A generalization of the invariant formulas of the $k$-chop integrals

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Abstract. The $k$-chop integrals are new conservative quantities of the full Kostant-Toda lattice. We generalize the fundamental formula.

Background

Let $G$ be the complex general linear group $GL_n(\mathbb{C})$. Let $B \subset G$ be the Borel subgroup of upper triangular matrices. Put $\mathfrak{g} = \text{Lie } G$ and $\mathfrak{b} = \text{Lie } B$. Let $\bar{\mathfrak{b}}$ be the opposite of $\mathfrak{b}$. Let $\Lambda$ be a shift matrix defined by $\Lambda = \sum_{i=1}^{n-1} E_{i,i+1}$, where $E_{i,j}$ is the $(i,j)$-matrix element. We define the affine space $\text{Lax}$ by $\Lambda + \bar{\mathfrak{b}}$. The matrix of $\text{Lax}$ is called Lax operator. The system of equations for $L \in \text{Lax}$

$$\frac{\partial L}{\partial t_j} = [(L^j)_+, L], \quad j = 1, \ldots, n - 1,$$  \hspace{1cm} (1.1)

where $(\cdot)_+$ is the projection from $\mathfrak{g}$ to $\mathfrak{b}$, is called the full Kostant-Toda lattice. There exists the Poisson structure on $\text{Lax}$, defined by $\{L_{i,j}, L_{k,\ell}\} = \delta_{j,k}L_{i,\ell} - \delta_{\ell,j}L_{i,k}$, where $L = \Lambda + (L_{i,j})$, $(L_{i,j}) \in \bar{\mathfrak{b}}$ [3]. For $L \in \text{Lax}$, we have

$$\left\{ \frac{1}{j+1} \text{tr} L^{j+1}, L_{k,\ell}\right\} = [(L^j)_+, L]_{k,\ell},$$ \hspace{1cm} (1.2)

where we mean $(X)_{k,\ell}$ is the $(k,\ell)$-component of $X$. Then we see that the Toda lattice is the system of Hamiltonian equations

$$\frac{\partial L}{\partial t_j} = \left\{ \frac{1}{j+1} \text{tr} L^{j+1}, L \right\}.$$ \hspace{1cm} (1.3)

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From (1.2) and (1.3), we have \( \partial/\partial t_i \text{tr} L^{j+1} = 0, i, j = 1, \ldots, n - 1 \). Then we see that \( \text{tr} L^{j+1}, j = 1, \ldots, n - 1 \) are conservative quantities of the full Kostant-Toda lattice. Put \( \det(\lambda - L) = \lambda^n + M_1(L)\lambda^{n-1} + \cdots + M_n(L) \). Since \( M_i(L) \) are polynomials of \( \text{tr} L, \ldots, \text{tr} L^n \), then \( M_i(L), i = 1, \ldots, n \) are also conservative quantities of the full Kostant-Toda lattice. The existence of these conservative quantities is guaranteed by the AdG-invariance of \( \text{tr} L^j \). For any \( X \in \text{Mat}_n(\mathbb{C}) \), put \( I_j(X) := \text{tr}X^j \), then we see that \( I_j(\text{Ad}gX) = I_j(X) \). For \( X \in \text{Mat}_n(\mathbb{C}) \), we define the truncated \((n - k) \times (n - k)\) matrix \( (X)_{(k)} \) by removing first \( k \) rows and last \( k \) columns from \( X \). If \( X = (x_{i,j})_{1 \leq i, j \leq n} \), \( (X)_{(k)} \) is

\[
\begin{pmatrix}
x_{k+1,1} & \cdots & x_{k+1,n-k} \\
\vdots & \ddots & \vdots \\
x_{n,1} & \cdots & x_{n,n-k}
\end{pmatrix}
\]

For \( L \in \text{Lax} \), put

\[
\det(\lambda - L)_{(k)} = F_{0,k}(L)\lambda^{n-2k} + F_{1,k}(L)\lambda^{2n-k-1} + \cdots + F_{n-k,k}(L). \tag{1.4}
\]

Let \( B_k \) be the Borel subgroup of \( GL_k(\mathbb{C}) \). Let \( P_k \) be the parabolic subgroup of \( G \) defined by

\[
P_k = \left\{ p = \begin{pmatrix} p_1 & * & * \\
O & p_2 & * \\
O & O & p_3 \end{pmatrix} \mid p_1, p_3 \in B_k, p_2 \in GL_{n-k}(\mathbb{C}) \right\}.
\]

Let \( p_1, \ldots, p_{k,k} \) and \( p_{n-k+1,n-k+1}, \ldots, p_{n,n} \) be diagonal components of \( p_1, p_3 \) respectively. Let \( \chi \) be the character of \( P_k \) defined by

\[
\chi(p) = p_{n-k+1,n-k+1} \cdots p_{n,n}/p_{1,1} \cdots p_{k,k}.
\]

In [1], they showed the relative invariant formula of \( \det(X)_{(k)} \) such as

\[
\det(\text{Ad}pX)_{(k)} = \chi(p)\det(X)_{(k)}, \tag{1.5}
\]

for \( p \in P_k \). Put

\[
Q_k(L, \lambda) = \det(\lambda - L)_{(k)}/F_{0,k} = \lambda^{n-2k} + I_{1,k}(L)\lambda^{n-2k-1} + \cdots + I_{n-2k,k}(L),
\]

where \( I_{i,k}(L) = F_{i,k}(L)/F_{0,k}(L), i = 1, \ldots, n - 2k \). Then it holds

\[
\det\{p(\lambda - L)p^{-1}\}_{(k)} = \chi(p)\det(\lambda - L)_{(k)} = \sum_{i=0}^{n-2k} \chi(p)F_{i,k}(L)\lambda^{n-2k-1}.
\]

It implies \( F_{i,k}(\text{Ad}pL) = \chi(p)F_{i,k}(L), i = 0, \ldots, n - 2k \). Then it holds that \( I_{i,k}(\text{Ad}pL) = \chi(p)F_{i,k}(L)/\chi(p)F_{0,k}(L) = I_{i,k}(L), i = 1, \ldots, n - 2k \) for \( p \in P_k \).
This invariance brings new conservative quantities of the full Kostant-Toda lattice $I_{i,k}(L)$ which are called $k$-chop integrals [2]. The result of this paper is a generalization of their formula. We extend $P_k$ and its character $\chi$ as follows,

$$P_k = \left\{ p = \begin{pmatrix} p_1 & * & * \\ O & p_2 & * \\ O & O & p_3 \end{pmatrix} | p_1, p_3 \in GL_k(\mathbb{C}), p_2 \in GL_{n-2k}(\mathbb{C}) \right\},$$

$$\chi(p) = \det p_3 / \det p_1.$$ The relative invariant formula is generalized as follows.

**Theorem** For $X \in \text{Mat}_n(\mathbb{C})$ and $p \in P_k$, it holds that

$$\det(\text{Ad} p X)_{(k)} = \chi(p) \det(X)_{(k)}.$$

1 Proof of the Theorem

Let $O_{\ell \times m}$ be the $\ell \times m$ zero matrix. Then we have

$$X_{(k)} = (O_{(n-k) \times k}, E_{n-k}) X \begin{pmatrix} E_{n-k} \\ O_{k \times (n-k)} \end{pmatrix}.$$  

Then we have

$$\det(pXp^{-1})_{(k)} = \det((O_{(n-k) \times k}, E_{n-k})pXp^{-1} \begin{pmatrix} E_{n-k} \\ O_{k \times (n-k)} \end{pmatrix}).$$

Note that

$$(O_{(n-k) \times k}, E_{n-k}) \begin{pmatrix} p_1 & *** & *** \\ O & p_2 & *** \\ O & O & p_3 \end{pmatrix} = (O_{(n-k) \times k}, \begin{pmatrix} p_2 & *** \\ O & p_3 \end{pmatrix})$$

$$= \begin{pmatrix} p_2 & *** \\ O & p_3 \end{pmatrix} (O_{(n-k) \times k}, E_{n-k}).$$

On the other hand, we see that

$$\begin{pmatrix} p_1^{-1} & *** & *** \\ O & p_2^{-1} & *** \\ O & O & p_3^{-1} \end{pmatrix} \begin{pmatrix} E_{n-k} \\ O_{k \times (n-k)} \end{pmatrix} = \begin{pmatrix} p_1^{-1} & *** \\ O & p_2^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} E_{n-k} \\ O_{k \times (n-k)} \end{pmatrix} \begin{pmatrix} p_1^{-1} & *** \\ O & p_2^{-1} \end{pmatrix}. $$
Then we have

\[ \det(pXp^{-1})_{(k)} = \det \left( \begin{array}{cc} p_2 & ** \\ O & p_3 \end{array} \right) \det \left( \begin{array}{cc} p_1^{-1} & ** \\ O & p_2^{-1} \end{array} \right) \det(X)_{(k)} \]

\[ = \det p_2 \det p_3 \det p_1^{-1} \det p_2^{-1} \det(X)_{(k)} = \frac{\det p_3}{\det p_1} \det(X)_{(k)}. \]

Q.E.D.

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