{2,4}^{(0)}(4)$ we consider $wt(\mathbf{u} + \mathbf{v} + \mathbf{w})$. It is clear that $wt(\mathbf{u}+\mathbf{v}+\mathbf{w}) = 4$, which is impossible in the binary Golay code. Thus we conclude that $\lambda_{2,4}^{(0)}(4) = 0$.
Proof of (ii). The number $\lambda_{4,4}^{(0)}(4)$ counts $\mathbf{w} \in \mathbf{C}_8$ satisfying $\mathbf{u} * \mathbf{w} = \mathbf{v} * \mathbf{w} = 4, \mathbf{u} * \mathbf{v} * \mathbf{w} = 0$ under the restriction $\mathbf{u} * \mathbf{v} = 4$. In this case we have $wt(\mathbf{u}+\mathbf{v}+\mathbf{w}) = 0$, meaning that $\mathbf{w} = \mathbf{u} + \mathbf{v}$ and $\lambda_{4,4}^{(0)}(4) = 1$.
As to $\lambda_{4,4}^{(1)}(4)$ we again consider $wt(\mathbf{u} + \mathbf{v} + \mathbf{w})$. We see that $wt(\mathbf{u} + \mathbf{v} + \mathbf{w}) = 4$, which is also impossible in the binary Golay code, and $\lambda_{4,4}^{(1)}(4) = 0$.
Proof of (iii). All numbers in (iii) imply that $\mathbf{v} * \mathbf{w} = 8$ and $\mathbf{v} = \mathbf{w}$. Since $\mathbf{u} * \mathbf{v} = 4$, it holds that $\mathbf{u} * \mathbf{v} * \mathbf{w} = 4$, and consequently $\lambda_{4,8}^{(0)}(4) = \lambda_{4,8}^{(1)}(4) = \lambda_{4,8}^{(2)}(4) = \lambda_{4,8}^{(3)}(4) = 0, \lambda_{4,8}^{(4)}(4) = 1$.

We want to prove that $\lambda_{i,j}^{(s)}(4)$ $(0 \leq s \leq 4)$ is constant for each fixed pair of $i,j \in I_*$, but we find that this can not be done straightforwardly even if we use the formulas (3.1)$\sim$(3.6) together with Lemma 4.3. The reader may understand this difficulty by viewing the proof of Proposition 4.12. To overcome this difficulty we make a more precise notation, that will work actually. Instead of $\Lambda_{i,j}^{(s)}(4)$ we use $\Lambda_{i,j}^{(s)}(4)(\mathbf{u}, \mathbf{v})$. The implications are the same but we are conscious about which pair $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$ satisfying $\mathbf{u} * \mathbf{v} = 4$ is used. A pair $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathbf{C}_8^2$ belongs to $\mathcal{L}_4$ if the condition $\mathbf{u} * \mathbf{v} = 4$ holds. It is easy to see that if $\langle \mathbf{u}, \mathbf{v} \rangle \in \mathcal{L}_4$ then so are $\langle \mathbf{u}, \mathbf{u} + \mathbf{v} \rangle, \langle \mathbf{u} + \mathbf{v}, \mathbf{v} \rangle$. With these facts we prove

**Lemma 4.10** *We employ the notations before, then it holds that*
*(i)* $\lambda_{2,2}^{(0)}(4)(\mathbf{u}, \mathbf{v}) = \lambda_{2,4}^{(2)}(4)(\mathbf{u}, \mathbf{u} + \mathbf{v}),$

*(ii)* $\lambda^{(2)}_{2,4}(4)(\mathbf{u},\mathbf{v}) = \lambda^{(0)}_{2,2}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})$,
*(iii)* $\lambda^{(3)}_{4,4}(4)(\mathbf{u},\mathbf{v}) = \lambda^{(1)}_{2,4}(4)(\mathbf{u},\mathbf{v})$,
*(iv)* $\lambda^{(4)}_{4,4}(4)(\mathbf{u},\mathbf{v}) = \lambda_{0,4}(4)(\mathbf{u},\mathbf{v})$.

*Proof.* Proof of (i). Let $\mathbf{w} \in \Lambda^{(0)}_{2,2}(4)(\mathbf{u},\mathbf{v})$, then by definition it holds that $\mathbf{u}*\mathbf{w} = 2, \mathbf{u}*\mathbf{v}*\mathbf{w} = 0, \mathbf{v}*\mathbf{w} = 2$. From these conditions we see that $\mathbf{u}*\mathbf{w} = 2, (\mathbf{u}+\mathbf{v})*\mathbf{w} = 4, \mathbf{u}*(\mathbf{u}+\mathbf{v})*\mathbf{w} = 2$ hold. This implies that $\mathbf{w} \in \Lambda^{(2)}_{2,4}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})$. Conversely from $\mathbf{w} \in \Lambda^{(2)}_{2,4}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})$ we can derive the consequence $\mathbf{w} \in \Lambda^{(0)}_{2,2}(4)(\mathbf{u},\mathbf{v})$. Thus $\Lambda^{(0)}_{2,2}(4)(\mathbf{u},\mathbf{v}) = \Lambda^{(2)}_{2,4}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})$, and $\lambda^{(0)}_{2,2}(4)(\mathbf{u},\mathbf{v}) = \lambda^{(2)}_{2,4}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})$.
Proof of (ii). Let $\mathbf{w} \in \Lambda^{(2)}_{2,4}(4)(\mathbf{u},\mathbf{v})$, then it holds that $\mathbf{u}*\mathbf{w} = \mathbf{u}*\mathbf{v}*\mathbf{w} = 2, \mathbf{v}*\mathbf{w} = 4$. By these conditions we see that $\mathbf{u}*\mathbf{w} = 2, (\mathbf{u}+\mathbf{v})*\mathbf{w} = 2, \mathbf{u}*(\mathbf{u}+\mathbf{v})*\mathbf{w} = 0$, and $\mathbf{w} \in \Lambda^{(0)}_{2,2}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})$. Conversely if $\mathbf{w} \in \Lambda^{(0)}_{2,2}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})$ then we can show that $\mathbf{w} \in \Lambda^{(2)}_{2,4}(4)(\mathbf{u},\mathbf{v})$. Thus we have $\lambda^{(2)}_{2,4}(4)(\mathbf{u},\mathbf{v}) = \lambda^{(0)}_{2,2}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})$.
Proof of (iii). We prove $\lambda^{(3)}_{4,4}(4)(\mathbf{u},\mathbf{v}) = \lambda^{(1)}_{2,4}(4)(\mathbf{u},\mathbf{v})$. Let $\mathbf{w} \in \Lambda^{(3)}_{4,4}(4)(\mathbf{u},\mathbf{v})$, then it holds that $\mathbf{u}*\mathbf{w} = \mathbf{v}*\mathbf{w} = 4, \mathbf{u}*\mathbf{v}*\mathbf{w} = 3, \mathbf{u}*\mathbf{v} = 4$. By these conditions we see that $\#\{h|\ u_h = 1, v_h = w_h = 0\} = 3, \#\{h|u_h = w_h = 1, v_h = 0\} = 1, \#\{h|u_h = v_h = w_h = 1\} = 3, \#\{h|\ u_h = v_h = 1, w_h = 0\} = 1, \#\{h|\ v_h = w_h = 1, u_h = 0\} = 1, \#\{h|\ u_h = 0, v_h = 1, w_h = 0\} = 3, \#\{h|\ u_h = v_h = 0, w_h = 1\} = 3, \#\{h|\ u_h = v_h = w_h = 0\} = 9$ holds. Using these relations we observe that $\mathbf{v}+\mathbf{w} \in \Lambda^{(1)}_{2,4}(4)$, and the correspondence $\mathbf{w} \leftrightarrow \mathbf{v}+\mathbf{w}$ is a one to one correspondence between two sets $\Lambda^{(3)}_{4,4}(4)$ and $\Lambda^{(1)}_{2,4}(4)$. Hence we get $\lambda^{(3)}_{4,4}(4) = \lambda^{(1)}_{2,4}(4)$.
The proof of $\lambda_{0,4}(4) = \lambda^{(4)}_{4,4}(4)$ is quite similar to that of $\lambda^{(3)}_{4,4}(4) = \lambda^{(1)}_{2,4}(4)$. We only point out that $\Lambda^{(4)}_{4,4}(4) \ni \mathbf{w} \leftrightarrow \mathbf{v}+\mathbf{w} \in \Lambda_{0,4}(4)$ is a one to one correspondence. And we omit the details.

**Lemma 4.11** *We let* $\mathbf{a} = \mathbf{u}\cup\mathbf{v}$ *with* $\mathbf{u}*\mathbf{v} = 4$. *Then it holds that*

$$\mathbf{w}*\mathbf{a} = \begin{cases} 4 & \textit{if}\ \mathbf{w} \in \Lambda^{(0)}_{2,2}(4)(\mathbf{u},\mathbf{v}) \\ 3 & \textit{if}\ \mathbf{w} \in \Lambda^{(1)}_{2,2}(4)(\mathbf{u},\mathbf{v}) \\ 2 & \textit{if}\ \mathbf{w} \in \Lambda^{(2)}_{2,2}(4)(\mathbf{u},\mathbf{v}) \end{cases}, \quad \mathbf{w}*\mathbf{a} = \begin{cases} 6 & \textit{if}\ \mathbf{w} \in \Lambda^{(0)}_{2,4}(4)(\mathbf{u},\mathbf{v}) \\ 5 & \textit{if}\ \mathbf{w} \in \Lambda^{(1)}_{2,4}(4)(\mathbf{u},\mathbf{v}) \\ 4 & \textit{if}\ \mathbf{w} \in \Lambda^{(2)}_{2,4}(4)(\mathbf{u},\mathbf{v}) \end{cases},$$

$$\mathbf{w}*\mathbf{a} = \begin{cases} 8 & \textit{if}\ \mathbf{w} \in \Lambda^{(0)}_{4,4}(4)(\mathbf{u},\mathbf{v}) \\ 7 & \textit{if}\ \mathbf{w} \in \Lambda^{(1)}_{4,4}(4)(\mathbf{u},\mathbf{v}) \\ 6 & \textit{if}\ \mathbf{w} \in \Lambda^{(2)}_{4,4}(4)(\mathbf{u},\mathbf{v}) \\ 5 & \textit{if}\ \mathbf{w} \in \Lambda^{(3)}_{4,4}(4)(\mathbf{u},\mathbf{v}) \\ 4 & \textit{if}\ \mathbf{w} \in \Lambda^{(4)}_{4,4}(4)(\mathbf{u},\mathbf{v}) \end{cases}.$$

*Proof.* Proofs of first six equations are very similar to those of Lemma 4.7 and we omit them. Proof of last 5 equations. Let $\mathbf{w} \in \Lambda^{(s)}_{4,4}(2)\ 0 \le s \le 4$, then as in Lemma 4.7 we see that $4 = \nu_1 + s, 4 = s + \nu_2$ and $\mathbf{w}*\mathbf{a} = \nu_1 + s + \nu_2 = 4 - s + s + 4 - s = 8 - s$. From this the last 5 equations follow.

**Proposition 4.12** *Let* $\mathbf{u},\mathbf{v} \in \mathbf{C}_8$ *satisfying* $\mathbf{u} * \mathbf{v} = 4$. *Let* $\lambda_{i,j}^{(s)}(4)$, *where* $i, j \in I_*, 0 \le s \le 4$ *defined above. Then we have*

$$\begin{aligned}
&\lambda_{0,0}(4)(\mathbf{u},\mathbf{v}) = 3, \lambda_{0,2}(4)(\mathbf{u},\mathbf{v}) = 24, \lambda_{0,4}(4)(\mathbf{u},\mathbf{v}) = 3, \lambda_{0,8}(4)(\mathbf{u},\mathbf{v}) = 0,\\
&\lambda_{2,0}(4)(\mathbf{u},\mathbf{v}) = 24, \lambda_{2,2}^{(0)}(4)(\mathbf{u},\mathbf{v}) = 72, \lambda_{2,2}^{(1)}(4)(\mathbf{u},\mathbf{v}) = 192, \lambda_{2,2}^{(2)}(4)(\mathbf{u},\mathbf{v}) = 24,\\
&\lambda_{2,2}^{(3)}(4)(\mathbf{u},\mathbf{v}) = 0, \lambda_{2,2}^{(4)}(4)(\mathbf{u},\mathbf{v}) = 0,\\
&\lambda_{4,0}(4)(\mathbf{u},\mathbf{v}) = 3, \lambda_{2,4}^{(0)}(4)(\mathbf{u},\mathbf{v}) = 0, \lambda_{2,4}^{(1)}(4)(\mathbf{u},\mathbf{v}) = 64, \lambda_{2,4}^{(2)}(4)(\mathbf{u},\mathbf{v}) = 72,\\
&\lambda_{2,4}^{(3)}(4)(\mathbf{u},\mathbf{v}) = 0, \lambda_{2,4}^{(4)}(4)(\mathbf{u},\mathbf{v}) = 0, \lambda_{8,0}(4) = 0,\\
&\lambda_{4,4}^{(0)}(4)(\mathbf{u},\mathbf{v}) = 1, \lambda_{4,4}^{(1)}(4)(\mathbf{u},\mathbf{v}) = 0, \lambda_{4,4}^{(2)}(4)(\mathbf{u},\mathbf{v}) = 72, \lambda_{4,4}^{(3)}(4)(\mathbf{u},\mathbf{v}) = 64, \lambda_{4,4}^{(4)}(4)(\mathbf{u},\mathbf{v}) = 3.
\end{aligned}$$

*Proof.* We use the vector $\mathbf{a} = \mathbf{u} \cup \mathbf{v}$. This time $wt(\mathbf{a}) = 12$. To determine $\lambda_{i,j}^{(s)}(4)$'s we utilize the formulas (3.1)$\sim$(3.5). In viewing Lemma 4.11 the formulas (3.1)$\sim$(3.5) go

$$\begin{aligned}
&\sum_{\mathbf{w}\in\mathbf{C}_8} (\mathbf{w}*\mathbf{a})^t\\
=\;& \sum_{\mathbf{w}\in\Lambda_{0,2}(4)\cup\Lambda_{2,0}(4)} (\mathbf{w}*\mathbf{a})^t + \sum_{\mathbf{w}\in\Lambda_{0,4}(4)\cup\Lambda_{4,0}(4)} (\mathbf{w}*\mathbf{a})^t\\
&+ \sum_{\mathbf{w}\in\Lambda_{2,2}^{(0)}(4)} (\mathbf{w}*\mathbf{a})^t + \sum_{\mathbf{w}\in\Lambda_{2,2}^{(1)}(4)} (\mathbf{w}*\mathbf{a})^t + \sum_{\mathbf{w}\in\Lambda_{2,2}^{(2)}(4)} (\mathbf{w}*\mathbf{a})^t\\
&+ \sum_{\mathbf{w}\in\Lambda_{2,4}^{(0)}(4)\cup\Lambda_{4,2}^{(0)}(4)} (\mathbf{w}*\mathbf{a})^t + \sum_{\mathbf{w}\in\Lambda_{2,4}^{(1)}(4)\cup\Lambda_{4,2}^{(1)}(4)} (\mathbf{w}*\mathbf{a})^t + \sum_{\mathbf{w}\in\Lambda_{2,4}^{(2)}(4)\cup\Lambda_{4,2}^{(2)}(4)} (\mathbf{w}*\mathbf{a})^t\\
&+ \sum_{\mathbf{w}\in\Lambda_{4,4}^{(0)}(4)} (\mathbf{w}*\mathbf{a})^t + \sum_{\mathbf{w}\in\Lambda_{4,4}^{(1)}(4)} (\mathbf{w}*\mathbf{a})^t + \sum_{\mathbf{w}\in\Lambda_{4,4}^{(2)}(4)} (\mathbf{w}*\mathbf{a})^t + \sum_{\mathbf{w}\in\Lambda_{4,4}^{(3)}(4)} (\mathbf{w}*\mathbf{a})^t\\
&+ \sum_{\mathbf{w}\in\Lambda_{4,4}^{(4)}(4)} (\mathbf{w}*\mathbf{a})^t + \sum_{\mathbf{w}\in\Lambda_{4,8}(4)\cup\Lambda_{8,4}(4)} (\mathbf{w}*\mathbf{a})^t\\
=\;& 2\cdot 2^t\cdot\lambda_{0,2}(4) + 2\cdot 4^t\cdot\lambda_{0,4}(4)\\
&+4^t\cdot\lambda_{2,2}^{(0)}(4) + 3^t\cdot\lambda_{2,2}^{(1)}(4) + 2^t\cdot\lambda_{2,2}^{(2)}(4)\\
&+2\cdot 6^t\cdot\lambda_{2,4}^{(0)}(4) + 2\cdot 5^t\cdot\lambda_{2,4}^{(1)}(4) + 2\cdot 4^t\cdot\lambda_{2,4}^{(2)}(4) + 8^t\cdot\lambda_{4,4}^{(0)}(4)\\
&+7^t\cdot\lambda_{4,4}^{(1)}(4) + 6^t\cdot\lambda_{4,4}^{(2)}(4) + 5^t\cdot\lambda_{4,4}^{(3)}(4)\\
&+4^t\cdot\lambda_{4,4}^{(4)}(4) + 2\cdot 8^t\cdot\lambda_{4,8}(4)\\
=\;& f_t(\mathbf{a}*\mathbf{a}),
\end{aligned}$$

where $f_t(\mathbf{a}*\mathbf{a})$ $(1 \le t \le 5)$ is the right handsides of the formulas (3.1)$\sim$(3.5) with $(\mathbf{a}*\mathbf{a}) = 12$.
In viewing Lemma 4.9 and Lemma 4.10,(iv),(v) the above five equations are sim-

plified to

$$
\text{(4.2)} \qquad \begin{cases} \lambda^{(1)}_{2,2}(4) &=& 192, \\ \lambda^{(1)}_{2,4}(4) &=& 64, \\ \lambda^{(2)}_{4,4}(4) &=& 72, \\ \lambda^{(2)}_{2,2}(4) &=& 72 - 2\lambda_{0,2}(4), \\ \lambda^{(0)}_{2,2}(4) &=& 225 - 3\lambda_{0,4}(4) - 2\lambda^{(2)}_{2,4}(4). \end{cases}
$$

The formula (3.6) unfortunately does not give additional linear condition. Here we use Lemma 4.4 to decrease the number of the unknowns:

$$
\begin{array}{rl} \multicolumn{2}{l}{\lambda_{2,0}(4) + \lambda_{2,2}(4) + \lambda_{2,4}(4) + \lambda_{2,8}(4)} \\ = & \lambda_{0,2}(4) + \lambda^{(0)}_{2,2}(4) + \lambda^{(1)}_{2,2}(4) + \lambda^{(2)}_{2,2}(4) \\ & +\lambda^{(0)}_{2,4}(4) + \lambda^{(1)}_{2,4}(4) + \lambda^{(2)}_{2,4}(4) \\ = & 448, \\ \multicolumn{2}{l}{\lambda_{4,0}(4) + \lambda_{4,2}(4) + \lambda_{4,4}(4) + \lambda_{4,8}(4)} \\ = & \lambda_{0,4}(4) + \lambda^{(0)}_{2,4}(4) + \lambda^{(1)}_{2,4}(4) + \lambda^{(2)}_{2,4}(4) \\ & +\lambda^{(0)}_{4,4}(4) + \lambda^{(1)}_{4,4}(4) + \lambda^{(2)}_{4,4}(4) \\ & +\lambda^{(3)}_{4,4}(4) + \lambda^{(4)}_{4,4}(4) + \lambda^{(4)}_{4,8}(4) \\ = & 280. \end{array}
$$

Substituting the known values and the identities (4.2) into the above two linear equations and rearranging the terms we have

$$
\begin{array}{rcl} \lambda_{0,2}(4) &=& 27 - \lambda_{0,4}(4), \\ \lambda^{(0)}_{2,2}(4) &=& 69 + \lambda_{0,4}(4), \\ \lambda^{(2)}_{2,2}(4) &=& 18 + 2\lambda_{0,2}(4), \\ \lambda^{(2)}_{2,4}(4) &=& 78 - 2\lambda_{0,2}(4). \end{array}
$$

The last 4 equations are written more precisely:

$$
\text{(4.3)} \qquad \begin{cases} \lambda_{0,2}(4)(\mathbf{u},\mathbf{v}) &=& 27 - \lambda_{0,4}(4)(\mathbf{u},\mathbf{v}), \\ \lambda^{(0)}_{2,2}(4)(\mathbf{u},\mathbf{v}) &=& 69 + \lambda_{0,4}(4)(\mathbf{u},\mathbf{v}), \\ \lambda^{(2)}_{2,2}(4)(\mathbf{u},\mathbf{v}) &=& 18 + 2\lambda_{0,4}(4)(\mathbf{u},\mathbf{v}), \\ \lambda^{(2)}_{2,4}(4)(\mathbf{u},\mathbf{v}) &=& 78 - 2\lambda_{0,4}(4)(\mathbf{u},\mathbf{v}). \end{cases}
$$

We now use (i),(ii) of Lemma 4.10. Combining (ii) of Lemma 4.10 with the second and the fourth relations of (4.3) we get

$$\begin{aligned}\lambda_{2,2}^{(0)}(4)(\mathbf{u},\mathbf{v}) &= 69+\lambda_{0,4}(4)(\mathbf{u},\mathbf{v}) \\ &= \lambda_{2,4}^{(2)}(4)(\mathbf{u},\mathbf{u}+\mathbf{v}) \\ \lambda_{2,4}^{(2)}(4)(\mathbf{u},\mathbf{u}+\mathbf{v}) &= 78-2\lambda_{0,4}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})\end{aligned}$$

Thus we obtain

$$\lambda_{0,4}(4)(\mathbf{u},\mathbf{v})+2\lambda_{0,4}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})=9, \tag{4.4}$$

The equation (4.4) yields by symmetric argument

$$\lambda_{0,4}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})+2\lambda_{0,4}(4)(\mathbf{u},\mathbf{v})=9. \tag{4.5}$$

The equations (4.4) and (4.5) give the solution:

$$\lambda_{0,4}(4)(\mathbf{u},\mathbf{u}+\mathbf{v})=\lambda_{0,4}(4)(\mathbf{u},\mathbf{v})=3.$$

All the other values except $\lambda_{0,0}(4)(\mathbf{u},\mathbf{v})$ of Proposition 4.12 are obtained by Lemma 4.9 and the equation (4.3). As to the value of $\lambda_{0,0}(4)(\mathbf{u},\mathbf{v})$ one may use the first relation of Lemma 4.4 and actually we obtain

$$\lambda_{0,0}(4)(\mathbf{u},\mathbf{v})=3.$$

## 4.6 Main Theorem

We prove

**Theorem 4.13** *The set $\mathbf{C}_8$ together with the relations $\mathcal{P}$ above forms an association scheme.*

*Proof.* We recall the set of relations $\mathcal{P}=\{P_8,P_0,P_2,P_4\}$ between the elements of $\mathbf{C}_8$, which are introduced directly after Lemma 4.3. One may immediately note that $P_8$ corresponds to $R_0$ in the definition of the association scheme. Viewing (i) of Lemma 4.1 we see that $\mathcal{P}$ satisfy the condition (i) of the association scheme. By the symmetry property of the intersection $*$ it holds that $P_8^T=P_8, P_0^T=P_0, P_2^T=P_2, P_4^T=P_4$, and therefore $\mathcal{P}$ satisfies (iii) of the association scheme.
It remains to show that $\mathcal{P}$ satisfies the condition (iv) of the association scheme.
(1) If $\langle\mathbf{u},\mathbf{v}\rangle\in P_8$ then $\mathbf{u}=\mathbf{v}$, and the number $p_{i,j}^{(8)}=\#\{\mathbf{w}\in\mathbf{C}_8|\ \mathbf{u}*\mathbf{w}=i,\mathbf{v}*\mathbf{w}=j\}$ is non-zero only if $i=j\in I_{ast}$. By Lemma 4.3 we conclude $p_{8,8}^{(8)}=1, p_{0,0}^{(8)}=30, p_{2,2}^{(8)}=448, p_{4,4}^{(8)}=280$.
(2) If $\langle\mathbf{u},\mathbf{v}\rangle\in P_0$ then the numbers $p_{i,j}^{(0)}=\#\{\mathbf{w}\in\mathbf{C}_8|\ \mathbf{u}*\mathbf{w}=i,\mathbf{v}*\mathbf{w}=j\}$ are treated in Section 4.3, and we see that $p_{i,j}^{(0)}$ are given by $\lambda_{i,j}(0)$. The precise values are discussed also in Section 4.3.

(3) If $\langle \mathbf{u}, \mathbf{v} \rangle \in P_2$ then the numbers $p^{(2)}_{i,j} = \#\{\mathbf{w} \in \mathbf{C}_8 |\ \mathbf{u} * \mathbf{w} = i, \mathbf{v} * \mathbf{w} = j\}$ are treated in Section 4.4. Some of $p^{(2)}_{i,j}$'s are derived with a refined process., and the summarized results are

$$\begin{array}{llll} p^{(2)}_{0,0} = \lambda_{0,0}(2) = 0, & p^{(2)}_{0,2} = \lambda_{0,2}(2) = 15, & p^{(2)}_{0,4} = \lambda_{0,4}(2) = 15, & p^{(2)}_{0,8} = \lambda_{0,8}(2) = 0, \\ p^{(2)}_{2,0} = p^{(2)}_{0,2} = 15, & \multicolumn{3}{l}{p^{(2)}_{2,2} = \lambda_{2,2}(2) = \lambda^{(0)}_{2,2}(2) + \lambda^{(1)}_{2,2}(2) + \lambda^{(2)}_{2,2}(2) = 252,} \\ \multicolumn{3}{l}{p^{(2)}_{2,4} = \lambda_{2,4}(2) = \lambda^{(0)}_{2,4}(2) + \lambda^{(1)}_{2,4}(2) + \lambda^{(2)}_{2,4}(2) = 180,} & p^{(2)}_{2,8} = \lambda_{2,8}(2) = 1, \\ p^{(2)}_{4,0} = \lambda_{4,0}(2) = 15, & p^{(2)}_{4,2} = \lambda_{4,2}(2) = 180, & & \\ \multicolumn{3}{l}{p^{(2)}_{4,4} = \lambda_{4,4}(2) = \lambda^{(0)}_{4,4}(2) + \lambda^{(1)}_{4,4}(2) + \lambda^{(2)}_{4,4}(2) = 85,} & p^{(2)}_{4,8} = \lambda_{4,8}(2) = 0. \end{array}$$

(4) If $\langle \mathbf{u}, \mathbf{v} \rangle \in P_4$ then the numbers $p^{(4)}_{i,j} = \#\{\mathbf{w} \in \mathbf{C}_8 |\ \mathbf{u} * \mathbf{w} = i, \mathbf{v} * \mathbf{w} = j\}$ are treated in Section 4.5. We could the values $p^{(4)}_{i,j}$ in a very complicated way. We summarize the results:

$$\begin{array}{llll} p^{(4)}_{0,0} = \lambda_{0,0}(4) = 3, & p^{(4)}_{0,2} = \lambda_{0,2}(4) = 24, & p^{(4)}_{0,4} = \lambda_{0,4}(4) = 3, & p^{(4)}_{0,8} = \lambda_{0,8}(4) = 0, \\ p^{(4)}_{2,0} = p^{(4)}_{0,2} = 24, & \multicolumn{3}{l}{p^{(4)}_{2,2} = \lambda_{2,2}(4) = \lambda^{(0)}_{2,2}(4) + \lambda^{(1)}_{2,2}(4) + \lambda^{(2)}_{2,2}(4) = 288,} \\ \multicolumn{3}{l}{p^{(4)}_{2,4} = \lambda_{2,4}(4) = \lambda^{(0)}_{2,4}(4) + \lambda^{(1)}_{2,4}(4) + \lambda^{(2)}_{2,4}(4) = 136,} & p^{(4)}_{4,8} = \lambda_{4,8}(4) = 0, \\ \multicolumn{4}{l}{p^{(4)}_{4,4} = \lambda_{4,4}(4) = \lambda^{(0)}_{4,4}(4) + \lambda^{(1)}_{4,4}(4) + \lambda^{(2)}_{4,4}(4) + \lambda^{(3)}_{4,4}(4) + \lambda^{(4)}_{4,4}(4) = 140,} \\ p^{(4)}_{8,0} = \lambda_{8,0}(4) = 0, & p^{(4)}_{8,2} = \lambda_{8,2}(4) = 0, & p^{(4)}_{8,4} = \lambda_{8,4}(4) = 1, & p^{(4)}_{8,8} = \lambda_{8,8}(4) = 0. \end{array}$$

By the processes above to determine the numbers $p^{(k)}_{i,j}$ $i, j, k \in I_*$ they do not depend on the choice of $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$ satisfying $\mathbf{u} * \mathbf{v} = k$, therefore the $(\mathbf{C}_8, \mathcal{P})$ satisfies the condition (iv) of the association scheme.

# 5 A Distance Regular Graph in $\mathcal{G}_{24}$

## 5.1 Some Basic Definitions of the Distance Regular Graph

We will define a graph structure on $\mathbf{C}_8$. For precise definitions of distance regular graphs one may refer [3]. Here we give some basic definitions of the graph. A graph $\mathbf{G}$ consists of two objects:
(i) one is a finite set $\mathcal{V}$ which is called a vertex set,
and
(ii) another is a finite set $\mathcal{E}$ which is called an edge set. Each edge connects two different vertices in $\mathcal{V}$.

We may assume that the cardinality of $\mathcal{V}$ is not less than 2 and the set $\mathcal{E}$ is not empty. A graph $\mathbf{G}$ is called simple if for each pair of vertices in $\mathcal{V}$ there is at most one edge which connects those vertices. In this section we only consider simple graphs. In a simple graph an edge is defined by a pair of vertices $v_1$ and $v_2$. Thus we may express an edge as $v_1v_2$. Two elements $v_1$ and $v_2$ in $\mathcal{V}$ are called adjacent if they are connected by an edge in $\mathcal{E}$. We write this relation for $v_1$ and $v_2$ as $v_1 \sim_a v_2$. The vertices which are adjacent to a vertex $v$ are called the neighbors of $v$. A path from a vertex $v_1$ to another vertex $v_n$ is a sequence of edges in $\mathcal{E}$:

$$v_1v_2, v_2v_3, \cdots, v_{n-1}v_n.$$

The number $n-1$ is called the length of the above path. The graph $\mathbf{G}$ is connected if for any two of the vertices $v_1$ and $v_2$ in $\mathcal{V}$ we can always find at least one path from $v_1$ to $v_2$. The degree or the valency $k(v)$ of a vertex $v \in \mathcal{V}$ is the number of the vertices that are adjacent to $v$. When $k(v)$ is a constant $k$ for all $v \in \mathcal{V}$ then the graph is called regular of valency $k$.
In a connected graph $\mathbf{G}$ we can define a graph distance. Let $v_1$ and $v_2$ in $\mathcal{V}$ then the graph distance $d(v_1, v_2)$ between $v_1$ and $v_2$ is the least length of all the path's joining $v_1$ and $v_2$. We understand that $d(u, v) = 0$ holds if and only if $u = v$. From here we suppose that the graph $\mathbf{G}$ is connected. In $\mathbf{G}$ the number

$$\delta = \delta(\mathbf{G}) = \max_{u,v \in \mathbf{G}} d(u, v)$$

is meaningful, and we call this $\delta$ the diameter of the graph $\mathbf{G}$. For a vertex $v \in \mathcal{V}$ we set

$$\Gamma_i(v) = \{u \in \mathcal{V} \mid d(u, v) = i\}.$$

We also consider $\Gamma_{i-1}(v)$ $(1 \leq i \leq \delta)$ and $\Gamma_{i+1}(v)$ $(0 \leq i \leq \delta - 1)$. The numbers defined by

$$\begin{aligned} c_i &= \#\{w \in \Gamma_{i-1}(v) | w \sim_a u\}, 1 \leq i \leq \delta, \\ b_i &= \#\{w \in \Gamma_{i+1}(v) | w \sim_a u\}, 0 \leq i \leq \delta - 1, \end{aligned}$$

are called intersection numbers of the graph $\mathbf{G}$. The graph $\mathbf{G}$ is called distance regular if $\mathbf{G}$ is regular and for any pair of vertices $v, u \in \mathcal{V}$ satisfying $d(u, v) = i$ there are precisely $c_i$ neighbors of $u$ in $\Gamma_{i-1}(v)$ and $b_i$ neighbors of $u$ in $\Gamma_{i+1}(v)$.

## 5.2 Graph Structure in $\mathbf{C}_8$

We will define a graph structure on $\mathbf{C}_8$. The vertices are the vectors of $\mathbf{C}_8$. Two codewords $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{C}_8$ are said to be adjacent if and only if $\mathbf{u} * \mathbf{v} = 0$ holds. For two codewords $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{C}_8$ we write $\mathbf{u} \sim_a \mathbf{v}$ if they are adjacent to each other. By Lemma 4.4 each vertex $\mathbf{u}$ has 30 adjacent vertices, and therefore the graph is regular and the valency is 30. In this section we must treat two kinds of distances. One is the Hamming distance of the code, which is denoted by $hd$. This distance is introduced in Section 1 as $d$. The another distance is the graph distance. A path from a vertex $\mathbf{u}$ to another vertex $\mathbf{v}$ is a sequence of vertices $\mathbf{u} = \mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_n = \mathbf{v}$ that satisfy the condition : $\mathbf{u}_i \sim_a \mathbf{u}_{i+1}$ for $0 \leq i \leq n-1$. $n$ is the length of the path. The graph distance $gd(\mathbf{u}, \mathbf{v})$ between two vertices $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$ is defined to be the length of the shortest path joining $\mathbf{u}$ and $\mathbf{v}$. The graph distance is denoted by $gd$. For a codeword $\mathbf{u} \in \mathbf{C}_8$ and a non negative integer $i$ the subset $\Gamma_i(\mathbf{u})$ of $\mathbf{C}_8$ is defined by

$$\Gamma_i(\mathbf{u}) = \{\mathbf{v} \mid gd(\mathbf{u}, \mathbf{v}) = i\}.$$

We now prove

**Proposition 5.1** *Let* $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$ *then the following two conditions are equivalent.*
*(i)* $hd(\mathbf{u}, \mathbf{v}) = 8$*,*
*(ii)* $gd(\mathbf{u}, \mathbf{v}) = 2$*.*

*Proof.* The condition (i) is equivalent to $\mathbf{u} * \mathbf{v} = 4$ by Proposition 4.2. First we assume the condition (i). By Proposition 4.12 there are 3 codewords $\mathbf{w} \in \mathbf{C}_8$ satisfying both $\mathbf{u} * \mathbf{w} = 0$ and $\mathbf{v} * \mathbf{w} = 0$, namely $gd(\mathbf{u}, \mathbf{w}) = 1$ and $gd(\mathbf{v}, \mathbf{w}) = 1$. This implies $gd(\mathbf{u}, \mathbf{v}) \leq 2$. Since $\mathbf{u}*\mathbf{v} = 4$, we can say that $gd(\mathbf{u}, \mathbf{v}) \geq 2$. Therefore $gd(\mathbf{u}, \mathbf{v}) = 2$ holds.
Next we assume the condition (ii). Then there is a codeword $\mathbf{w} \in \mathbf{C}_8$ such that $gd(\mathbf{u}, \mathbf{w}) = 1$ and $gd(\mathbf{v}, \mathbf{w}) = 1$. The last conditions imply that $\mathbf{u} * \mathbf{w} = 0$ and $\mathbf{v} * \mathbf{w} = 0$. Concerning the value of $\mathbf{u} * \mathbf{v}$ there are three possibilities: 0,2 and 4. One may note that $\mathbf{u}*\mathbf{v} = 8$ is impossible. Suppose that $\mathbf{u}*\mathbf{v} = 2$ then we see that $wt(\mathbf{u} + \mathbf{v} + \mathbf{w}) = 20$, which is impossible in $\mathcal{G}_{24}$. If $\mathbf{u} * \mathbf{v} = 0$, then $gd(\mathbf{u}, \mathbf{v}) = 1$, which contradicts to the condition (ii). Therefore $\mathbf{u} * \mathbf{v} = 4$ is the only possibility. Then by Proposition 4.2 $hd(\mathbf{u}, \mathbf{v}) = 8$.

**Proposition 5.2** *Let* $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$ *then the following two conditions are equivalent.*
*(i)* $hd(\mathbf{u}, \mathbf{v}) = 12$*,*
*(ii)* $gd(\mathbf{u}, \mathbf{v}) = 3$*.*

*Proof.* First we assume (i) holds. By Proposition 4.2 (i) implies $\mathbf{u} * \mathbf{v} = 2$. This shows that $gd(\mathbf{u}, \mathbf{v}) \geq 3$. By Proposition 4.8 there is a codeword $\mathbf{w} \in \mathbf{C}_8$ such that $\mathbf{u} * \mathbf{w} = 0, \mathbf{v} * \mathbf{w} = 4$. Thus $gd(\mathbf{u}, \mathbf{w}) = 1$ and $gd(\mathbf{v}, \mathbf{w}) = 2$, and $gd(\mathbf{u}, \mathbf{v}) \leq 3$. Consequently $gd(\mathbf{u}, \mathbf{v}) = 3$.
Next we assume that (ii) holds. Then there is a path of length 3 and there is no shorter path joining $\mathbf{u}$ and $\mathbf{v}$. We set that the sequence of codewords $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in \mathbf{C}_8$ forms a shortest path of length 3. By this setting it must hold that $\mathbf{u} * \mathbf{u}_1 = 0, \mathbf{u}_1 * \mathbf{u}_2 = 0, \mathbf{u}_2 * \mathbf{v} = 0$, $gd(\mathbf{u}, \mathbf{u}_1) = 2$ and $gd(\mathbf{u}_1, \mathbf{v}) = 2$. We now discuss on the intersection $\mathbf{u} * \mathbf{u}_2$. If $\mathbf{u} * \mathbf{u}_2 = 0$, then $\mathbf{u} \sim_a \mathbf{u}_2$ and the length of the path $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in \mathbf{C}_8$ is less than 3, which contradicts our setting. If $\mathbf{u} * \mathbf{u}_2 = 2$, then by Proposition 4.2 and the proof of (i) of the present proposition $gd(\mathbf{u}, \mathbf{u}_2) = 3$. This contradicts to our setting. Hence the equation $\mathbf{u} * \mathbf{u}_2 = 4$ must hold. By the same reasoning we have $\mathbf{u}_1 * \mathbf{v} = 4$.
Finally we consider the value of $\mathbf{u} * \mathbf{v}$. If $\mathbf{u} * \mathbf{v} = 4$, then by Proposition 4.2 $hd(\mathbf{u}, \mathbf{v}) = 8$ and by Proposition 5.1 $gd(\mathbf{u}, \mathbf{v}) = 2$. This contradicts to the present assumption (ii). If $\mathbf{u} * \mathbf{v} = 0$, then $gd(\mathbf{u}, \mathbf{v}) = 1$ which also contradicts to the present assumption (ii). Therefore we must have $\mathbf{u} * \mathbf{v} = 2$, and by Proposition 4.2 $hd(\mathbf{u}, \mathbf{v}) = 12$.

Here we summarize some results scattered in many places in this paper as a proposition,

**Proposition 5.3** *Let* $\mathbf{u}, \mathbf{v} \in \mathbf{C}_8$ *then the following hold.*

$$\begin{array}{ccccc}
\mathbf{u} * \mathbf{v} = 8 & \Leftrightarrow & hd(\mathbf{u}, \mathbf{v}) = 0 & \Leftrightarrow & gd(\mathbf{u}, \mathbf{v}) = 0, \\
\mathbf{u} * \mathbf{v} = 0 & \Leftrightarrow & hd(\mathbf{u}, \mathbf{v}) = 16 & \Leftrightarrow & gd(\mathbf{u}, \mathbf{v}) = 1, \\
\mathbf{u} * \mathbf{v} = 2 & \Leftrightarrow & hd(\mathbf{u}, \mathbf{v}) = 12 & \Leftrightarrow & gd(\mathbf{u}, \mathbf{v}) = 3, \\
\mathbf{u} * \mathbf{v} = 4 & \Leftrightarrow & hd(\mathbf{u}, \mathbf{v}) = 8 & \Leftrightarrow & gd(\mathbf{u}, \mathbf{v}) = 2.
\end{array}$$

**Theorem 5.4** *The set $\mathbf{C}_8$ with the adjacent relation above is a distance regular graph.*

*Proof.* By the definition of adjacency for a given codeword $\mathbf{u}$, there 30 codewords that are adjacent to $\mathbf{u}$, by Propositions 4.3 and 5.1 there are 280 codewords that have distance 2 from $\mathbf{u}$, and by Propositions 4.3 and 5.2 there are 448 codewords that have distance 3 from $\mathbf{u}$. These are 758 in number. Therefore $\mathbf{C}_8$ is a connected regular graph of diameter 3, since there is no codeword which has distance greater than 3 from any codeword.
We now determine the intersection array. We fix a codeword $\mathbf{u} \in \mathbf{C}_8$. Let $\Gamma_i(\mathbf{u})$ $0 \leq i \leq 3$ be the subset of $\mathbf{C}_8$ defined before. For an element $\mathbf{v} \in \mathbf{C}_8$ with $gd(\mathbf{v}, \mathbf{u}) = i$ two numbers are associated :$c_i = \#\{\mathbf{w} \in \Gamma_{i-1}(\mathbf{u}) | gd(\mathbf{w}, \mathbf{v}) = 1\}$, and $b_i = \#\{\mathbf{w} \in \Gamma_{i+1}(\mathbf{u}) | gd(\mathbf{w}, \mathbf{v}) = 1\}$. $c_0$ is meaningless. $c_1$ is the number of $\mathbf{w}$'s satisfying $gd(\mathbf{u}, \mathbf{w}) = 0, gd(\mathbf{w}, \mathbf{v}) = 1$ under the condition $gd(\mathbf{v}, \mathbf{u}) = 1$. This implies that $\mathbf{u} = \mathbf{w}$, and $c_1 = 1$. $c_2$ is the number of $\mathbf{w}$'s satisfying $gd(\mathbf{u}, \mathbf{w}) = 1, gd(\mathbf{w}, \mathbf{v}) = 1$ under the condition $gd(\mathbf{v}, \mathbf{u}) = 2$. Looking Proposition 5.3 and Proposition 4.12 we see that $c_2 = \lambda_{0,0}(4) = 3$. $c_3$ is the number of $\mathbf{w}$'s satisfying $gd(\mathbf{u}, \mathbf{w}) = 2, gd(\mathbf{w}, \mathbf{v}) = 1$ under the condition $gd(\mathbf{v}, \mathbf{u}) = 2$. By Propositions 5.3 and 4.8 we see that $c_3 = \lambda_{2,0}(2) = 15$.
$b_0$ is the number of $\mathbf{w}$'s satisfying $gd(\mathbf{u}, \mathbf{w}) = 1, gd(\mathbf{w}, \mathbf{v}) = 1$ under the condition $gd(\mathbf{v}, \mathbf{u}) = 0$. The last condition implies that $\mathbf{v} = \mathbf{u}$. Hence the condition on $\mathbf{w}$ is simply $\mathbf{u} * \mathbf{w} = 0$ by Proposition 5.3. By Proposition 4.4 $b_0 = 30$. $b_1$ is the number of $\mathbf{w}$'s satisfying $gd(\mathbf{u}, \mathbf{w}) = 2, gd(\mathbf{w}, \mathbf{v}) = 1$ under the condition $gd(\mathbf{v}, \mathbf{u}) = 1$. By Proposition 5.3 we see that $b_1 = \lambda_{4,0}(0) = 28$, which is discussed directly before Lemma 4.5. $b_2$ is the number of $\mathbf{w}$'s satisfying $gd(\mathbf{u}, \mathbf{w}) = 3, gd(\mathbf{w}, \mathbf{v}) = 1$ under the condition $gd(\mathbf{v}, \mathbf{u}) = 2$. By Propositions 5.3 and 4.12 we have $b_2 = \lambda_{2,0}(4) = 24$. Since the conditions on $b_3$ contain $gd(\mathbf{u}, \mathbf{w}) = 4$ and therefore $b_3 = 0$. Thus the graph $\mathbf{C}_8$ is a distance regular graph.

**Note 1** *The intersection array of the distance regular graph proved in Theorem 5.4 is $\iota(\mathbf{C}_8) = \{b_0, b_1, b_2; c_1, c_2, c_3\} = \{30, 28, 24; 1, 3, 15\}$, which is identical to the one given in Brouwer-Cohen-Neumaier [3], Chapter 11, Section 4.*

**Note 2** *It is known that a distance regular graph is also an association scheme [3], Chapter 4. It may be possible to compute intersection numbers $q^k_{i,j}$ $(i, jk \in \{0, 1, 2, 3\})$ associated with the association scheme for present distance regular graph along with the explanation in [3], Chapter 4, but here we can give $q^k_{i,j}$'s by utilizing Proposition 5.3 and the results in Section 4. We observe that $q^k_{i,j} = \#\{\mathbf{w} \in \mathbf{C}_8 \mid gd(\mathbf{u}, \mathbf{w}) = i, gd(\mathbf{v}, \mathbf{w}) = j\}$ under the condition $gd(\mathbf{u}, \mathbf{v}) = k$. Below we give tables of $q^k_{i,j}$ $(k = 1, 2, 3)$ without precise explanation.*

$q^{(1)}_{i,j}$

| $i \backslash j$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 28 | 0 |
| 2 | 0 | 28 | 28 | 224 |
| 3 | 0 | 0 | 224 | 224 |

$q^{(2)}_{i,j}$

| $i \backslash j$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 |
| 1 | 0 | 3 | 3 | 24 |
| 2 | 1 | 3 | 140 | 136 |
| 3 | 0 | 24 | 136 | 288 |

$q^{(3)}_{i,j}$

| $i \backslash j$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 15 | 15 |
| 2 | 0 | 15 | 85 | 180 |
| 3 | 1 | 15 | 180 | 252 |

**Note 3** *By the previous note we may show that the association scheme obtained in Section 4 and that of Section 5 are isomorphic.*

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