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A generalization of the invariant formulas of the k-chop integrals

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Abstract. The k-chop integrals are new conservative quantities of the full Kostant-Toda lattice. We generalize the fundamental formula.

Background

Let G be the complex general linear group $GL_n(\mathbb{C})$. Let $B \subset G$ be the Borel subgroup of upper triangular matrices. Put $\mathfrak{g} = \mathrm{Lie}\ G$ and $\mathfrak{b} = \mathrm{Lie}\ B$. Let $\bar{\mathfrak{b}}$ be the opposite of $\mathfrak{b}$. Let Λ be a shift matrix defined by $\Lambda = \sum_{i=1}^{n-1} E_{i,i+1}$, where $E_{i,j}$ is the (i,j)-matrix element. We define the affine space Lax by $Lax = \Lambda + \bar{\mathfrak{b}}$. The matrix of Lax is called Lax operator. The system of equations for $L \in Lax$

$$\frac{\partial L}{\partial t_j} = [(L^j)_+, L],\ j = 1, \dots, n-1, \tag{1.1}$$

where $(*)_+$ is the projection from $\mathfrak{g}$ to $\mathfrak{b}$, is called the full Kostant-Toda lattice. There exists the Poisson structure on Lax, defined by $\{L_{i,j}, L_{k,\ell}\} = \delta_{j,k}L_{i,\ell} - \delta_{\ell,i}L_{k,j}$, where $L = \Lambda + (L_{i,j}),\ (L_{i,j}) \in \bar{\mathfrak{b}}$ [3]. For $L \in Lax$, we have

$$\{\frac{1}{j+1}\mathrm{tr}L^{j+1}, L_{k,\ell}\} = ([(L^j)_+, L])_{k,\ell}, \tag{1.2}$$

where we mean $(X)_{k,\ell}$ is the (k,ℓ)-component of X. Then we see that the Toda lattice is the system of Hamiltonian equations

$$\frac{\partial L}{\partial t_j} = \{\frac{1}{j+1}\mathrm{tr}L^{j+1}, L\}. \tag{1.3}$$

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From (1.2) and (1.3), we have $\partial/\partial t_i \mathrm{tr} L^{j+1} = 0, i, j = 1, \ldots, n-1$. Then we see that $\mathrm{tr} L^{j+1}$, $j = 1, \ldots n-1$ are conservative quantities of the full Kostant-Toda lattice. Put $\det(\lambda - L) = \lambda^n + M_1(L)\lambda^{n-1} + \cdots + M_n(L)$. Since $M_i(L)$ are polynomials of $\mathrm{tr}L, \ldots, \mathrm{tr}L^n$, then $M_i(L)$, $i = 1, \ldots, n$ are also conservative quantities of the full Kostant-Toda lattice. The existence of these conservative quantities is guaranteed by the $\mathrm{Ad}G$-invariance of $\mathrm{tr}L^j$. For any $X \in Mat_n(\mathbb{C})$, put $I_j(X) := \mathrm{tr}X^j$, then we see that $I_j(\mathrm{Ad}gX) = I_j(X)$. For $X \in Mat_n(\mathbb{C})$, we define the truncated $(n-k) \times (n-k)$ matrix $(X)_{(k)}$ by removing first k rows and last k columns from X. If $X = (x_{i,j})_{1 \leq i,j \leq n}$, $(X)_{(k)}$ is

$$\begin{pmatrix} x_{k+1,1} & \cdots & x_{k+1,n-k} \\ \vdots & \cdots & \vdots \\ x_{n,1} & \cdots & x_{n,n-k} \end{pmatrix}.$$

For $L \in Lax$, put

$$\det(\lambda - L)_{(k)} = F_{0,k}(L)\lambda^{n-2k} + F_{1,k}(L)\lambda^{2n-k-1} + \cdots + F_{n-2k,k}(L). \tag{1.4}$$

Let B_k be the Borel subgroup of $GL_k(\mathbb{C})$. Let P_k be the parabolic subgroup of G defined by

$$P_k = \left\{ p = \begin{pmatrix} p_1 & * & * \\ O & p_2 & * \\ O & O & p_3 \end{pmatrix} | p_1, p_3 \in B_k, p_2 \in GL_{n-2k}(\mathbb{C}) \right\}.$$

Let $p_{1,1}, \ldots, p_{k,k}$ and $p_{n-k+1,n-k+1}, \ldots, p_{n,n}$ be diagonal components of p_1, p_3 respectively. Let χ be the character of P_k defined by

$$\chi(p) = p_{n-k+1,n-k+1} \cdots p_{n,n} / p_{1,1} \cdots p_{k,k}.$$

In [1], they showed the relative invariant formula of $\det(X)_{(k)}$ such as

$$\det(\mathrm{Ad}pX)_{(k)} = \chi(p)\det(X)_{(k)}, \tag{1.5}$$

for $p \in P_k$. Put

$$Q_k(L, \lambda) = \det(\lambda - L)_{(k)}/F_{0,k} = \lambda^{n-2k} + I_{1,k}(L)\lambda^{n-2k-1} + \cdots + I_{n-2k,k}(L),$$

where $I_{i,k}(L) = F_{i,k}(L)/F_{0,k}(L), i = 1, \ldots, n-2k$. Then it holds

$$\det\{p(\lambda - L)p^{-1}\}_{(k)} = \chi(p)\det(\lambda - L)_{(k)} = \sum_{i=0}^{n-2k} \chi(p)F_{i,k}(L)\lambda^{n-2k-i}.$$

It implies $F_{i,k}(\mathrm{Ad}pL) = \chi(p)F_{i,k}(L)$, $i = 0, \ldots, n-2k$. Then it holds that $I_{i,k}(\mathrm{Ad}pL) = \chi(p)F_{i,k}(L)/\chi(p)F_{0,k}(L) = I_{i,k}(L)$, $i = 1, \ldots, n-2k$ for $p \in P_k$.

This invariance brings new conservative quantities of the full Kostant-Toda lattice $I_{i,k}(L)$ which are called k-chop integrals [2]. The result of this paper is a generalization of their formula. We extend P_k and its character χ as follows,

$$P_k = \left\{ p = \begin{pmatrix} p_1 & * & * \\ O & p_2 & * \\ O & O & p_3 \end{pmatrix} | p_1, p_3 \in GL_k(\mathbb{C}), p_2 \in GL_{n-2k}(\mathbb{C}) \right\},$$

$\chi(p) = \det p_3 / \det p_1$. The relative invariant formula is generalized as follows.

Theorem *For $X \in Mat_n(\mathbb{C})$ and $p \in P_k$, it holds that*

$$\det(\mathrm{Ad} pX)_{(k)} = \chi(p)\det(X)_{(k)}.$$

1 Proof of the Theorem

Let $O_{\ell\times m}$ be the $\ell \times m$ zero matrix. Then we have

$$X_{(k)} = (O_{(n-k)\times k}, E_{n-k})X \begin{pmatrix} E_{n-k} \\ O_{k\times(n-k)} \end{pmatrix}.$$

Then we have

$$\det(pXp^{-1})_{(k)} = \det(O_{(n-k)\times k}, E_{n-k})pXp^{-1} \begin{pmatrix} E_{n-k} \\ O_{k\times(n-k)} \end{pmatrix}.$$

Note that

$$(O_{(n-k)\times k}, E_{n-k}) \begin{pmatrix} p_1 & *** & *** \\ O & p_2 & *** \\ O & O & p_3 \end{pmatrix} = (O_{(n-k)\times k}, \begin{pmatrix} p_2 & *** \\ O & p_3 \end{pmatrix})$$

$$= \begin{pmatrix} p_2 & *** \\ O & p_3 \end{pmatrix} (O_{(n-k)\times k}, E_{n-k}).$$

On the other hand, we see that

$$\begin{pmatrix} p_1^{-1} & *** & *** \\ O & p_2^{-1} & *** \\ O & O & p_3^{-1} \end{pmatrix} \begin{pmatrix} E_{n-k} \\ O_{k\times(n-k)} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} p_1^{-1} & *** \\ O & p_2^{-1} \end{pmatrix} \\ O_{k\times(n-k)} \end{pmatrix}$$

$$= \begin{pmatrix} E_{n-k} \\ O_{k\times(n-k)} \end{pmatrix} \begin{pmatrix} p_1^{-1} & *** \\ O & p_2^{-1} \end{pmatrix}.$$

Then we have

$$\det(pXp^{-1})_{(k)} = \det \begin{pmatrix} p_2 & *** \\ O & p_3 \end{pmatrix} \det \begin{pmatrix} p_1^{-1} & *** \\ O & p_2^{-1} \end{pmatrix} \det(X)_{(k)}$$

$$= \det p_2 \det p_3 \det p_1^{-1} \det p_2^{-1} \det(X)_{(k)} = \frac{\det p_3}{\det p_1} \det(X)_{(k)}.$$

Q.E.D.

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