Kumamoto J. Math.
Vol.**29** (2016), 1–33

Power moments for the double zeta-function

Soichi Ikeda, Isao Kiuchi and Kaneaki Matsuoka

(Received May 18, 2015)
(Accepted November 11, 2015)

Abstract. We investigate the fourth power moment and the even power moments of the double zeta-function of the Euler–Zagier type, which is defined by analytic continuation of the double series $\zeta_2(s_1, s_2) = \sum_{1\le m<n} m^{-s_1} n^{-s_2}$ for $\mathrm{Re}\ \mathrm{s}_2 > 1$ and $\mathrm{Re}\ \mathrm{s}_1 + \mathrm{Re}\ \mathrm{s}_2 > 2$, under the region $0 < \mathrm{Re}\ \mathrm{s}_1 < 1$, $0 < \mathrm{Re}\ \mathrm{s}_2 < 1$ and $0 < \mathrm{Re}\ \mathrm{s}_1 + \mathrm{Re}\ \mathrm{s}_2 < 2$. We provide the Ω results of the double zeta-function, and calculate the double integral under certain conditions.

1 Introduction

Let $s_j = \sigma_j + it_j$ $(j = 1, 2)$ be complex variables with σ_j, $t_j \in \mathbb{R}$, and let $\zeta(s)$ be the Riemann zeta-function. The Euler–Zagier type double zeta-function is defined by

$$\zeta_2(s_1, s_2) = \sum_{1\le m<n} \frac{1}{m^{s_1} n^{s_2}},$$

which is absolutely convergent for $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$. One can easily verify that the reciprocity law

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1 + s_2) + \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) \tag{1.1}$$

holds for $\sigma_1 > 1$ and $\sigma_2 > 1$ (see T. M. Apostol and T. H. Vu [2]). The mean value formula of the Riemann zeta-function given by F. V. Atkinson [3] (see also A. Ivić [9]), which was applied to the analytic continuation of $\zeta_2(s_1, s_2)$ to the region $\{(s_1, s_2) \in \mathbb{C}^2 \mid 0 < \sigma_1 < 1, \quad 0 < \sigma_2 < 1\}$. The function $\zeta_2(s_1, s_2)$ is continued meromorphically to $\mathbb{C}^2$, which is holomorphic in $\{(s_1, s_2) \in \mathbb{C}^2 | s_2 \neq 1, s_1 + s_2 \notin \{2, 1, 0, -2, -4, -6, \ldots\}\}$ was proved in S. Akiyama, S. Egami and Y. Tanigawa [1], and also the analytic continuation of (1.1) was obtained by J. Q. Zhao [25] independently. Some analytic properties of $\zeta_2(s_1, s_2)$ were considered by Akiyama, Egami and Tanigawa, H. Ishikawa and K. Matsumoto [8], I. Kiuchi and

Mathematical Subject Classification (2010): Primary 11M41; Secondary 11M06
Key words: Double zeta-functions, Fourth power moment, Higher power moment, Riemann zeta-function

Y. Tanigawa [16], I. Kiuchi, Y. Tanigawa and W. Zhai [17], K. Matsumoto [19], [20], [21], [22], Y. Komori, K. Matsumoto and H. Tsumura [18], and others.

We assume certain conditions for the order of magnitude between the variables t_1 and t_2. The first purpose of this paper is to derive the formulas of the fourth power moment of the double zeta-function $\zeta_2(s_1, s_2)$ with respect to the variable t_2 under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 2$. Secondly, we shall obtain the formulas of the even power moments of the double zeta-function $\zeta_2(s_1, s_2)$ with respect to the variable t_1 under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 2$, and the formulas of that with respect to the variable t_2, too. Furthermore, we shall consider an average order of magnitude of difference for the double integral $\int_2^N \int_2^T \left\{|\zeta_2(s_1, s_2)|^{2k} - |\zeta_2(s_2, s_1)|^{2k}\right\} dt_1 dt_2$ under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $\sigma_1 + \sigma_2 = 1$ for any fixed positive integer k.

Notations. When $g(x)$ is a positive function of x for $x \geq x_0$, $f(x) = \Omega(g(x))$ means that $f(x) = o(g(x))$ does not hold as $x \to \infty$. In what follows, ε denotes any arbitrarily small positive number, not necessarily the same ones at each occurrence.

2 Mean square formula

Before the introduction of our results, let us recall the mean value formulas of the double zeta-function $\zeta_2(s_1, s_2)$ with respect to the variable t_1 under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 2$. We also recall the mean value formulas of the double zeta-function $\zeta_2(s_1, s_2)$ concerning the variable t_2 in place of the variable t_1 within the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 2$.

2.1 Mean square formula – concerning t_1

A new type of mean value formulas of the double zeta-function $\zeta_2(s_1, s_2)$, namely, $\int_2^T |\zeta_2(s_1, s_2)|^2 dt_2$, which was first considered by K. Matsumoto and H. Tsumura [23] for a fixed complex number s_1 and any large positive number $T > 2$, who derived useful approximate formulas for $\zeta_2(s_1, s_2)$ and obtained the following interesting formula. For $\frac{1}{2} < \sigma_2 \leq 1$ and $\frac{3}{2} < \sigma_1 + \sigma_2 \leq 2$, they showed that

$$\int_2^T |\zeta_2(s_1, s_2)|^2 dt_2 = \left(\sum_{n=2}^{\infty} \left|\sum_{m=1}^{n-1} \frac{1}{m^{s_1}}\right|^2 \frac{1}{n^{2\sigma_2}}\right) T$$

$$+ \begin{cases} O\left(T^{4-2\sigma_1-2\sigma_2} \log T\right) + O\left(T^{\frac{1}{2}}\right) & \text{if } \frac{1}{2} < \sigma_1 < 1,\ \frac{1}{2} < \sigma_2 < 1, \\ O\left(T^{2-2\sigma_1} \log^2 T\right) + O\left(T^{\frac{1}{2}}\right) & \text{if } \frac{1}{2} < \sigma_1 < 1,\ \sigma_2 = 1, \\ O\left(T^{2-2\sigma_1} \log^3 T\right) + O\left(T^{\frac{1}{2}}\right) & \text{if } \sigma_1 = 1,\ \frac{1}{2} < \sigma_2 < 1, \\ O\left(T^{\frac{1}{2}}\right) & \text{if } \sigma_1 = 1,\ \sigma_2 = 1, \\ O\left(T^{2-2\sigma_1-2\sigma_2} \log T\right) + O\left(T^{\frac{1}{2}}\right) & \text{if } 1 < \sigma_1 < \frac{3}{2},\ \frac{1}{2} < \sigma_2 < 1, \end{cases} \tag{2.1}$$

where the point (s_1, s_2) does not encounter the hyperplane $s_1 + s_2 = 2$, which is a singular locus of the function $\zeta_2(s_1, s_2)$. Here the coefficient of the main term on the right-hand side of (2.1) is convergent for $\sigma_1 \geq 1$ and $\sigma_2 > \frac{1}{2}$, or for $\sigma_1 < 1$ and $\sigma_1 + \sigma_2 > \frac{3}{2}$. S. Ikeda, K. Matsuoka and Y. Nagata [6] studied the integral (2.1) under the region $\sigma_2 \geq \frac{1}{2}$ and $\sigma_1 + \sigma_2 \geq \frac{3}{2}$, and also considered the integral $\int_2^T |\zeta_2(s_1, s_2)|^2 dt_1$ with respect to the variable t_1 for a fixed complex number s_2. For any large positive number $T > 2$, they showed that

$$\int_2^T |\zeta_2(s_1, s_2)|^2 dt_1 = \left(\sum_{m=1}^{\infty} \frac{1}{m^{2\sigma_1}} \left| \zeta(s_2) - \sum_{n=1}^{m} \frac{1}{n^{s_2}} \right|^2 \right) T + \begin{cases} O\left(T^{4-2\sigma_1-2\sigma_2}\right) & \text{if } \frac{3}{2} < \sigma_1 + \sigma_2 < 2, \\ O\left(\log^2 T\right) & \text{if } \sigma_1 + \sigma_2 = 2, \end{cases} \tag{2.2}$$

where the coefficient of the main term on the right-hand side of (2.2) converges if $\sigma_1 + \sigma_2 > \frac{3}{2}$. In particular, Ikeda, Matsuoka and Nagata deduced that the asymptotic formula

$$\int_2^T |\zeta_2(s_1, s_2)|^2 dt_1 = \frac{1}{|s_2 - 1|^2} T \log T + O(T) \tag{2.3}$$

holds for the region $\sigma_1 + \sigma_2 = \frac{3}{2}$ and $\sigma_2 > \frac{1}{2}$. Matsumoto and Tsumura conjectured that the form of the main term of the mean square formula for the double zeta-function would not be CT with a constant, most probably, some log-factor would appear on the line $\sigma_1 + \sigma_2 = \frac{3}{2}$. This result (2.3) implied that the conjecture of Matsumoto and Tsumura on the line $\sigma_1 + \sigma_2 = \frac{3}{2}$ was true.

Some mean value formulas obtained by I. Kiuchi and M. Minamide [14] were inspired by Matsumoto and Tsumura, and Ikeda, Matsuoka and Nagata. For the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < \frac{3}{2}$, Kiuchi and Minamide have been considered that one divide evaluations of the integral $\int_2^T |\zeta_2(s_1, s_2)|^2 dt_1$ with respect to the variable t_1 into five cases, namely; for any sufficiently large positive number $T > 2$,

$$\int_2^T |\zeta_2(s_1, s_2)|^2 dt_1 = \frac{\zeta(2)}{4\pi |s_2 - 1|^2} T^2 + O\left(t_2^{-\frac{1}{2}} (\log t_2) T^{\frac{3}{2}}\right) \tag{2.4}$$

with $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $\sigma_1 + \sigma_2 = 1$ and $2 \leq t_2 \leq \frac{T^{\frac{1}{3}}}{\log T}$,

$$\int_2^T |\zeta_2(s_1, s_2)|^2 dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2} + O\left(t_2^{-\frac{1}{2}} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) \tag{2.5}$$

with $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $1 < \sigma_1 + \sigma_2 < \frac{3}{2}$ and $2 \leq t_2 \leq T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$,

$$\int_2^T |\zeta_2(s_1, s_2)|^2 dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2} + O\left(t_2^{\frac{1}{2}-\sigma_1-\sigma_2} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) \tag{2.6}$$

with $0<\sigma_1<1,\ 0<\sigma_2<1,\ \frac{1}{2}<\sigma_1+\sigma_2<1$ and $2\le t_2\le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$,

$$\int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 = \frac{\zeta(3)}{12\pi^2|s_2-1|^2}T^3 + \begin{cases} O\left(T^2\right) & \text{if } \sqrt{\log T}\le t_2\le T^{\frac{1}{2}}, \\ O\left(t_2^{-1}T^2\sqrt{\log T}\right) & \text{if } 2\le t_2\le\sqrt{\log T} \end{cases} \tag{2.7}$$

with $0<\sigma_1<1$, $0<\sigma_2<1$ and $\sigma_1+\sigma_2=\frac{1}{2}$, and

$$\int_2^T |\zeta_2(s_1,s_2)|^2\, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3}\,\frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2}\,T^{4-2\sigma_1-2\sigma_2} + \begin{cases} O\left(t_2^{\frac{1}{2}-\sigma_1-\sigma_2}T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) & \text{if } T^{\frac{1-2\sigma_1-2\sigma_2}{3-2\sigma_1-2\sigma_2}}\le t_2\le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}, \\ O\left(t_2^{-1}T^{3-2\sigma_1-2\sigma_2}\right) & \text{if } 2\le t_2\le T^{\frac{1-2\sigma_1-2\sigma_2}{3-2\sigma_1-2\sigma_2}} \end{cases} \tag{2.8}$$

with $0<\sigma_1<1,\ 0<\sigma_2<1$ and $0<\sigma_1+\sigma_2<\frac{1}{2}$. Note that the O-constants depend on σ_1 and σ_2 in these results. Thus, they deduced the following Ω results

$$\zeta_2(\sigma_1+it_1,\sigma_2+it_2)=\Omega\left(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\right) \tag{2.9}$$

for $1\le\sigma_1+\sigma_2<\frac{3}{2}$, $2\le t_1\le T$ and $2\le t_2\le T^{1-\frac{2}{3}(\sigma_1+\sigma_2)-\varepsilon}$, and

$$\zeta_2(\sigma_1+it_1,\sigma_2+it_2)=\Omega\left(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\right) \tag{2.10}$$

for $0<\sigma_1+\sigma_2<1$, $2\le t_1\le T$ and $2\le t_2\le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}-\varepsilon}$, with ε being any small positive constant.

The formulas (2.9) and (2.10) provide a certain improvement on the Ω results of Kiuchi, Tanigawa and Zhai [17].

Ikeda, Matsuoka and Nagata made use of the mean value theorems for Dirichlet polynomials and suitable approximate formulas derived from the Euler–Maclaurin summation formula to obtain the formulas (2.2) and (2.3). However, Kiuchi and Minamide [14] used some mean value formulas of the Riemann zeta-function for the region $-1<\sigma<\frac{3}{2}$ and a weak form of the approximate formula of Kiuchi, Tanigawa and Zhai for $\zeta_2(s_1,s_2)$ to obtain the mean value formulas (2.4)–(2.8) of the double zeta-function. In (2.2) and (2.3), s_2 is a constant, but t_2 is not a constant in (2.4)–(2.8). This fact is one of some important properties, because the analytic properties of $\zeta_2(s_1,s_2)$ depend on both s_1 and s_2.

2.2 Mean square formula – concerning t_2

As mentioned earlier, Matsumoto and Tsumura first obtained the mean square formula (2.1) of the double zeta-function $\zeta_2(s_1,s_2)$ concerning the variable t_2.

The mean square formulas of the double zeta-function $\zeta_2(s_1,s_2)$ concerning the variable t_2 in place of the variable t_1 within the region $0<\sigma_1<1$, $0<\sigma_2<1$ and $0<\sigma_1+\sigma_2<\frac{3}{2}$ were recently studied by S. Ikeda, I. Kiuchi and K. Matsuoka [7], who made use of the reciprocity law (1.1) and the method of Kiuchi and Minamide to obtain the mean square formulas of the double zeta-function $\zeta_2(s_2,s_1)$ under the region $0<\sigma_1<1$, $0<\sigma_2<1$ and $0<\sigma_1+\sigma_2<\frac{3}{2}$. Ikeda, Kiuchi and Matsuoka showed that for any sufficiently large positive number $T>2$, the integral $\int_2^T|\zeta_2(s_2,s_1)|^2\,dt_1$ was evaluated by

$$\int_2^T|\zeta_2(s_2,s_1)|^2\,dt_1=\frac{\zeta(2)}{4\pi|s_2-1|^2}\,T^2+O\left(t_2^{-\frac{1}{2}}(\log t_2)T^{\frac{3}{2}}\right) \tag{2.11}$$

with $\frac{1}{2}<\sigma_1<1$, $0<\sigma_2<\frac{1}{2}$, $\sigma_1+\sigma_2=1$ and $2\le t_2\le\frac{T^{\frac{1}{3}}}{\log T}$,

$$\int_2^T|\zeta_2(s_2,s_1)|^2\,dt_1=\frac{\zeta(2)}{4\pi|s_2-1|^2}\,T^2 +\begin{cases} O\left(t_2^{-\frac{1}{2}}(\log t_2)T^{\frac{3}{2}}\right) & \text{if }\ \log^{\frac{3}{2}}T\le t_2\le\dfrac{T^{\frac{1}{3}}}{\log T},\\ O\left(t_2^{-\frac{5}{6}}(\log t_2)T^{\frac{3}{2}}(\log T)^{\frac{1}{2}}\right) & \text{if }\ 2\le t_2\le\log^{\frac{3}{2}}T\end{cases} \tag{2.12}$$

with $\sigma_1=\sigma_2=\frac{1}{2}$ and

$$\begin{aligned}\int_2^T|\zeta_2(s_2,s_1)|^2\,dt_1&=\frac{\zeta(2)}{4\pi|s_2-1|^2}\,T^2+O\left(t_2^{-\frac{1}{2}}(\log t_2)T^{\frac{3}{2}}\right)\\&+O\left(t_2^{\frac{2}{3}\sigma_1}(\log t_2)^4T^{2-2\sigma_1}\right)+O\left(t_2^{-1+\frac{1}{3}\sigma_1}(\log t_2)^2T^{2-\sigma_1}\right)\end{aligned} \tag{2.13}$$

with $0<\sigma_1<\frac{1}{2}$, $\frac{1}{2}<\sigma_2<1$, $\sigma_1+\sigma_2=1$ and $2\le t_2\le\frac{T^{\frac{1}{3}}}{\log T}$. They observed that the average order of magnitude of (2.4) and (2.11) are the same one with $\frac{1}{2}<\sigma_1<1$, $0<\sigma_2<\frac{1}{2}$ and $\sigma_1+\sigma_2=1$, but (2.11) does not coincide with (2.4); namely

$$\int_2^T|\zeta_2(s_1,s_2)|^2\,dt_1-\int_2^T|\zeta_2(s_2,s_1)|^2\,dt_1=O\left(t_2^{-\frac{1}{2}}(\log t_2)T^{\frac{3}{2}}\right). \tag{2.14}$$

Integrating (2.14) by the variable t_2, we have

$$\frac{1}{TN}\int_2^N\int_2^T\left\{|\zeta_2(s_1,s_2)|^2-|\zeta_2(s_2,s_1)|^2\right\}dt_1dt_2=O\left(\sqrt{\frac{T}{N}}\log N\right) \tag{2.15}$$

for $2<N\le\frac{T^{\frac{1}{3}}}{\log T}$. Furthermore, from (2.11) and (2.12) they derived an alternative proof of the Ω results in Kiuchi, Tanigawa and Zhai [17] within $\sigma_1+\sigma_2=1$, $\frac{1}{2}\le\sigma_1<1$, $0<\sigma_2\le\frac{1}{2}$ and $2\le t_1\le T$; namely

$$\zeta_2(\sigma_2+it_2,\sigma_1+it_1)=\Omega\left(\frac{t_1^{\frac{1}{2}}}{t_2}\right)$$

for $2 \le t_2 \le T^{\frac{1}{3}-\varepsilon}$, with ε being any small positive constant. Ikeda, Kiuchi and Matsuoka considered the mean square formula of the double zeta-function $\zeta_2(s_2, s_1)$ with respect to the variable t_1 under the region $0 < \sigma_1 < 1,\ 0 < \sigma_2 < 1$ and $1 < \sigma_1 + \sigma_2 < \frac{3}{2}$, they showed that for any sufficiently large positive number $T > 2$,

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2} + O\left(t_2^{-\frac{1}{2}} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) \tag{2.16}$$

with $\frac{1}{2} < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $1 < \sigma_1 + \sigma_2 < \frac{3}{2}$ and $2 \le t_2 \le T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$,

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_2-2} \frac{\zeta(3-2\sigma_2)}{(3-2\sigma_2)|s_2-1|^2} T^{3-2\sigma_2} + O\left(t_2^{-\frac{1}{2}} T^{2-\sigma_2}\right) + O\left(t_2^{-\frac{2}{3}-\frac{1}{3}\sigma_2} (\log t_2)^2 T^{2-\sigma_2} (\log T)^{\frac{1}{2}}\right) \tag{2.17}$$

with $\sigma_1 = \frac{1}{2}$, $\frac{1}{2} < \sigma_2 < 1$, and $2 \le t_2 \le T^{\frac{2}{3}-\frac{2}{3}\sigma_2}$, and

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-4\sigma_2} + O\left(t_2^{-\frac{1}{2}} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) + O\left(t_2^{-\frac{2}{3}-\frac{1}{3}\sigma_2} (\log t_2)^2 T^{3-2\sigma_1-\sigma_2}\right) + O\left(t_2^{\frac{2}{3}(1-\sigma_2)} (\log t_2)^4 T^{2-2\sigma_1}\right) \tag{2.18}$$

with $0 < \sigma_1 < \frac{1}{2}$, $\frac{1}{2} < \sigma_2 < 1$, $1 < \sigma_1 + \sigma_2 < \frac{3}{2}$ and $2 \le t_2 \le T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$. They also considered the mean square formula of the double zeta-function $\zeta_2(s_2, s_1)$ with respect to the variable t_1 under the region $0 < \sigma_1 < 1,\ \ 0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 1$, namely,

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2} + O\left(t_2^{-\frac{1}{2}-\frac{2}{3}\sigma_2} (\log t_2) T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) + O\left(t_2^{1-\frac{4}{3}\sigma_2} (\log t_2)^2 T\right) + O\left(t_2^{\frac{1}{2}-\sigma_1-\sigma_2} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) \tag{2.19}$$

with $\frac{1}{2} < \sigma_1 < 1$, $0 < \sigma_2 < \frac{1}{2}$, $\frac{1}{2} < \sigma_1 + \sigma_2 < 1$ and $2 \le t_2 \le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$,

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_2-2} \frac{\zeta(3-2\sigma_2)}{(3-2\sigma_2)|s_2-1|^2} T^{3-2\sigma_2} + O\left(t_2^{-\sigma_2} T^{2-\sigma_2}\right) + O\left(t_2^{1-\frac{4}{3}\sigma_2} (\log t_2)^2 T \log T\right) + O\left(t_2^{-\frac{1}{2}-\frac{2}{3}\sigma_2} (\log t_2) T^{2-\sigma_2} (\log T)^{\frac{1}{2}}\right) \tag{2.20}$$

with $\sigma_1 = \frac{1}{2}$, $0 < \sigma_2 < \frac{1}{2}$, and $2 \le t_2 \le T^{\frac{1-\sigma_2}{2-\sigma_2}}$,

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2}$$
$$+ O\left(t_2^{-\frac{1}{2}-\frac{2}{3}\sigma_2} (\log t_2) T^{3-2\sigma_1-\sigma_2}\right) + O\left(t_2^{1-\frac{4}{3}\sigma_2} (\log t_2)^2 T^{2-2\sigma_1}\right)$$
$$+ O\left(t_2^{\frac{1}{2}-\sigma_1-\sigma_2} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) \tag{2.21}$$

with $0 < \sigma_1 < \frac{1}{2}$, $0 < \sigma_2 < \frac{1}{2}$, $\frac{1}{2} < \sigma_1 + \sigma_2 < 1$ and $2 \le t_2 \le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$,

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_1-2} \frac{\zeta(3-2\sigma_1)}{(3-2\sigma_1)|s_2-1|^2} T^{3-2\sigma_1}$$
$$+ O\left(t_2^{-\frac{5}{6}} (\log t_2) T^{\frac{5}{2}-2\sigma_1}\right) + O\left(t_2^{\frac{1}{3}} (\log t_2)^2 T^{2-2\sigma_1}\right)$$
$$+ O\left(t_2^{-\sigma_1} T^{2-\sigma_1}\right) \tag{2.22}$$

with $0 < \sigma_1 < \frac{1}{2}$, $\sigma_2 = \frac{1}{2}$, $\frac{1}{2} < \sigma_1 + \sigma_2 < 1$ and $2 \le t_2 \le T^{\frac{1-\sigma_1}{2-\sigma_1}}$,

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2}$$
$$+ O\left(t_2^{-\frac{2}{3}-\frac{1}{3}\sigma_2} (\log t_2)^2 T^{3-2\sigma_1-\sigma_2}\right) + O\left(t_2^{\frac{2}{3}(1-\sigma_2)} (\log t_2)^4 T^{2-2\sigma_1}\right)$$
$$+ O\left(t_2^{\frac{1}{2}-\sigma_1-\sigma_2} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) \tag{2.23}$$

with $0 < \sigma_1 < \frac{1}{2}$, $\frac{1}{2} < \sigma_2 < 1$, $\frac{1}{2} < \sigma_1 + \sigma_2 < 1$ and $2 \le t_2 \le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$,

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = \frac{\zeta(3)}{12\pi^2|s_2-1|^2} T^3 + O\left(t_2^{-\frac{1}{2}-\frac{2}{3}\sigma_2} (\log t_2) T^{\frac{5}{2}-2\sigma_1}\right)$$
$$+ O\left(t_2^{1-\frac{4}{3}\sigma_2} (\log t_2)^2 T^{2-2\sigma_1}\right) + O\left(T^2\right) + O\left(t_2^{-1} T^2 \sqrt{\log T}\right) \tag{2.24}$$

with $0 < \sigma_1 < \frac{1}{2}$, $0 < \sigma_2 < \frac{1}{2}$, $\sigma_1 + \sigma_2 = \frac{1}{2}$ and $2 \le t_2 \le T^{\frac{1}{2}}$, and

$$\int_2^T |\zeta_2(s_2, s_1)|^2 \, dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3} \frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2} T^{4-2\sigma_1-2\sigma_2}$$
$$+ O\left(t_2^{-\frac{1}{2}-\frac{2}{3}\sigma_2} (\log t_2) T^{3-\sigma_1-\sigma_2}\right) + O\left(t_2^{1-\frac{4}{3}\sigma_2} (\log t_2)^2 T^{2-2\sigma_1}\right)$$
$$+ O\left(t_2^{\frac{1}{2}-\sigma_1-\sigma_2} T^{\frac{5}{2}-\sigma_1-\sigma_2}\right) + O\left(t_2^{-1} T^{3-2\sigma_1-2\sigma_2}\right) \tag{2.25}$$

with $0 < \sigma_1 < \frac{1}{2}$, $0 < \sigma_2 < \frac{1}{2}$, $0 < \sigma_1 + \sigma_2 < \frac{1}{2}$ and $2 \le t_2 \le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. Note that the O-constants of the above depend on σ_1 and σ_2. In short, t_2 is not a constant. This fact is important observation, because the analytic properties of $\zeta_2(s_1, s_2)$ depend on both s_1 and s_2. From (2.16) and (2.17), they showed that

$$\zeta_2(\sigma_2 + it_2, \sigma_1 + it_1) = \Omega\left(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\right) \tag{2.26}$$

with $1<\sigma_1+\sigma_2<\frac{3}{2}$, $\frac{1}{2}<\sigma_1<1$, $0<\sigma_2<1$, $2\le t_1\le T$, $2\le t_2\le T^{1-\frac{2}{3}(\sigma_1+\sigma_2)-\varepsilon}$ and ε being any small positive constant.

3 Fourth power moment

In this section, we shall derive the fourth power moment of the double zeta-function $\zeta_2(s_1,s_2)$ with respect to the variable t_2.

3.1 Fourth power moment – concerning t_1

The fourth power moment of the double zeta-function $\zeta_2(s_1,s_2)$ within the region $0<\sigma_1<1$, $0<\sigma_2<1$ and $0<\sigma_1+\sigma_2<2$ with respect to the variable t_1, which first studied by I. Kiuchi and M. Minamide [14], who used some mean value formulas of the Riemann zeta-function under the region $-1<\sigma<\frac{3}{2}$ and a weak form of the approximate formula of Kiuchi, Tanigawa and Zhai [17] for $\zeta_2(s_1,s_2)$ to obtain the following formulas (3.1)–(3.6) below. For $0<\sigma_1<1$, $0<\sigma_2<1$ and $2\le t_1\le T$, they proved that for any sufficiently large positive number $T>2$,

$$\int_2^T |\zeta_2(s_1,s_2)|^4\,dt_1=\frac{\zeta^4(2)}{12\pi^2\zeta(4)}\frac{T^3}{|s_2-1|^4}+O\left(t_2^{-\frac{5}{2}}(\log t_2)T^{\frac{5}{2}}\right) \tag{3.1}$$

with $\sigma_1+\sigma_2=1$ and $2\le t_2\le\frac{T^{\frac{1}{3}}}{\log T}$,

$$\int_2^T |\zeta_2(s_1,s_2)|^4\,dt_1=\frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2}\,\frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)}\,\frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4}+O\left(t_2^{-\frac{5}{2}}T^{\frac{11}{2}-3\sigma_1-3\sigma_2}\right) \tag{3.2}$$

with $1<\sigma_1+\sigma_2<\frac{3}{2}$ and $2\le t_2\le T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$,

$$\int_2^T |\zeta_2(s_1,s_2)|^4\,dt_1=\frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2}\frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)}\frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4}+\begin{cases} O\left(t_2^{-4}T^{6-3\sigma_1-3\sigma_2+\varepsilon}\right) & \text{if } 2\le t_2\le T^{\frac{1}{5-2\sigma_1-2\sigma_2}+\varepsilon}, \\ O\left(t_2^{-\frac{3}{2}-\sigma_1-\sigma_2}T^{\frac{11}{2}-3\sigma_1-3\sigma_2}\right) & \text{if } T^{\frac{1}{5-2\sigma_1-2\sigma_2}+\varepsilon}\le t_2\le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}} \end{cases} \tag{3.3}$$

with $\frac{1}{2}\le\sigma_1+\sigma_2<1$,

$$\begin{aligned}\int_2^T |\zeta_2(s_1,s_2)|^4\,dt_1=&\frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2}\frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)}\frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4}\\&+O\left(t_2^{6-4\sigma_1-4\sigma_2}T\right)+O\left(t_2^{-3}T^{6-4\sigma_1-4\sigma_2}\right)\\&+O\left(t_2^{-\frac{3}{2}-\sigma_1-\sigma_2}T^{\frac{11}{2}-3\sigma_1-3\sigma_2}\right)+O\left(t_2^{-4}T^{6-3\sigma_1-3\sigma_2+\varepsilon}\right)\end{aligned} \tag{3.4}$$

with $0 < \sigma_1 + \sigma_2 < \frac{1}{2}$ and $2 \leq t_2 \leq T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$,

$$\int_2^T |\zeta_2(s_1, s_2)|^4 \, dt_1 = O\left(t_2^2 T\right) \tag{3.5}$$

with $\frac{3}{2} < \sigma_1 + \sigma_2 < 2$, and

$$\int_2^T |\zeta_2(s_1, s_2)|^4 \, dt_1 \\ = \begin{cases} \dfrac{1}{2\pi^2} \dfrac{T \log^4 T}{|s_2 - 1|^4} + O\left(t_2^{-\frac{5}{2}} T \log^3 T\right) & \text{if} \quad 2 \leq t_2 \leq (\log T)^{\frac{2}{3}}, \\ O\left(t_2^2 T\right) & \text{if} \quad t_2 \geq (\log T)^{\frac{2}{3}} \end{cases} \tag{3.6}$$

with $\sigma_1 + \sigma_2 = \frac{3}{2}$. Furthermore, Kiuchi and Minamide derived the Ω results of the double zeta-function $\zeta_2(s_1, s_2)$ from the above formulas, which hold that under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $2 \leq t_1 \leq T$, we have

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) = \Omega\left(\frac{t_1^{\frac{3}{2} - \sigma_1 - \sigma_2}}{t_2}\right) \tag{3.7}$$

with $1 \leq \sigma_1 + \sigma_2 < \frac{3}{2}$ and $2 \leq t_2 \leq T^{1 - \frac{2}{3}(\sigma_1 + \sigma_2) - \varepsilon}$, and

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) = \Omega\left(\frac{t_1^{\frac{3}{2} - \sigma_1 - \sigma_2}}{t_2}\right) \tag{3.8}$$

with $0 < \sigma_1 + \sigma_2 < 1$, $2 \leq t_2 \leq T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2} - \varepsilon}$, and ε being any small positive constant. The Ω results (3.7) and (3.8) coincide with (2.9) and (2.10), respectively.

3.2 Fourth power moment – concerning t_2

The first purpose of the present paper is to prove the fourth power moments of the double zeta-function $\zeta_2(s_1, s_2)$ concerning the variable t_2 instead of the variable t_1 under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 2$. We make use of the reciprocity law (1.1) and the method of the authors [7] to obtain the fourth power moments for the double zeta-function $\zeta_2(s_2, s_1)$ under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 2$. Then we have the following.

Theorem 1 *Suppose that $2 \leq t_1 \leq T$ and $2 \leq t_2 \leq \frac{T^{\frac{1}{3}}}{\log T}$. For any sufficiently large positive number $T > 2$, we have*

$$\int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 = \frac{\zeta^4(2)}{12\pi^2 \zeta(4)} \frac{T^3}{|s_2 - 1|^4} + O\left(t_2^{-\frac{5}{2}} (\log t_2) T^{\frac{5}{2}}\right) \tag{3.9}$$

with $\frac{1}{2} < \sigma_1 < 1$, $0 < \sigma_2 < \frac{1}{2}$ and $\sigma_1 + \sigma_2 = 1$.

For $2 \le t_1 \le T$ and $2 \le t_2 \le \frac{T^{\frac{1}{3}}}{\log T}$, we observe that the main terms on the right-hand side of (3.1) and (3.9) are the same one with $\frac{1}{2} < \sigma_1 < 1$, $0 < \sigma_2 < \frac{1}{2}$ and $\sigma_1 + \sigma_2 = 1$, but (3.9) does not coincide with (3.1), namely

$$\int_2^T |\zeta_2(s_1, s_2)|^4 \, dt_1 - \int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 = O\left(t_2^{-\frac{5}{2}} (\log t_2) T^{\frac{5}{2}}\right). \tag{3.10}$$

Integrating (3.10) by the variable t_2, then

$$\frac{1}{TN} \int_2^N \int_2^T \left\{ |\zeta_2(s_1, s_2)|^4 - |\zeta_2(s_2, s_1)|^4 \right\} dt_1 dt_2 = O\left(\frac{T^{\frac{3}{2}}}{N}\right) \tag{3.11}$$

for $2 < N \le \frac{T^{\frac{1}{3}}}{\log T}$.

Theorem 2 *Suppose that $2 \le t_1 \le T$ and $\sigma_1 = \sigma_2 = \frac{1}{2}$. Then, for any sufficiently large positive number $T > 2$, we have*

$$\begin{aligned}\int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 &= \frac{\zeta^4(2)}{12\pi^2 \zeta(4)} \frac{T^3}{|s_2 - 1|^4} \\ &+ \begin{cases} O\left(t_2^{-\frac{5}{2}} (\log t_2) T^{\frac{5}{2}}\right) & \text{if } \log^3 T \le t_2 \le \frac{T^{\frac{1}{3}}}{\log T}, \\ O\left(t_2^{-\frac{17}{6}} (\log t_2) T^{\frac{5}{2}} \log T\right) & \text{if } 2 \le t_2 \le \log^3 T. \end{cases}\end{aligned} \tag{3.12}$$

Theorem 3 *Suppose that $2 \le t_1 \le T$, $2 \le t_2 \le \frac{T^{\frac{1}{3}}}{\log T}$, $0 < \sigma_1 < \frac{1}{2}$, $\frac{1}{2} < \sigma_2 < 1$ and $\sigma_1 + \sigma_2 = 1$. Then, for any sufficiently large positive number $T > 2$, we have*

$$\begin{aligned}\int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 &= \frac{\zeta^4(2)}{12\pi^2 \zeta(4)} \frac{T^3}{|s_2 - 1|^4} \\ &+ \begin{cases} O\left(t_2^{-\frac{5}{2}} (\log t_2) T^{\frac{5}{2}}\right) & \text{if } T^{3\frac{1-2\sigma_2}{1+2\sigma_2}} (\log T)^{\frac{6}{1+2\sigma_2}} \le t_2 \le \frac{T^{\frac{1}{3}}}{\log T}, \\ O\left(t_2^{-\frac{8}{3} - \frac{\sigma_2}{3}} \log^2 t_2 \; T^{3-\sigma_1}\right) & \text{if } 2 \le t_2 \le T^{3\frac{1-2\sigma_2}{1+2\sigma_2}} (\log T)^{\frac{6}{1+2\sigma_2}}. \end{cases}\end{aligned} \tag{3.13}$$

From Theorems 1 and 2, we immediately derive an alternative proof of one of the Ω results in Kiuchi, Tanigawa and Zhai [17], namely

Corollary 1 *Let $\sigma_1 + \sigma_2 = 1$, $\frac{1}{2} \le \sigma_1 < 1$, $0 < \sigma_2 \le \frac{1}{2}$ and $2 \le t_1 \le T$. We have*

$$\zeta_2(\sigma_2 + it_2, \sigma_1 + it_1) = \Omega\left(\frac{t_1^{\frac{1}{2}}}{t_2}\right) \tag{3.14}$$

with $2 \le t_2 \le T^{\frac{1}{3} - \varepsilon}$ and ε being any small positive constant.

Under the condition $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $1 < \sigma = \sigma_1 + \sigma_2 < \frac{3}{2}$, we shall consider the fourth power moments for the double zeta-function $\zeta_2(s_2, s_1)$ with respect to the variable t_1.

Theorem 4 *Suppose that* $2 \leq t_1 \leq T$, $2 \leq t_2 \leq T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$, $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ *and* $1 < \sigma_1 + \sigma_2 < \frac{3}{2}$. *Then, for any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 = \frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2} \frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)} \frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4} + O\left(t_2^{-\frac{5}{2}} T^{\frac{11}{2}-3\sigma_1-3\sigma_2}\right) \tag{3.15}$$

with $\frac{1}{2} < \sigma_1 < 1$ *and* $0 < \sigma_2 < 1$,

$$\int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 = \frac{(2\pi)^{4\sigma_2-4}}{5-4\sigma_2} \frac{\zeta^4(3-2\sigma_2)}{\zeta(6-4\sigma_2)} \frac{T^{5-4\sigma_2}}{|s_2-1|^4} + O\left(t_2^{-\frac{5}{2}} T^{4-3\sigma_2}\right) + O\left(t_2^{-\frac{8}{3}-\frac{1}{3}\sigma_2} \log^2 t_2 \, T^{4-3\sigma_2} \log T\right) \tag{3.16}$$

with $\sigma_1 = \frac{1}{2}$ *and* $\frac{1}{2} < \sigma_2 < 1$, *and*

$$\int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 = \frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2} \frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)} \frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4} + O\left(t_2^{-\frac{5}{2}} T^{\frac{11}{2}-3\sigma_1-3\sigma_2}\right) + O\left(t_2^{-\frac{8}{3}-\frac{1}{3}\sigma_2} \log^2 t_2 \, T^{6-4\sigma_1-3\sigma_2}\right) \tag{3.17}$$

with $0 < \sigma_1 < \frac{1}{2}$ *and* $\frac{1}{2} < \sigma_2 < 1$.

Furthermore, we shall consider the fourth power moments of the double zeta-function $\zeta_2(s_2, s_1)$ with respect to the variable t_1 under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 1$.

Theorem 5 *Suppose that* $2 \leq t_1 \leq T$, $2 \leq t_2 \leq T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$, $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ *and* $\frac{1}{2} < \sigma_1 + \sigma_2 < 1$. *Then, for any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 \tag{3.18}$$

$$= \frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2} \frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)} \frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4} + O\left(t_2^{-4} T^{6-3\sigma_1-3\sigma_2+\epsilon}\right) + O\left(t_2^{-\frac{3}{2}-\sigma_1-\sigma_2} T^{\frac{11}{2}-3\sigma_1-3\sigma_2}\right) \tag{3.19}$$

with $\frac{1}{2} < \sigma_1 < 1$ *and* $0 < \sigma_2 < \frac{1}{2}$,

$$\int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 \tag{3.20}$$

$$= \frac{(2\pi)^{4\sigma_2-4}}{5-4\sigma_2} \frac{\zeta^4(3-2\sigma_2)}{\zeta(6-4\sigma_2)} \frac{T^{5-4\sigma_2}}{|s_2-1|^4} + O\left(t_2^{-4} T^{\frac{9}{2}-3\sigma_2+\epsilon}\right) + O\left(t_2^{-2-\sigma_2} T^{4-3\sigma_2}\right) + O\left(t_2^{-\frac{5}{2}-\frac{2}{3}\sigma_2} \log t_2 \, T^{4-3\sigma_2} \log T\right)$$

with $\sigma_1=\frac{1}{2}$ *and* $0<\sigma_2<\frac{1}{2}$, *and*

$$\begin{aligned}&\int_2^T |\zeta_2(s_2,s_1)|^4\,dt_1 \qquad (3.21)\\&=\frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2}\frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)}\frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4}+O\left(t_2^{-4}T^{6-3\sigma_1-3\sigma_2+\epsilon}\right)\\&+O\left(t_2^{-\frac{3}{2}-\sigma_1-\sigma_2}T^{\frac{11}{2}-3\sigma_1-3\sigma_2}\right)+O\left(t_2^{-\frac{5}{2}-\frac{\sigma_2}{2}}\log^2 t_2T^{6-4\sigma_1-3\sigma_2}\right)\end{aligned}$$

with $0<\sigma_1<\frac{1}{2}$ *and* $0<\sigma_2<1$.

Theorem 6 *Suppose that* $2\le t_1\le T$, $2\le t_2\le T^{\frac{1}{2}}$, $0<\sigma_1<\frac{1}{2}$, $0<\sigma_2<\frac{1}{2}$ *and* $\sigma_1+\sigma_2=\frac{1}{2}$. *Then, for any sufficiently large positive number* $T>2$, *we have*

$$\begin{aligned}\int_2^T |\zeta_2(s_2,s_1)|^4\,dt_1&=\frac{\zeta^4(3)T^5}{5(2\pi)^4\zeta(6)|s_2-1|^4}+O\left(t_2^{-4}T^{\frac{9}{2}+\epsilon}\right) \qquad (3.22)\\&+O\left(t_2^{-2}T^4\right)+O\left(t_2^{-\frac{5}{2}-\frac{2}{3}\sigma_2}\log t_2\ T^{4+\sigma_2}\right).\end{aligned}$$

Theorem 7 *Suppose that* $2\le t_1\le T$, $2\le t_2\le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$, $0<\sigma_1<\frac{1}{2}$, $0<\sigma_2<\frac{1}{2}$ *and* $0<\sigma_1+\sigma_2<\frac{1}{2}$. *Then, for any sufficiently large positive number* $T>2$, *we have*

$$\begin{aligned}&\int_2^T |\zeta_2(s_2,s_1)|^4\,dt_1=\frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2}\frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)}\frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4} \qquad (3.23)\\&+O\left(t_2^{-3}T^{6-4\sigma_1-4\sigma_2}\right)+O\left(t_2^{-\frac{3}{2}-\sigma_1-\sigma_2}T^{\frac{11}{2}-3\sigma_1-3\sigma_2}\right)+O\left(t_2^{-4}T^{6-3\sigma_1-3\sigma_2+\epsilon}\right)\\&+O\left(t_2^{6-4\sigma_1-4\sigma_2}T\right)+O\left(t_2^{-\frac{5}{2}-\frac{\sigma_2}{2}}(\log t_2)T^{6-4\sigma_1-3\sigma_2}\right).\end{aligned}$$

As an application of Theorems 4–7, we give the Ω results of the double zeta-function $\zeta_2(s_1,s_2)$. Then we deduce the following formulas.

Corollary 2 *Let* $0<\sigma_1+\sigma_2<\frac{3}{2}$, $\sigma_1+\sigma_2\ne1$, $0<\sigma_1<1$, $0<\sigma_2<1$ *and* $2\le t_1\le T$. *We have*

$$\zeta_2(\sigma_2+it_2,\sigma_1+it_1)=\Omega\left(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\right) \qquad (3.24)$$

for $1<\sigma_1+\sigma_2<\frac{3}{2}$ *and* $2\le t_2\le T^{1-\frac{2}{3}(\sigma_1+\sigma_2)-\varepsilon}$, *and*

$$\zeta_2(\sigma_2+it_2,\sigma_1+it_1)=\Omega\left(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\right) \qquad (3.25)$$

for $0<\sigma_1+\sigma_2<1$ *and* $2\le t_2\le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}-\varepsilon}$, *with* ε *being any small positive constant.*

Now, we suppose that $\frac{1}{2} < \sigma_1 < 1$, $\frac{1}{2} < \sigma_2 < 1$ and $2 \leq t_1 \leq T$. In a similar manner to the method of proofs of Theorems 1–7, we have

$$\int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 = O\left(t_2^2 T\right) \tag{3.26}$$

for $\frac{3}{2} < \sigma_1 + \sigma_2 < 2$,

$$\int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 = \frac{1}{2\pi^2} \frac{T \log^4 T}{|s_2 - 1|^4} + O\left(t_2^{-\frac{5}{2}} T \log^3 T\right) \tag{3.27}$$

for $\sigma_1 + \sigma_2 = \frac{3}{2}$ and $2 \leq t_2 \leq \log^{\frac{2}{3}} T$, and

$$\int_2^T |\zeta_2(s_2, s_1)|^4 \, dt_1 = O\left(t_2^2 T\right) \tag{3.28}$$

for $\sigma_1 + \sigma_2 = \frac{3}{2}$ and $t_2 \geq \log^{\frac{2}{3}} T$.

To improve the formulas (3.26)–(3.28), we must obtain a sharper estimate for the function $E(s_1, s_2)$ in Lemma 1 below, but this is very difficult.

4 Even power moments

Let $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 2$, and let k (≥ 3) be any fixed positive integer. We shall consider the $2k$-th power moments of the double zeta-function $\zeta_2(s_1, s_2)$ with respect to the variable t_1. Furthermore, we shall consider the $2k$-th power moments of the double zeta-function $\zeta_2(s_1, s_2)$ with respect to the variable t_2, too.

4.1 Even power moments – concerning t_1

To state our theorem, let k be any fixed positive integer larger than or equal to 3, and let $d_k(n)$ be the number of ways n can be written as a product of k factors. Furthermore, the Dirichlet series $D_k(\sigma)$ is given by $D_k(\sigma) = \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{2\sigma}}$ and $D_k = D_k(1)$. We derive the $2k$-th power moments of the double zeta-function $\zeta_2(s_1, s_2)$ with respect to the variable t_1 in the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 2$, whose proof makes use of the method of Kiuchi and Minamide [14]. Then we derive the following formula.

Theorem 8 *Suppose that $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $\sigma_1 + \sigma_2 = 1$. For any sufficiently large positive number $T > 2$, we have*

$$\int_2^T |\zeta_2(s_1, s_2)|^{2k} \, dt_1 = \frac{D_k}{(2\pi)^k (k+1)} \frac{T^{k+1}}{|s_2 - 1|^{2k}} + O\left(t_2^{\frac{3}{2} - 2k} (\log t_2) T^{k + \frac{1}{2}}\right) \tag{4.1}$$

with $2 \leq t_2 \leq \frac{T^{\frac{1}{3}}}{\log T}$.

Inserting $t_2 = \frac{T^{\frac{1}{3}}}{\log T}$ into (4.1), the right-hand side of the formula (4.1) is estimated by $T^{\frac{1}{3}k+1}(\log T)^{2k}$, if we can take $t_2 = \frac{T^{\frac{1}{2}}}{\log T}$, so estimates $T^{\frac{5}{4}}(\log T)^{2k}$. The main term of this theorem is not $T(\log T)^A$ $(A > 0)$, but T^{k+1}, since the analytic behaviour of the double zeta-function $\zeta_2(s_1, s_2)$ depends on both s_1 and s_2. This observation is in fact to support of the Ω result of Kiuchi, Tanigawa and Zhai.

As two applications of (4.1), we give the evaluation of the double integral $\int_2^N \int_2^T |\zeta_2(s_1, s_2)|^{2k} dt_1 dt_2$ and the Ω result of the double zeta-function. Then we deduce the following.

Corollary 3 *Let* $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ *and* $\sigma_1 + \sigma_2 = 1$. *We have*

$$\lim_{T\to\infty} \frac{1}{T^{k+1}} \int_2^N \int_2^T |\zeta_2(s_1, s_2)|^{2k} dt_1 dt_2$$
$$= \frac{D_k}{(2\pi)^k (k+1)} \int_2^\infty \frac{1}{|s_2 - 1|^{2k}} dt_2 + O\left(N^{1-2k}\right) \tag{4.2}$$

for $2 \le N \le \frac{T^{\frac{1}{3}}}{\log T}$.

Hence, this observation may be regarded as certain average order of magnitude of the double zeta-function, which is

$$\frac{D_k}{(2\pi)^k (k+1)} \int_2^\infty \frac{1}{|s_2 - 1|^{2k}} dt_2$$

for $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $\sigma_1 + \sigma_2 = 1$ and $2 \le N \le \frac{T^{\frac{1}{3}}}{\log T}$.

Corollary 4 *Suppose that* $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $\sigma_1 + \sigma_2 = 1$, $2 \le t_1 \le T$ *and* $2 \le t_2 \le T^{\frac{1}{3} - \varepsilon}$ *with* ε *being any small positive constant. Then we have*

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) = \Omega\left(\frac{t_1^{\frac{1}{2}}}{t_2}\right). \tag{4.3}$$

This corollary imply an improvement on the result of Kiuchi, Tanigawa and Zhai.

Theorem 9 *Suppose that* $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ *and* $1 < \sigma_1 + \sigma_2 < 2$. *For any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(s_1, s_2)|^{2k} \, dt_1 = O\left(t_2^{-2k} T^{\frac{k}{3}(7 - 4(\sigma_1 + \sigma_2)) + 1} \log^{2k} T\right) \tag{4.4}$$

with $1 < \sigma_1 + \sigma_2 \le \frac{3}{2}$ *and* $2 \le t_2 \le T^{\frac{1}{9}(7 - 4(\sigma_1 + \sigma_2))} \log^{\frac{2}{3}} T$, *and*

$$\int_2^T |\zeta_2(s_1, s_2)|^{2k} \, dt_1 = O\left(t_2^{-2k} T^{\frac{2k}{3}(2 - \sigma_1 - \sigma_2) + 1} \log^{4k} T\right) \tag{4.5}$$

with $\frac{3}{2} < \sigma_1 + \sigma_2 < 2$ *and* $2 \le t_2 \le T^{\frac{2}{9}(2 - \sigma_1 - \sigma_2)} \log^{\frac{4}{3}} T$.

Theorem 10 *Suppose that* $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $2 \leq t_1 \leq T$ *and* $2 \leq t_2 \leq T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. *For any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(s_1,s_2)|^{2k}\, dt_1 = \frac{(2k)^{3k-2k(\sigma_1+\sigma_2)} D_k(2-\sigma_1-\sigma_2)}{(3k+1-k(\sigma_1+\sigma_2))|s_2-1|^{2k}}\, T^{3k+1-2k(\sigma_1+\sigma_2)} \tag{4.6}$$
$$+ O\left(t_2^{\frac{5}{2}-2k-\sigma_1-\sigma_2} T^{3k-\frac{1}{2}-(2k-1)(\sigma_1+\sigma_2)}\right) + O\left(t_2^{1-2k} T^{3k-\frac{1}{6}-(2k-\frac{2}{3})(\sigma_1+\sigma_2)} \log^2 T\right)$$
$$+ O\left(t_2^{-2k} T^{3k-(2k-1)(\sigma_1+\sigma_2)}\right)$$

with $\frac{1}{2} < \sigma_1 + \sigma_2 < 1$, *and*

$$\int_2^T |\zeta_2(s_1,s_2)|^{2k}\, dt_1 = \frac{(2k)^{3k-2k(\sigma_1+\sigma_2)} D_k(2-\sigma_1-\sigma_2)}{(3k+1-k(\sigma_1+\sigma_2))|s_2-1|^{2k}}\, T^{3k+1-2k(\sigma_1+\sigma_2)} \tag{4.7}$$
$$+ O\left(t_2^{\frac{5}{2}-2k-\sigma_1-\sigma_2} T^{3k-\frac{1}{2}-(2k-1)(\sigma_1+\sigma_2)}\right) + O\left(t_2^{1-2k} T^{3k-(2k-\frac{1}{3})(\sigma_1+\sigma_2)} \log T\right)$$
$$+ O\left(t_2^{-2k} T^{3k-(2k-1)(\sigma_1+\sigma_2)}\right)$$

with $0 < \sigma_1 + \sigma_2 \leq \frac{1}{2}$.

From Theorem 10, we deduce the following.

Corollary 5 *Suppose that* $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $0 < \sigma_1 + \sigma_2 < 1$, $2 \leq t_1 \leq T$ *and* $2 \leq t_2 \leq T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}-\varepsilon}$ *with* ε *being any small positive constant. Then we have*

$$\zeta_2(\sigma_1+it_1, \sigma_2+it_2) = \Omega\left(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\right). \tag{4.8}$$

4.2 Even power moment – concerning t_2

Secondly, we shall consider the $2k$-th power moments of the double zeta-function $\zeta_2(s_1,s_2)$ concerning the variable t_2 in place of the variable t_1 within the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $0 < \sigma_1 + \sigma_2 < 2$. Thus, by using the reciprocity law (1.1) and the method of Ikeda, Kiuchi and Matsuoka [7], it is sufficient to calculate the integral $\int_2^T |\zeta(s_2,s_1)|^{2k} dt_1$ within the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $\sigma_1 + \sigma_2 = 1$. Then we derive the following formula.

Theorem 11 *Suppose that* $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ *and* $\sigma_1 + \sigma_2 = 1$. *Then, for any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(s_2,s_1)|^{2k}\, dt_1 = \frac{D_k}{(2\pi)^k(k+1)}\, \frac{T^{k+1}}{|s_2-1|^{2k}} \tag{4.9}$$

$$
+\begin{cases} O\left(t_2^{\frac{4}{3}-2k-\frac{1}{3}\sigma_2}(\log t_2)^2T^{k+1-\frac{2}{3}\sigma_1}\log T\right)+O\left(t_2^{\frac{3}{2}-2k}(\log t_2)T^{k+\frac{1}{2}}\right) \\ \quad \text{if } 0<\sigma_1<\frac{1}{2},\ \frac{1}{2}<\sigma_2<1,\ \sigma_1+\sigma_2=1 \text{ and } 2\le t_2\le T^{\frac{2\sigma_1}{4-\sigma_2}}(\log T)^{-3}, \\ O\left(t_2^{\frac{7}{6}-2k}(\log t_2)T^{k+\frac{2}{3}}\log T\right)+O\left(t_2^{\frac{3}{2}-2k}(\log t_2)T^{k+\frac{1}{2}}\right) \\ \quad \text{if } \sigma_1=\sigma_2=\frac{1}{2} \text{ and } 2\le t_2\le T^{\frac{2}{7}}(\log T)^{-3}, \\ O\left(t_2^{\frac{3}{2}-2k-\frac{2}{3}\sigma_2}(\log t_2)T^{k+\frac{5}{6}-\frac{1}{3}\sigma_1}\log^2 T\right)+O\left(t_2^{\frac{3}{2}-2k}(\log t_2)T^{k+\frac{1}{2}}\right) \\ \quad \text{if } \frac{1}{2}<\sigma_1<1,\ 0<\sigma_2<\frac{1}{2},\ \sigma_1+\sigma_2=1 \text{ and } 2\le t_2\le T^{\frac{1+2\sigma_1}{9-4\sigma_2}}(\log T)^{-3}. \end{cases}
$$

We observe that the main terms on the right-hand side of (4.1) and (4.9) are the same one with $0<\sigma_1<1$, $0<\sigma_2<1$ and $\sigma_1+\sigma_2=1$, but (4.9) does not coincide with (4.1), namely

$$
\int_2^T |\zeta_2(s_2,s_1)|^{2k}dt_1-\int_2^T |\zeta_2(s_1,s_2)|^{2k}dt_1
$$
$$
=\begin{cases} O\left(t_2^{\frac{3}{2}-2k}(\log t_2)T^{k+\frac{1}{2}}\right)+O\left(t_2^{\frac{4}{3}-2k-\frac{1}{3}\sigma_2}(\log t_2)^2T^{k+1-\frac{2}{3}\sigma_1}\log T\right) \\ \quad \text{if } \ 0<\sigma_1<\frac{1}{2},\ \frac{1}{2}<\sigma_2<1, \text{ and } 2\le t_2\le T^{\frac{2\sigma_1}{4-\sigma_2}}(\log T)^{-3}, \\ O\left(t_2^{\frac{3}{2}-2k}(\log t_2)T^{k+\frac{1}{2}}\right)+O\left(t_2^{\frac{7}{6}-2k}(\log t_2)T^{k+\frac{2}{3}}\log T\right) \\ \quad \text{if } \sigma_1=\sigma_2=\frac{1}{2} \text{ and } 2\le t_2\le T^{\frac{2}{7}}(\log T)^{-3}, \\ O\left(t_2^{\frac{3}{2}-2k}(\log t_2)T^{k+\frac{1}{2}}\right)+O\left(t_2^{\frac{3}{2}-2k-\frac{2}{3}\sigma_2}(\log t_2)T^{k+\frac{5}{6}-\frac{1}{3}\sigma_1}\log T\right) \\ \quad \text{if } \ \frac{1}{2}<\sigma_1<1,\ 0<\sigma_2<\frac{1}{2}, \text{ and } 2<t_2<T^{\frac{1+2\sigma_1}{9-4\sigma_2}}(\log T)^{-3}. \end{cases}
$$

Integrating the above by the variable t_2, we have

$$
\int_2^N\int_2^T \left\{|\zeta_2(s_2,s_1)|^{2k}-|\zeta_2(s_1,s_2)|^{2k}\right\}dt_1dt_2
$$
$$
=\begin{cases} O\left(T^{k+\frac{1}{2}}\right)+O\left(T^{k+1-\frac{2}{3}\sigma_1}\log T\right) \\ \quad \text{if } \ 0<\sigma_1<\frac{1}{2},\ \frac{1}{2}<\sigma_2<1, \text{ and } 2\le N\le T^{\frac{2\sigma_1}{4-\sigma_2}}(\log T)^{-3}, \\ O\left(T^{k+\frac{2}{3}}\log T\right) \\ \quad \text{if } \ \sigma_1=\sigma_2=\frac{1}{2} \text{ and } 2\le N\le T^{\frac{2}{7}}(\log T)^{-3}, \\ O\left(T^{k+\frac{1}{2}}\right)+O\left(T^{k+\frac{5}{6}-\frac{1}{3}\sigma_1}\log T\right) \\ \quad \text{if } \ \frac{1}{2}<\sigma_1<1,\ 0<\sigma_2<\frac{1}{2}, \text{ and } 2\le N\le T^{\frac{1+2\sigma_1}{9-4\sigma_2}}(\log T)^{-3}. \end{cases}
$$

Hence the double integral for $|\zeta_2(s_1,s_2)|^2-|\zeta_2(s_2,s_1)|^2$ may be regarded as certain average of the double zeta-function, namely

$$\frac{1}{TN}\int_2^N\int_2^T\left\{|\zeta_2(s_2,s_1)|^{2k}-|\zeta_2(s_1,s_2)|^{2k}\right\}dt_1dt_2$$
$$=O\left(\frac{T^{k-\frac12}}{N}+\frac{T^{k-\frac23\sigma_1}\log T}{N}+\frac{T^{k-\frac16-\frac13\sigma_1}\log T}{N}\right) \tag{4.10}$$

under the conditions $0<\sigma_1\le\frac12$, $\frac12\le\sigma_2<1$, $\sigma_1+\sigma_2=1$ and $2\le N\le T^{\frac{2\sigma_1}{4-\sigma_2}}(\log T)^{-3}$, or $\frac12\le\sigma_1<1$, $0<\sigma_2\le\frac12$, $\sigma_1+\sigma_2=1$ and $2\le N\le T^{\frac{1+2\sigma_1}{9-4\sigma_2}}(\log T)^{-3}$.

Corollary 6 *Suppose that* $0<\sigma_1\le\frac12$, $\frac12\le\sigma_2<1$, $\sigma_1+\sigma_2=1$, $2\le t_1\le T$ *and* $2\le t_2\le T^{\frac{2\sigma_1}{4-\sigma_2}-\varepsilon}$, *or* $\frac12\le\sigma_1<1$, $0<\sigma_2\le\frac12$, $\sigma_1+\sigma_2=1$, $2\le t_1\le T$ *and* $2\le t_2\le T^{\frac{1+2\sigma_1}{9-4\sigma_2}-\varepsilon}$ *with* ε *being any small positive constant. Then we have*

$$\zeta_2(\sigma_2+it_2,\sigma_1+it_1)=\Omega\left(\frac{t_1^{\frac12}}{t_2}\right). \tag{4.11}$$

Furthermore, we shall consider the $2k$-th power moments for the double zeta-function $\zeta_2(s_2,s_1)$ with respect to the variable t_1 within the region $0<\sigma_1<1$, $0<\sigma_2<1$ and $\sigma_1+\sigma_2\neq1$.

Theorem 12 *Suppose that* $0<\sigma_1<1$, $0<\sigma_2<1$, $1<\sigma_1+\sigma_2<2$ *and* $2\le t_1\le T$. *Then, for any sufficiently large positive number* $T>2$, *we have*

$$\int_2^T|\zeta_2(s_2,s_1)|^{2k}dt_1=O\left(t_2^{-2k}T^{\frac73k+1-\frac43k(\sigma_1+\sigma_2)}\log^{2k}T\right) \tag{4.12}$$

with $0<\sigma_1\le\frac12$, $\frac12<\sigma_2<1$, $1<\sigma_1+\sigma_2\le\frac32$ *and* $2\le t_2\le T^{\frac{2-2\sigma_2}{4-\sigma_2}}(\log T)^{-6}$, *or* $\frac12<\sigma_1<1$, $\frac12<\sigma_2<1$, $1<\sigma_1+\sigma_2\le\frac32$ *and* $2\le t_2\le T^{\frac{5-2\sigma_1-4\sigma_2}{8-2\sigma_2}}(\log T)^{-3}$, *or* $\frac12<\sigma_1<1$, $0<\sigma_2\le\frac12$, $1<\sigma_1+\sigma_2\le\frac32$ *and* $2\le t_2\le T^{\frac{5-2\sigma_1-4\sigma_2}{9-4\sigma_2}}(\log T)^{-6}$.

Furthermore, for any sufficiently large positive number $T>2$, *we have*

$$\int_2^T|\zeta_2(s_2,s_1)|^{2k}dt_1=O\left(t_2^{-2k}T^{\frac{4k}{3}+1-\frac{2k}{3}(\sigma_1+\sigma_2)}\log^{4k}T\right) \tag{4.13}$$

with $\frac12<\sigma_1<1$, $\frac12<\sigma_2<1$, $\frac32<\sigma_1+\sigma_2<2$ *and* $2\le t_2\le T^{\frac{1+2\sigma_1-2\sigma_2}{8-2\sigma_2}}(\log T)^{-1}$.

Theorem 13 *Suppose that* $0<\sigma_1<1$, $0<\sigma_2<1$, $0<\sigma_1+\sigma_2<1$, $2\le t_1\le T$ *and* $2\le t_2\le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. *Then,, for any sufficiently large positive number* $T>2$, *we have*

$$\int_2^T|\zeta_2(s_2,s_1)|^{2k}dt_1=\frac{(2k)^{3k-2k(\sigma_1+\sigma_2)}D_k(2-\sigma_1-\sigma_2)}{(3k+1-k(\sigma_1+\sigma_2))|s_2-1|^{2k}}T^{3k+1-2k(\sigma_1+\sigma_2)} \tag{4.14}$$
$$+O\left(t_2^{\frac52-2k-\sigma_1-\sigma_2}T^{3k-\frac12-(2k-1)(\sigma_1+\sigma_2)}\right)+O\left(t_2^{1-2k}T^{3k-\frac16-(2k-\frac23)(\sigma_1+\sigma_2)}\log^2T\right)$$
$$+O\left(t_2^{-2k}T^{3k-(2k-1)(\sigma_1+\sigma_2)}\right)$$

$$+\begin{cases} O\left(t_2^{\frac{4}{3}-2k-\frac{1}{3}\sigma_2}(\log t_2)^2 T^{3k-\frac{2}{3}\sigma_1-(2k-1)(\sigma_1+\sigma_2)}\log T\right) \\ \qquad \text{if } 0<\sigma_1<\frac{1}{2} \text{ and } \frac{1}{2}<\sigma_2<1, \\ O\left(t_2^{\frac{3}{2}-2k-\frac{2}{3}\sigma_2}(\log t_2) T^{3k-\frac{2}{3}\sigma_1-(2k-1)(\sigma_1+\sigma_2)}\log T\right) \\ \qquad \text{if } 0<\sigma_1\le\frac{1}{2} \text{ and } 0<\sigma_2\le\frac{1}{2}, \\ O\left(t_2^{\frac{3}{2}-2k-\frac{2}{3}\sigma_2}(\log t_2) T^{3k-\frac{1}{6}-\frac{1}{3}\sigma_1-(2k-1)(\sigma_1+\sigma_2)}\right) \\ \qquad \text{if } \frac{1}{2}<\sigma_1<1 \text{ and } 0<\sigma_2<\frac{1}{2} \end{cases}$$

with $\frac{1}{2}<\sigma_1+\sigma_2<1$, *and*

$$\int_2^T |\zeta_2(s_2,s_1)|^{2k}\,dt_1 = \frac{(2k)^{3k-2k(\sigma_1+\sigma_2)}D_k(2-\sigma_1-\sigma_2)}{(3k+1-k(\sigma_1+\sigma_2))|s_2-1|^{2k}}\,T^{3k+1-2k(\sigma_1+\sigma_2)} \quad (4.15)$$
$$+O\left(t_2^{\frac{5}{2}-2k-\sigma_1-\sigma_2}T^{3k-\frac{1}{2}-(2k-1)(\sigma_1+\sigma_2)}\right)+O\left(t_2^{1-2k}T^{3k-(2k-\frac{1}{3})(\sigma_1+\sigma_2)}\log T\right)$$
$$+O\left(t_2^{-2k}T^{3k-(2k-1)(\sigma_1+\sigma_2)}\right)+O\left(t_2^{\frac{3}{2}-2k-\frac{2}{3}\sigma_2}(\log t_2)T^{3k-\frac{2}{3}\sigma_1-(2k-1)(\sigma_1+\sigma_2)}\log T\right)$$

with $0<\sigma_1+\sigma_2\le\frac{1}{2}$.

From Theorem 13, we deduce the following.

Corollary 7 *Suppose that* $0<\sigma_1<1$, $0<\sigma_2<1$, $0<\sigma_1+\sigma_2<1$, $2\le t_1\le T$ *and* $2\le t_2\le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}-\varepsilon}$ *with* ε *being any small positive constant. Then we have*

$$\zeta_2(\sigma_2+it_2,\sigma_1+it_1)=\Omega\left(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\right). \quad (4.16)$$

The Ω result of Kiuchi, Tanigawa and Zhai [17] is improved by the formula (4.16), which coincide with (4.8).

5 Some Lemmas

To prove our theorems, we use Lemma 1, which is a weak form of the approximate function of the double zeta-function $\zeta_2(s_1,s_2)$ given by Kiuchi, Tanigawa and Zhai [17]. Kiuchi and Minamide used Lemma 2 of the cases $k=1$ and $k=2$ to prove the mean value formulas (2.4)–(2.8) and the fourth power moments (3.1)–(3.6) of the double zeta-function $\zeta_2(s_1,s_2)$ with respect to the variable t_1 under the region $0<\sigma_1<1$, $0<\sigma_2<1$ and $0<\sigma_1+\sigma_2<\frac{3}{2}$. The authors used Lemma 3 below of the case $k=1$ to obtain the mean value formulas (2.11)–(2.13) and (2.16)–(2.25) of the double zeta-function $\zeta_2(s_2,s_1)$ with respect to the variable t_1 under the region $0<\sigma_1<1$, $0<\sigma_2<1$ and $0<\sigma_1+\sigma_2<\frac{3}{2}$.

5.1 Preliminaries

We start from a weak form of the approximate formula of Kiuchi, Tanigawa and Zhai. The next lemma gives us a simple form of Lemma 1 in [15] (see also (2.2) and (2.3) in [14]).

Lemma 1 *Suppose that* $0 < \sigma_1 < 1$ *and* $0 < \sigma_2 < 1$. *We have*

$$\zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} - \frac{1}{2}\zeta(s_1 + s_2) + E(s_1, s_2) \tag{5.1}$$

where the error term $E(s_1, s_2)$ *is estimated as*

$$E(s_1, s_2) = \begin{cases} O\left(|t_2|^{\frac{3}{2}-\sigma_1-\sigma_2}\right) & \text{if } 0 < \sigma_1 + \sigma_2 < 1, \\ O\left(|t_2|^{\frac{1}{2}} \log |t_2|\right) & \text{if } \sigma_1 + \sigma_2 = 1, \\ O\left(|t_2|^{\frac{1}{2}}\right) & \text{if } \sigma_1 + \sigma_2 > 1. \end{cases} \tag{5.2}$$

Note that this error term $E(s_1, s_2)$ is *independent of* t_1. We use Lemma 1 and Hölder's inequality to obtain the $2k$-th power moments of the double zeta-function $\zeta_2(s_1, s_2)$ with respect to the variable t_1. Then we have the following.

Lemma 2 *Suppose that* $0 < \sigma_1 < 1$ *and* $0 < \sigma_2 < 1$. *We have*

$$\int_2^T |\zeta_2(s_1, s_2)|^{2k} dt_1 = I_1 + I_2 + I_3$$
$$+ O\left(I_1^{1-\frac{1}{2k}} I_2^{\frac{1}{2k}} + I_1^{1-\frac{1}{2k}} I_3^{\frac{1}{2k}} + I_2^{1-\frac{1}{2k}} I_3^{\frac{1}{2k}} + I_2^{1-\frac{1}{2k}} I_1^{\frac{1}{2k}} + I_3^{1-\frac{1}{2k}} I_2^{\frac{1}{2k}} + I_3^{1-\frac{1}{2k}} I_1^{\frac{1}{2k}}\right) \tag{5.3}$$

for any fixed positive integer k, *where the integrals* I_1, I_2 *and* I_3 *are defined by*

$$I_1 = \frac{1}{|s_2 - 1|^{2k}} \int_2^T |\zeta(s_1 + s_2 - 1)|^{2k} dt_1,$$
$$I_2 = \frac{1}{4^k} \int_2^T |\zeta(s_1 + s_2)|^{2k} dt_1$$

and

$$I_3 = \int_2^T |E(s_1, s_2)|^{2k} dt_1,$$

respectively.

As a similar method for the proof of Lemma 3.2 in [7], we establish the following Lemma 3. Now, using the reciprocity law (1.1), namely

$$\zeta_2(s_2, s_1) = \zeta(s_1)\zeta(s_2) - \zeta(s_1 + s_2) - \zeta_2(s_1, s_2)$$

and Hölder's inequality, we deduce the following formula

$$\int_2^T |\zeta_2(s_2, s_1)|^{2k} dt_1 = J_1 + J_2 + J_3 + O\left(J_1^{1-\frac{1}{2k}} J_2^{\frac{1}{2k}} + J_2^{1-\frac{1}{2k}} J_3^{\frac{1}{2k}} + J_3^{1-\frac{1}{2k}} J_1^{\frac{1}{2k}}\right)$$

within the region $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$, and the integrals $J_1 = \int_2^T |\zeta_2(s_1, s_2)|^{2k} dt_1$, $J_2 = |\zeta(s_2)|^2 \int_2^T |\zeta(s_1)|^{2k} dt_1$ and $J_3 = \int_2^T |\zeta(s_1 + s_2)|^{2k} dt_1$. Using the analytic property of the power moments for the Riemann zeta-function (see Lemmas 4 and 5 below), there is a positive constant $c = c(\alpha, \beta)$ depending on α and β such that the inequality

$$\int_2^{T+\xi} |\zeta(\alpha + it)|^{2k} dt \le c \int_2^T |\zeta(\beta + it)|^{2k} dt \tag{5.4}$$

holds for $0 < \beta < \alpha < 2$ and $2 \le \xi \le T$. This constant c is independent of ξ. From the inequality (5.4) we obtain $J_3 = O(J_2)$. Hence, it completes the proof of Lemma 3.

Lemma 3 *We have*

$$\int_2^T |\zeta_2(s_2, s_1)|^{2k} dt_1 = J_1 + J_2 + O\left(J_1^{1-\frac{1}{2k}} J_2^{\frac{1}{2k}} + J_2^{1-\frac{1}{2k}} J_1^{\frac{1}{2k}}\right) \tag{5.5}$$

with the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ *and* $0 < \sigma_1 + \sigma_2 < 2$, *and the integrals*

$$J_1 = \int_2^T |\zeta_2(s_1, s_2)|^{2k} dt_1 \tag{5.6}$$

and

$$J_2 = |\zeta(s_2)|^{2k} \int_2^T |\zeta(s_1)|^{2k} dt_1. \tag{5.7}$$

To deal with the integral (5.7) when $k = 2$, we need the following Lemma 4, which is the fourth power moments of the Riemann zeta-function within the region $-1 < \sigma < 2$.

Lemma 4 *For any positive number* $T > 2$, *we have*

$$\int_2^T |\zeta(\sigma + it)|^4 dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} T + O(T^{2-\sigma+\varepsilon}) + O(1) \tag{5.8}$$

with $\sigma > 1$,

$$\int_2^T |\zeta(1 + it)|^4 dt = \frac{\zeta^4(2)}{\zeta(4)} T + O(\log^4 T) \tag{5.9}$$

with $\sigma = 1$,

$$\int_2^T |\zeta(\sigma + it)|^4 dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} T + O\left(T^{2-2\sigma} \log^3 T\right) \tag{5.10}$$

with $\frac{1}{2} < \sigma < 1$,

$$\int_2^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = T\sum_{k=0}^{4} c_k(\log T)^{4-k} + O\left(T^{\frac{7}{8}+\varepsilon}\right) \tag{5.11}$$

with $\sigma = \frac{1}{2}$, $c_0 = \frac{1}{2\pi^2}$, *and the other constants* c_k *being computable,*

$$\int_2^T |\zeta(\sigma+it)|^4 dt = \frac{(2\pi)^{4\sigma-2}}{3-4\sigma}\frac{\zeta^4(2-2\sigma)}{\zeta(4-4\sigma)}T^{3-4\sigma} + O\left(T^{2-2\sigma}\log^3 T\right) \tag{5.12}$$

with $0 < \sigma < \frac{1}{2}$,

$$\int_2^T |\zeta(it)|^4 dt = \frac{\zeta^4(2)}{12\pi^2\zeta(4)}T^3 + O\left(T^2\log^4 T\right) \tag{5.13}$$

with $\sigma = 0$, *and*

$$\int_2^T |\zeta(\sigma+it)|^4 dt = \frac{(2\pi)^{4\sigma-2}}{3-4\sigma}\frac{\zeta^4(2-2\sigma)}{\zeta(4-4\sigma)}T^{3-4\sigma} + O\left(T^{3-3\sigma+\varepsilon}\right) \tag{5.14}$$

with $-1 < \sigma < 0$.

Proof. The formulas (5.8) and (5.11) follow from the standard texts (see [9], [10] or [24]). From [4] and [11], we get the formulas (5.9) and (5.10), respectively. The formulas (5.12)–(5.14) derive from Lemma 3 in [15].

To calculate the integral (5.7), we need the following Lemma 5, which is the $2k$-th power moments of the Riemann zeta-function within the region $-1 < \sigma < 2$. We again denote $d_k(n)$ by the number of ways of expressing n as a product of k factors, $D_k(\sigma) = \sum_{n=1}^{\infty}\frac{d_k(n)^2}{n^{2\sigma}}$ and $D_k(1) = D_k$.

Lemma 5 *For any positive number* $T > 2$, *we have*

$$\int_2^T |\zeta(\sigma+it)|^{2k} dt = D_k(\sigma)T + O\left(T^{2-\sigma+\varepsilon}\right) + O(1) \tag{5.15}$$

with $\sigma > 1$,

$$\int_2^T |\zeta(1+it)|^{2k} dt = D_kT + O(\log^{k^2} T), \tag{5.16}$$

with $\sigma = 1$,

$$\int_2^T |\zeta(\sigma+it)|^{2k} dt = O\left(T^{\frac{2}{3}k(1-\sigma)+1}\log^{4k} T\right) \tag{5.17}$$

with $\frac{1}{2} < \sigma < 1$,

$$\int_2^T |\zeta(\sigma+it)|^{2k} dt = O\left(T^{k(1-\frac{4}{3}\sigma)+1}\log^{2k} T\right) \tag{5.18}$$

with $0 < \sigma \le \frac{1}{2}$,

$$\int_2^T |\zeta(it)|^{2k} dt = \frac{D_k}{(2\pi)^k (k+1)} \ T^{k+1} + O\left(T^k \log^{k^2} T\right), \tag{5.19}$$

with $\sigma = 0$, *and*

$$\int_2^T |\zeta(\sigma+it)|^{2k} dt = \frac{D_k(1-\sigma)}{(2\pi)^{k(1-2\sigma)}(k+1-2\sigma k)} \ T^{k+1-2\sigma k} + O\left(T^{k+1-(2k-1)\sigma}\right) \tag{5.20}$$

with $-1 < \sigma < 0$.

Proof. The formula (5.15) follows from Theorem 1.10 in A. Ivić[9]. The formula (5.16) follows from [4]. We use the functional equation of the Riemann zeta-function (see [24] or [9]), the equality $|\zeta(1-it)| = |\zeta(1+it)|$ and the formula $|\chi(it)|^{2k} = \left(\frac{t}{2\pi}\right)^k + O\left(t^{k-1}\right)$ $(t \ge t_0 > 0)$, (5.16) and integration by parts to obtain

$$\begin{aligned} \int_2^T |\zeta(it)|^{2k} dt &= \int_2^T |\chi(it)|^{2k} |\zeta(1+it)|^{2k} dt \\ &= \left(\frac{1}{2\pi}\right)^k \int_2^T t^k |\zeta(1+it)|^{2k} dt + O\left(\int_2^T t^{k-1} |\zeta(1+it)|^{2k} dt\right) \\ &= \frac{1}{(2\pi)^k} \ \frac{D_k}{k+1} \ T^{k+1} + O\left(T^k \log^{k^2} T\right). \end{aligned}$$

Similarly as the above, we have the formula (5.20). Next, by using Lemma 6 below, we have the estimates (5.17) and (5.18) of the power moments of the Riemann zeta-function.

Furthermore, we need an upper bound for the Riemann zeta-function in the critical strip to prove our theorems, which follows from Lemma 2.5 in Kiuchi and Tanigawa [16] (see also [9] or [24]).

Lemma 6 *For* $t \ge t_0 > 1$ *uniformly in* σ, *we have*

$$\zeta(\sigma+it) = \begin{cases} O\left(t^{\frac{1}{3}(1-\sigma)} \log^2 t\right) & \text{if } \frac{1}{2} < \sigma \le 1, \\ O\left(t^{\frac{1}{2}-\frac{2}{3}\sigma} \log t\right) & \text{if } 0 \le \sigma \le \frac{1}{2}. \end{cases} \tag{5.21}$$

5.2 The function $E(s_1, s_2)$

As for the function $E(s_1, s_2)$ mentioned in Kiuchi and Minamide [14], they showed that another expression of the function $E(s_1, s_2)$ is given by

$$E(s_1, s_2) = \chi(s_2) \sum_{ml \le \frac{|t_2|}{2\pi}} \frac{1}{m^{1-s_2}} \cdot \frac{1}{l^{s_1}} + O\left(|t_2|^{\max(0, 1-\sigma_1-\sigma_2)+\varepsilon}\right). \tag{5.22}$$

The first term on the right-hand side of (5.22) can be written as follows:

$$\chi(s_2)\sum_{m\le M}\frac{1}{m^{1-s_2}}\sum_{l\le L}\frac{1}{l^{s_1}}+\cdots$$

with $M\ge 1,\ L\ge 1$ and $ML=\frac{|t_2|}{2\pi}$. We use the approximate functional equation of the Riemann zeta-function (see Theorem 4.13 in [24]) and the simplest form on the approximation to the Riemann zeta-function (see Theorem 4.11 in [24]) to obtain

$$\chi(s_2)\sum_{m\le M}\frac{1}{m^{1-s_2}}=\zeta(s_2)-\sum_{l\le L}\frac{1}{l^{s_2}}+O\left(L^{-\sigma_2}\log|t_2|\right)+O\left(|t_2|^{\frac{1}{2}-\sigma_2}M^{\sigma_2-1}\right)$$

with $0<\sigma_2<1$, and

$$\sum_{l\le L}\frac{1}{l^{s_1}}=\zeta(s_1)+\frac{L^{1-s_1}}{1-s_1}+O\left(L^{-\sigma_1}\right)$$

with $0<\sigma_1<1$, respectively. Taking the product of the above formulas and using (5.22), the function $E(s_1,s_2)$ is given by

$$E(s_1,s_2)=\zeta(s_2)\zeta(s_1)-\zeta(s_1)\sum_{l\le L}\frac{1}{l^{s_2}}+\frac{L^{1-s_1}}{1-s_1}\zeta(s_2)-\frac{L^{1-s_1}}{1-s_1}\sum_{l\le L}\frac{1}{l^{s_2}}+O\left(\cdots\right).$$

Roughly speaking, in the case where $|t_1|\asymp|t_2|$, the true order of magnitude for the function $E(s_1,s_2)$ may be regarded as $|E(s_1,s_2)|\asymp|\zeta(s_1)|\,|\zeta(s_2)|$. But in the case where $|t_2|\ll|t_1|^{\alpha}$ $(0<\alpha<\frac{1}{3})$, the order of magnitude of the function $E(s_1,s_2)$ is smaller than that of the first term on the right-hand side of (5.1), hence (5.22) implies the error term of (5.1).

6 Proofs of fourth power moment

Taking $k=2$ in Lemma 3, we have

$$\int_2^T|\zeta_2(s_2,s_1)|^4dt_1=K_1+K_2+O\left(K_1^{\frac{3}{4}}K_2^{\frac{1}{4}}\right)+O\left(K_2^{\frac{3}{4}}K_1^{\frac{1}{4}}\right)\qquad(6.1)$$

where $K_1 = \int_2^T |\zeta_2(s_1,s_2)|^4 dt_1$ and $K_2 = |\zeta(s_2)|^4 \int_2^T |\zeta(s_1)|^4 dt_1$. By using Lemmas 4 and 6, we have the following estimations

$$K_2 = \begin{cases} O\left(t_2^{2-\frac{8}{3}\sigma_2}\log^4 t_2\ T^{3-4\sigma_1}\right) & \text{if } 0<\sigma_1<\frac{1}{2},\ 0<\sigma_2\le\frac{1}{2}, \\ O\left(t_2^{\frac{4}{3}(1-\sigma_2)}\log^8 t_2\ T^{3-4\sigma_1}\right) & \text{if } 0<\sigma_1<\frac{1}{2},\ \frac{1}{2}<\sigma_2<1, \\ O\left(t_2^{2-\frac{8}{3}\sigma_2}\log^4 t_2\ T\log^4 T\right) & \text{if } \sigma_1=\frac{1}{2},\ 0<\sigma_2\le\frac{1}{2}, \\ O\left(t_2^{\frac{4}{3}(1-\sigma_2)}\log^8 t_2\ T\log^4 T\right) & \text{if } \sigma_1=\frac{1}{2},\ \frac{1}{2}<\sigma_2<1, \\ O\left(t_2^{2-\frac{8}{3}\sigma_2}\log^4 t_2\ T\right) & \text{if } \frac{1}{2}<\sigma_1<1,\ 0<\sigma_2\le\frac{1}{2}, \\ O\left(t_2^{\frac{4}{3}(1-\sigma_2)}\log^8 t_2\ T\right) & \text{if } \frac{1}{2}<\sigma_1<1,\ \frac{1}{2}<\sigma_2<1\ . \end{cases} \tag{6.2}$$

Throughout this section, we use (6.2) to evaluate the fourth power moments of the double zeta-function $\zeta_2(s_1,s_2)$.

6.1 Proofs of Theorems 1–3

We shall evaluate the integral $\int_2^T |\zeta_2(s_2,s_1)|^4 dt_1$ under the region $0<\sigma_1<1$, $0<\sigma_2<1$ and $\sigma_1+\sigma_2=1$.

When the case $\frac{1}{2}<\sigma_1<1$ and $0<\sigma_2<\frac{1}{2}$, we use (3.1) and (6.2) to obtain

$$K_1^{\frac{3}{4}}K_2^{\frac{1}{4}} = O\left((t_2^{-4}T^3)^{\frac{3}{4}}(t_2^{2-\frac{8}{3}\sigma_2}\log^4 t_2\ T)^{\frac{1}{4}}\right) = O\left(t_2^{-\frac{5}{2}}\log t_2\ T^{\frac{5}{2}}\right),$$

and $K_2 = O\left(K_1^{\frac{3}{4}}K_2^{\frac{1}{4}}\right)$. Since all error terms on the right-hand side of (6.1) are absorbed into $O\left(t_2^{-\frac{5}{2}}\log t_2\ T^{\frac{5}{2}}\right)$, we have Theorem 1.

Similarly, in the case $\sigma_1=\sigma_2=\frac{1}{2}$, we have

$$K_1^{\frac{3}{4}}K_2^{\frac{1}{4}} = O\left((t_2^{-4}T^3)^{\frac{3}{4}}(t_2^{\frac{2}{3}}\log^4 t_2\ T\log^4 T)^{\frac{1}{4}}\right) = O\left(t^{-\frac{17}{6}}\log t_2\ T^{\frac{5}{2}}\log T\right).$$

All error terms on the right-hand side of (6.1) are absorbed into $O\left(t_2^{-\frac{5}{2}}\log t_2\ T^{\frac{5}{2}}\right)$ if $\log^3 T\le t_2\le \frac{T^{\frac{1}{3}}}{\log T}$, or into $O\left(t_2^{-\frac{17}{6}}\log t_2\ T^{\frac{5}{2}}\log T\right)$ if $2\le t_2\le\log^3 T$. Hence we have Theorem 2.

Similarly, in the case $0<\sigma_1<\frac{1}{2}$ and $0<\sigma_2<\frac{1}{2}$, we have $K_1^{\frac{3}{4}}K_2^{\frac{1}{4}} = O\left(t_2^{-\frac{8}{3}-\frac{\sigma_2}{3}}\log^2 t_2\ T^{3-\sigma_1}\right)$. Hence, all error terms on the right-hand side of (6.1) are absorbed into $O\left(t_2^{-\frac{5}{2}}\log t_2\ T^{\frac{5}{2}}\right)$ if $T^{3\frac{1-2\sigma_2}{1+2\sigma_2}}(\log T)^{\frac{6}{1+2\sigma_2}}\le t_2\le\frac{T^{\frac{1}{3}}}{\log T}$, or into $O\left(t_2^{-\frac{8}{3}-\frac{\sigma_2}{3}}\log^2 t_2\ T^{3-\sigma_1}\right)$ if $2\le t_2\le T^{3\frac{1-2\sigma_2}{1+2\sigma_2}}(\log T)^{\frac{6}{1+2\sigma_2}}$. This implies Theorem 3. $\square$

6.2 Proof of Theorem 4

As in the proof of Theorems 1–3, we shall evaluate the integral $\int_2^T |\zeta_2(s_2, s_1)|^4 dt_1$ under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $1 < \sigma_1 + \sigma_2 < \frac{3}{2}$ and $2 \le t_2 \le T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$. In the case where $\frac{1}{2} < \sigma_1 < 1$ and $0 < \sigma_2 \le \frac{1}{2}$, we use (3.2) and (6.2) to obtain

$$\begin{aligned} K_1{}^{\frac{3}{4}} K_2{}^{\frac{1}{4}} &= O\left(\left(t_2^{-4} T^{7-4\sigma_1-4\sigma_2}\right)^{\frac{3}{4}} \left(t_2^{2-\frac{8}{3}\sigma_2} \log^4 t_2\ T\right)^{\frac{1}{4}} \right) \\ &= O\left(t_2^{-\frac{5}{2}-\frac{2}{3}\sigma_2} \log t_2\ T^{\frac{11}{2}-3\sigma_1-3\sigma_2} \right). \end{aligned}$$

Similarly, in the case $\frac{1}{2} < \sigma_1 < 1$ and $\frac{1}{2} < \sigma_2 < 1$, we have

$$\begin{aligned} K_1^{\frac{3}{4}} K_2^{\frac{1}{4}} &= O\left(\left(t_2^{-4} T^{7-4\sigma_1-4\sigma_2}\right)^{\frac{3}{4}} \left(t_2^{\frac{4}{3}(1-\sigma_2)} \log^8 t_2\ T\right)^{\frac{1}{4}} \right) \\ &= O\left(t^{-\frac{8}{3}-\frac{1}{3}\sigma_2} \log^2 t_2\ T^{\frac{11}{2}-3\sigma_1-3\sigma_2} \right). \end{aligned}$$

Hence, we have the formula (3.15).

Similarly, in the case $\sigma_1 = \frac{1}{2}$ and $\frac{1}{2} < \sigma_2 < 1$, we have

$$K_1{}^{\frac{3}{4}} K_2{}^{\frac{1}{4}} = O\left(t^{-\frac{8}{3}-\frac{1}{3}\sigma_2} \log^2 t_2\ T^{4-3\sigma_2} \log T \right).$$

Hence, we have the formula (3.16).

Similarly, in the case $0 < \sigma_1 < \frac{1}{2}$ and $\frac{1}{2} < \sigma_2 < 1$, we have

$$K_1{}^{\frac{3}{4}} K_2{}^{\frac{1}{4}} = O\left(t^{-\frac{8}{3}-\frac{1}{3}\sigma_2} \log^2 t_2\ T^{6-4\sigma_1-3\sigma_2} \right),$$

hence, we obtain the formula (3.17). $\square$

6.3 Proofs of Theorems 5 and 6

Under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $\frac{1}{2} < \sigma_1 + \sigma_2 = 1$ and $2 \le t_2 \le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$, we shall evaluate the integral $\int_2^T |\zeta_2(s_2, s_1)|^4 dt_1$. In the case where $1/2 \le \sigma_1 < 1$ and $0 < \sigma_2 < \frac{1}{2}$, we use (3.3) and (6.2) to obtain

$$\begin{aligned} K_1{}^{\frac{3}{4}} K_2{}^{\frac{1}{4}} &= O\left(\left(t_2^{-4} T^{7-4\sigma_1-4\sigma_2}\right)^{\frac{3}{4}} \left(t_2^{2-\frac{8}{3}\sigma_2} \log^4 t_2\ T\right)^{\frac{1}{4}} \right) \\ &= O\left(t_2^{-\frac{5}{2}-\frac{2}{3}\sigma_2} \log t_2\ T^{\frac{11}{2}-3\sigma_1-3\sigma_2} \right), \end{aligned}$$

From (3.3), (6.1) and (6.2) we have the formula (3.18). Similarly, in the cases where $\sigma_1 = \frac{1}{2}$ and $0 < \sigma_2 < \frac{1}{2}$, and $0 < \sigma_1 < \frac{1}{2}$ and $0 < \sigma_2 < 1$, the error term $K_1{}^{\frac{3}{4}} K_2{}^{\frac{1}{4}}$ are estimated as

$$O\left(t_2^{-\frac{5}{2}-\frac{2}{3}\sigma_2} \log t_2\ T^{4-3\sigma_2} \log T \right)$$

and

$$O\left(\left(t_2^{-\frac{8}{3}-\frac{1}{3}\sigma_2}+t_2^{-\frac{5}{2}-\frac{2}{3}\sigma_2}\right)\log^2 t_2\ T^{6-4\sigma_1-3\sigma_2}\right),$$

respectively. Hence, we have the formulas (3.20) and (3.21).

In a similar manner to above, we shall evaluate the integral $\int_2^T |\zeta_2(s_2,s_1)|^4 dt_1$ under the region $0<\sigma_1<\frac{1}{2}$, $0<\sigma_2<\frac{1}{2}$, $\sigma_1+\sigma_2=\frac{1}{2}$ and $2\le t_2\le T^{\frac{1}{2}}$. We use (3.3) and (6.2) to obtain $K_1^{\frac{3}{4}}K_2^{\frac{1}{4}}=O\left(t_2^{-\frac{5}{2}-\frac{2}{3}\sigma_2}\log t_2\ T^{4+\sigma_2}\right)$. This completes the proof of Theorem 6. □

6.4 Proof of Theorem 7

Lastly, we shall evaluate the integral $\int_2^T |\zeta_2(s_2,s_1)|^4 dt_1$ under the region $0<\sigma_1<\frac{1}{2}$, $0<\sigma_2<\frac{1}{2}$, $0<\sigma_1+\sigma_2<\frac{1}{2}$ and $2\le t_2\le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. We use (3.4) and (6.2) to obtain

$$K_1^{\frac{3}{4}}K_2^{\frac{1}{4}}=O\left(t_2^{-\frac{5}{2}-\frac{2}{3}\sigma_2}\log t_2\ T^{6-4\sigma_1-3\sigma_2}\right).$$

From (3.4), (6.1) and (6.2), this proves Theorem 7. □

7 Proofs of even power moment

7.1 Proofs of Theorem 8 and Corollary 3

Let $2\le t_2\le \frac{T^{\frac{1}{3}}}{\log T}$ and let k be any fixed positive integer. We shall evaluate the integral $\int_2^T |\zeta_2(s_1,s_2)|^{2k} dt_1$ under the region $0<\sigma_1<1$, $0<\sigma_2<1$ and $\sigma_1+\sigma_2=1$. From the formula (5.19) in Lemma 5 and Lemma 2, we obtain

$$\begin{aligned}&\int_2^T |\zeta(i(t_2+t_1))|^{2k} dt_1\\ &\quad=\frac{D_k}{(2\pi)^k(k+1)}\left\{(T+t_2)^{k+1}-(t_2+2)^{k+1}+O\left(T^k\log^{k^2}T\right)\right\}\\ &\quad=\frac{D_k}{(2\pi)^k(k+1)}T^{k+1}+O\left(t_2T^k\right)+O\left(T^k\log^{k^2}T\right).\end{aligned}$$

Hence, we have

$$I_1=\frac{D_k}{(2\pi)^k(k+1)}\ \frac{T^{k+1}}{|s_2-1|^{2k}}\ +O\left(t_2^{1-2k}T^k\right)+O\left(t_2^{-2k}T^k\log^{k^2}T\right). \tag{7.1}$$

From the formula (5.16) and the estimate (5.2) in Lemma 1, we have

$$I_2=\frac{D_k}{4^k}T+O\left(\log^{k^2}T\right) \tag{7.2}$$

and

$$I_3=O\left(Tt_2^k\log^{2k}t_2\right). \tag{7.3}$$

Substituting (7.1)–(7.3) into (5.3), we observe that all error terms on the right-hand side of (5.3) are absorbed into $O\left(t_2^{\frac{3}{2}-2k}\log t_2\, T^{k+\frac{1}{2}}\right)$. This completes the proof of Theorem 8.

Next, as an application of (4.1), we have

$$\int_2^N\int_2^T|\zeta_2(s_1,s_2)|^{2k}dt_1dt_2=\frac{D_k}{(2\pi)^k(k+1)}\left(\int_2^N\frac{1}{|s_2-1|^{2k}}dt_2\right)T^{k+1}+O\left(T^{k+\frac{1}{2}}\int_2^N t_2^{\frac{3}{2}-2k}\log t_2dt_2\right)$$

for $2\le N\le\frac{T^{\frac{1}{3}}}{\log T}$. It follows from $0<\sigma_2<1$ and any fixed positive integer k (≥ 3) that

$$\int_2^N\frac{1}{|s_2-1|^{2k}}dt_2=\int_2^\infty\frac{1}{|s_2-1|^{2k}}dt_2+O\left(N^{1-2k}\right)$$

and

$$\int_2^N t_2^{\frac{3}{2}-2k}\log t_2dt_2=\int_2^\infty t_2^{\frac{3}{2}-2k}\log t_2dt_2+O\left(N^{\frac{1}{2}-2k}\log N\right)$$

hold. Hence, we have

$$\frac{1}{T^{k+1}}\int_2^N\int_2^T|\zeta_2(s_1,s_2)|^{2k}dt_1dt_2=\frac{D_k}{(2\pi)^k(k+1)}\int_2^\infty\frac{1}{|s_2-1|^{2k}}dt_2+O\left(T^{-\frac{1}{2}}+N^{1-2k}\right)$$

for $2\le N\le\frac{T^{\frac{1}{3}}}{\log T}$. This proves Corollary 3. □

7.2 Proof of Theorem 9

We shall evaluate the integral $\int_2^T|\zeta_2(s_1,s_2)|^{2k}dt_1$ under the region $0<\sigma_1<1$, $0<\sigma_2<1$ and $1<\sigma_1+\sigma_2\le\frac{3}{2}$. As in the proof of Theorem 8, it follows from Lemmas 2 and 5 that

$$\begin{aligned}I_1&=\frac{1}{|s_2-1|^{2k}}\int_2^T|\zeta(\sigma_1+\sigma_2-1+i(t_2+t_1))|^{2k}dt_1\\&=O\left(t_2^{-2k}T^{\frac{k}{3}(7-4(\sigma_1+\sigma_2))+1}\log^{2k}T\right)\end{aligned}\tag{7.4}$$

holds. From Lemmas 1, 2 and 5, we have $I_2=O(T)$ and $I_3=O\left(Tt_2^k\right)$. Substituting (7.4), $I_2=O(T)$ and $I_3=O\left(Tt_2^k\right)$ into (5.3), we observe that the right-hand side of (5.3) is estimated by $O\left(t_2^{-2k}T^{\frac{k}{3}(7-4(\sigma_1+\sigma_2))+1}\log^{2k}T\right)$ for $2\le t_2\le T^{\frac{1}{9}(7-4(\sigma_1+\sigma_2))}(\log T)^{\frac{2}{3}}$. This completes the proof of (4.4).

Next, under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $\frac{3}{2} < \sigma_1 + \sigma_2 < 2$, we shall evaluate the integral $\int_2^T |\zeta_2(s_1, s_2)|^{2k} dt_1$. From Lemmas 1, 2 and 5, we have $I_2 = O(T)$ and $I_3 = O\left(Tt_2^k\right)$. It follows from Lemmas 2 and 5 that

$$I_1 = O\left(t_2^{-2k} T^{\frac{2k}{3}(2-\sigma_1-\sigma_2)+1} \log^{4k} T\right) \tag{7.5}$$

holds. Substituting (7.5), $I_2 = O(T)$ and $I_3 = O\left(Tt_2^k\right)$ into (5.3), the right-hand side of (5.3) is estimated as $O\left(t_2^{-2k} T^{\frac{2k}{3}(2-\sigma_1-\sigma_2)+1} \log^{4k} T\right)$ for $2 \le t_2 \le T^{\frac{2}{9}(2-\sigma_1-\sigma_2)} \log^{\frac{4}{3}} T$. This proves (4.5). □

7.3 Proofs of Theorem 10

We shall evaluate the integral $\int_2^T |\zeta_2(s_1, s_2)|^{2k} dt_1$ under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $\frac{1}{2} < \sigma_1 + \sigma_2 < 1$. From Lemmas 1, 2 and 5, we have

$$\begin{aligned} I_1 = &\frac{(2\pi)^{2k(\sigma_1+\sigma_2)-3k} D_k(2-\sigma_1-\sigma_2)}{3k+1-2k(\sigma_1+\sigma_2)} \frac{T^{3k+1-2k(\sigma_1+\sigma_2)}}{|s_2-1|^{2k}} \\ &+ O\left(t_2^{1-2k} T^{3k-2k(\sigma_1+\sigma_2)}\right) + O\left(t_2^{-2k} T^{3k-(2k-1)(\sigma_1+\sigma_2)}\right), \end{aligned} \tag{7.6}$$

$$I_2 = O\left(T^{\frac{2k}{3}(1-\sigma_1-\sigma_2)+1} \log^{4k} T\right), \tag{7.7}$$

and

$$I_3 = O\left(Tt_2^{3k-2k(\sigma_1+\sigma_2)}\right). \tag{7.8}$$

Substituting (7.6)–(7.8) into (5.3), we observe that all error terms on the right-hand side of (5.3) are

$$\begin{aligned} &O\left(t_2^{\frac{5}{2}-2k-\sigma_1-\sigma_2} T^{3k-\frac{1}{2}-(2k-1)(\sigma_1+\sigma_2)}\right) + O\left(t_2^{1-2k} T^{3k-\frac{1}{6}-(2k-\frac{2}{3})(\sigma_1+\sigma_2)} \log^2 T\right) \\ &\quad + O\left(t_2^{-2k} T^{3k-(2k-1)(\sigma_1+\sigma_2)}\right) \end{aligned}$$

with $2 \le t_2 \le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. Hence, this completes the proof of the formula (4.6).

In a similar manner to the above, we can obtain the formula (4.7). □

7.4 Proof of Theorem 11

We shall evaluate the integral $\int_2^T |\zeta_2(s_2, s_1)|^{2k} dt_1$ under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $\sigma_1 + \sigma_2 = 1$ and $2 \le t_1 \le T$. We use Lemmas 3, 5 and 6 to obtain

$$J_2 = \begin{cases} O\left(t_2^{\frac{2}{3}k(1-\sigma_2)} (\log t_2)^{4k}\ T^{k+1-\frac{4}{3}k\sigma_1} \log^{2k} T\right) & \text{if } 0 < \sigma_1 < \frac{1}{2} \text{ and } \frac{1}{2} < \sigma_2 < 1, \\ O\left(t_2^{\frac{k}{3}} (\log t_2)^{2k}\ T^{\frac{k}{3}+1} \log^{2k} T\right) & \text{if } \sigma_1 = \sigma_2 = \frac{1}{2}, \\ O\left(t_2^{k-\frac{4}{3}k\sigma_2} (\log t_2)^{2k}\ T^{\frac{2}{3}k+1-\frac{2}{3}k\sigma_1} \log^{4k} T\right) & \text{if } \frac{1}{2} < \sigma_1 < 1 \text{ and } 0 < \sigma_2 < \frac{1}{2}. \end{cases} \tag{7.9}$$

From (4.1) and (5.6), we have

$$J_1 = \frac{D_k}{(2\pi)^k(k+1)} \frac{T^{k+1}}{|s_2-1|^{2k}} + O\left(t_2^{\frac{3}{2}-2k}(\log t_2)T^{k+\frac{1}{2}}\right) \tag{7.10}$$

with $2 \le t_2 \le \frac{T^{\frac{1}{3}}}{\log T}$. Substituting (7.9) and (7.10) into (5.5), all error terms on the right-hand side of (5.5) are absorbed into

$$O\left(t_2^{\frac{4}{3}-2k-\frac{1}{3}\sigma_2}(\log t_2)^2 T^{k+1-\frac{2}{3}\sigma_1}\log T\right) + O\left(t_2^{\frac{3}{2}-2k}(\log t_2)T^{k+\frac{1}{2}}\right),$$

if $0 < \sigma_1 < \frac{1}{2}$, $\frac{1}{2} < \sigma_2 < 1$, $\sigma_1 + \sigma_2 = 1$ and $2 \le t_2 \le T^{\frac{2\sigma_1}{4-\sigma_2}}(\log T)^{-3}$, or into

$$O\left(t_2^{\frac{7}{6}-2k}(\log t_2)T^{k+\frac{2}{3}}\log T\right) + O\left(t_2^{\frac{3}{2}-2k}(\log t_2)T^{k+\frac{1}{2}}\right),$$

if $\sigma_1 = \sigma_2 = \frac{1}{2}$ and $2 \le t_2 \le T^{\frac{2}{7}}(\log T)^{-3}$, or into

$$O\left(t_2^{\frac{3}{2}-2k-\frac{2}{3}\sigma_2}(\log t_2)T^{k+\frac{5}{6}-\frac{1}{3}\sigma_1}\log^2 T\right) + O\left(t_2^{\frac{3}{2}-2k}(\log t_2)T^{k+\frac{1}{2}}\right)$$

if $\frac{1}{2} < \sigma_1 < 1$, $0 < \sigma_2 < \frac{1}{2}$, $\sigma_1 + \sigma_2 = 1$ and $2 \le t_2 \le T^{\frac{1+2\sigma_1}{9-4\sigma_2}}(\log T)^{-3}$. Hence, it completes the proof of Theorem 11. □

7.5 Proof of Theorem 12

We shall evaluate the integral $\int_2^T |\zeta_2(s_2,s_1)|^{2k}dt_1$ under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $1 < \sigma_1 + \sigma_2 \le \frac{3}{2}$. From Lemmas 3, 5 and 6, we have

$$J_2 = \begin{cases} O\left(t_2^{\frac{2k}{3}-\frac{2k}{3}\sigma_2}(\log t_2)^{4k}T^{k+1-\frac{4}{3}k\sigma_1}\log^{2k} T\right) & \text{if } 0 < \sigma_1 \le \frac{1}{2} \text{ and } \frac{1}{2} < \sigma_2 < 1, \\ O\left(t_2^{\frac{2k}{3}-\frac{2k}{3}\sigma_2}(\log t_2)^{4k}T^{\frac{2}{3}k+1-\frac{2}{3}k\sigma_1}\log^{4k} T\right) & \text{if } \frac{1}{2} < \sigma_1 < 1 \text{ and } \frac{1}{2} < \sigma_2 < 1, \\ O\left(t_2^{k-\frac{4}{3}k\sigma_2}(\log t_2)^{2k}T^{\frac{2}{3}k+1-\frac{2}{3}k\sigma_1}\log^{4k} T\right) & \text{if } \frac{1}{2} < \sigma_1 < 1 \text{ and } 0 < \sigma_2 \le \frac{1}{2}. \end{cases} \tag{7.11}$$

From (4.4) and (5.6), we have

$$J_1 = O\left(t_2^{-2k}T^{\frac{k}{3}(7-4(\sigma_1+\sigma_2))+1}\log^{2k} T\right) \tag{7.12}$$

with $2 \le t_2 \le T^{\frac{1}{9}(7-4(\sigma_1+\sigma_2))}\log^{\frac{2}{3}} T$. Substituting (7.11) and (7.12) into (5.5), we have

$$\int_2^T |\zeta_2(s_2,s_1)|^{2k}\,dt_1 = O\left(t_2^{-2k}T^{\frac{7k}{3}+1-\frac{4k}{3}(\sigma_1+\sigma_2)}\log^{2k} T\right)$$

with $0 < \sigma_1 \le \frac{1}{2}$, $\frac{1}{2} < \sigma_2 < 1$, $1 < \sigma_1 + \sigma_2 \le \frac{3}{2}$ and $2 \le t_2 \le T^{\frac{2-2\sigma_2}{4-\sigma_2}}(\log T)^{-6}$, or $\frac{1}{2} < \sigma_1 < 1$, $\frac{1}{2} < \sigma_2 < 1$, $1 < \sigma_1 + \sigma_2 \le \frac{3}{2}$ and $2 \le t_2 \le T^{\frac{5-2\sigma_1-4\sigma_2}{8-2\sigma_2}}(\log T)^{-3}$, or

$\frac{1}{2} < \sigma_1 < 1$, $0 < \sigma_2 \le \frac{1}{2}$, $1 < \sigma_1 + \sigma_2 \le \frac{3}{2}$ and $2 \le t_2 \le T^{\frac{5-2\sigma_1-4\sigma_2}{9-4\sigma_2}} (\log T)^{-6}$. This proves the formula (4.12).

Next, under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and $\frac{3}{2} < \sigma_1 + \sigma_2 < 2$, we shall evaluate the integral $\int_2^T |\zeta_2(s_2, s_1)|^{2k} dt_1$. From Lemmas 3, 5 and 6, we have

$$J_2 = O\left(t_2^{\frac{2k}{3} - \frac{2k}{3}\sigma_2} (\log t_2)^{4k} T^{k+1-\frac{4k}{3}\sigma_1} \log^{2k} T\right) \tag{7.13}$$

with $\frac{1}{2} < \sigma_1 < 1$, $\frac{1}{2} < \sigma_2 < 1$ and $\frac{3}{2} < \sigma_1 + \sigma_2 < 2$. From (4.5) and (5.6), we have

$$J_1 = O\left(t_2^{-2k} T^{\frac{4}{3}k+1-\frac{2}{3}k(\sigma_1+\sigma_2)} \log^{4k} T\right) \tag{7.14}$$

with $2 \le t_2 \le T^{\frac{1}{9}(7-4(\sigma_1+\sigma_2))} \log^{\frac{4}{3}} T$. Substituting (7.11) and (7.12) into (5.5), we have

$$\int_2^T |\zeta_2(s_2, s_1)|^{2k} dt_1 = O\left(t_2^{-2k} T^{\frac{4k}{3}+1-\frac{2k}{3}(\sigma_1+\sigma_2)} \log^{2k} T\right)$$

with $\frac{1}{2} < \sigma_1 < 1$, $\frac{1}{2} < \sigma_2 < 1$, $\frac{3}{2} < \sigma_1 + \sigma_2 < 2$ and $2 \le t_2 \le T^{\frac{1+2\sigma_1-2\sigma_2}{8-2\sigma_2}} (\log T)^{-1}$. This proves the formula (4.13). □

7.6 Proof of Theorem 13

Under the region $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, $\frac{1}{2} < \sigma_1 + \sigma_2 < 1$ and $2 \le t_2 \le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$, we shall evaluate the integral $\int_2^T |\zeta_2(s_2, s_1)|^{2k} dt_1$. It follows from Lemmas 3, 5 and 6 that

$$J_2 = \begin{cases} O\left(t_2^{\frac{2k}{3} - \frac{2k}{3}\sigma_2} (\log t_2)^{4k} T^{k+1-\frac{4}{3}k\sigma_1} \log^{2k} T\right) & \text{if } 0 < \sigma_1 < \frac{1}{2} \text{ and } \frac{1}{2} < \sigma_2 < 1, \\ O\left(t_2^{k-\frac{4k}{3}\sigma_2} (\log t_2)^{2k} T^{k+1-\frac{4k}{3}\sigma_1} \log^{2k} T\right) & \text{if } 0 < \sigma_1 \le \frac{1}{2} \text{ and } 0 < \sigma_2 \le \frac{1}{2}, \\ O\left(t_2^{k-\frac{4k}{3}\sigma_2} (\log t_2)^{2k} T^{\frac{2k}{3}+1-\frac{2k}{3}\sigma_1} \log^{4k} T\right) & \text{if } \frac{1}{2} < \sigma_1 < 1 \text{ and } 0 < \sigma_2 < \frac{1}{2}. \end{cases} \tag{7.15}$$

From (4.6) and (5.6), we have

$$\begin{aligned} J_1 = {} & \frac{(2k)^{3k-2k(\sigma_1+\sigma_2)} D_k(2-\sigma_1-\sigma_2)}{(3k+1-k(\sigma_1+\sigma_2))|s_2-1|^{2k}} T^{3k+1-2k(\sigma_1+\sigma_2)} \qquad (7.16) \\ & + O\left(t_2^{\frac{5}{2}-2k-\sigma_1-\sigma_2} T^{3k-\frac{1}{2}-(2k-1)(\sigma_1+\sigma_2)}\right) + O\left(t_2^{1-2k} T^{3k-\frac{1}{6}-(2k-\frac{2}{3})(\sigma_1+\sigma_2)} \log^2 T\right) \\ & + O\left(t_2^{-2k} T^{3k-(2k-1)(\sigma_1+\sigma_2)}\right). \end{aligned}$$

From (7.15) and (7.16), the error term $J_1^{1-\frac{1}{2k}} J_2^{\frac{1}{2k}}$ is estimated as

$$\begin{cases} O\left(t_2^{\frac{4}{3}-2k-\frac{1}{3}\sigma_2}(\log t_2)^2 T^{3k-\frac{2}{3}\sigma_1-(2k-1)(\sigma_1+\sigma_2)}\log T\right) & \text{if } 0<\sigma_1<\frac{1}{2} \text{ and } \frac{1}{2}<\sigma_2<1, \\ O\left(t_2^{\frac{3}{2}-2k-\frac{2}{3}\sigma_2}(\log t_2) T^{3k-\frac{2}{3}\sigma_1-(2k-1)(\sigma_1+\sigma_2)}\log T\right) & \text{if } 0<\sigma_1\le\frac{1}{2} \text{ and } 0<\sigma_2\le\frac{1}{2}, \\ O\left(t_2^{\frac{3}{2}-2k-\frac{2}{3}\sigma_2}(\log t_2) T^{3k-\frac{1}{6}-\frac{1}{3}\sigma_1-(2k-1)(\sigma_1+\sigma_2)}\log^2 T\right) & \text{if } \frac{1}{2}<\sigma_1<1 \text{ and } 0<\sigma_2<\frac{1}{2}. \end{cases} \tag{7.17}$$

We substitute (7.15), (7.16), (7.17) into (5.5) to obtain the formula (4.14).

As for the integral $\int_2^T |\zeta_2(s_2,s_1)|^{2k} dt_1$ under the region $0<\sigma_1<\frac{1}{2}$, $0<\sigma_2<\frac{1}{2}$, $0<\sigma_1+\sigma_2\le\frac{1}{2}$ and $2\le t_2\le T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$, in a similar manner to the above, we can obtain the formula (4.15). □

Acknowledgements. The authors would like to thank the referee for his/her careful reading of the earlier version of this paper, giving them many valuable suggestions.

References

[1] S. Akiyama, S. Egami and Y. Tanigawa, An analytic continuation of multiple zeta-functions and their values at non-positive integers, Acta Arith. **98** (2001), 107–116.

[2] T. M. Apostol and T. H. Vu, Dirichlet series related to the Riemann zeta-function, Journal of Number Theory **19** (1984), 85–102.

[3] F. V. Atkinson, The mean-value of the Riemann zeta-function, Acta Math. **81** (1949), 353–376.

[4] R. Balasubramanian, A. Ivić and K. Ramachandra, The mean square of the Riemann zeta-function on the line $\sigma = 1$, L'Enseign Math. (2) **38** (1992), 13–25.

[5] D. R. Heath-Brown, The fourth power moment of the Riemann zeta-function, Proc. London Math. Soc. **38** (3), (1979), 385–422.

[6] S. Ikeda, K. Matsuoka and Y. Nagata, On certain mean values of the double zeta-function, Nagoya. Math. J. **217** (2015), 161–190.

[7] S. Ikeda, I. Kiuchi and K. Matsuoka, The mean values of the double zeta-function, preprint.

[8] H. Ishikawa and K. Matsumoto, On the estimation of the order of Euler-Zagier multiple zeta-functions, Illinois J. Math. **47** (2003), 1151–1166.

[9] A. Ivić, *The Riemann Zeta-Function*, John Wiley and Sons, New York, 1985 (2nd ed. Dover, 2003).

[10] A. Ivić, *Mean Values of the Riemann Zeta Function*, Lecture Note Ser.82. Tata Institute of Fundamental Research, Bombay. Berlin–Heidelberg–New York; Springer 1991.

[11] A. Ivić, Some problems on mean values of the Riemann zeta-function, J. Théor. Nombres Bordeaux. **8** (1996), 101–123.

[12] A. Ivić, On mean values of some zeta-functions in the critical strip, J. Théor. Nombres Bordeaux. **15** (2003), 163–178.

[13] A. Ivić and K. Matsumoto , On the error term in the mean square formula for the Riemann zeta-function in the critical strip, Monatsh. Math. **121** (1996), 213–229.

[14] I. Kiuchi and M. Minamide, Mean square formula for the double zeta-function, to appear in Funct. Approx. Comment. Math.

[15] I. Kiuchi and M. Minamide, The fourth power moment of the double zeta-function, to appear in Publs. Inst. Math. (Belgrade).

[16] I. Kiuchi and Y. Tanigawa, Bounds for double zeta-functions, Ann. Sc. Norm. Sup. Pisa, Cl. Sci. Ser. V **5** (2006), 445–464.

[17] I. Kiuchi, Y. Tanigawa and W. Zhai, Analytic properties of double zeta-functions, Indag. Math. **21** (2011), 16–29.

[18] Y. Komori, K. Matsumoto and H. Tsumura, Functional equations and functional relations for the Euler double zeta-function and its generalization of Eisenstein type, Publ. Math. Debrecen **77** (2010), 15–31.

[19] K. Matsumoto, On the analytic continuation of various multiple zeta-functions, In: "Number Theory for the Millennium, Proc. Millennial Conf. Number Theory", Vol. II, M. A. Bennett et al. (eds.), A K Peters 2002, 417–440.

[20] K. Matsumoto, The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions I, J. Number Theory **101** (2003), 223–243.

[21] K. Matsumoto, Asymptotic expansions of double zeta-functions and Barnes, of Shintani, and Eisenstein series, Nagoya. Math. J. **172** (2003), 59–102.

[22] K. Matsumoto, Functional equations for double zeta-functions, Math. Proc. Cambridge Philos. Soc. **136** (2004), 1–7.

[23] K. Matsumoto and H. Tsumura, Mean value theorems for the double zeta-function, J. Math. Soc. Japan **67** (2015), 383–406.

[24] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Second Edition, Edited and with a preface by D. R. Heath–Brown, The Clarendon Press, Oxford University Press, New York, 1986.

[25] J. Q. Zhao, Analytic continuation of multiple zeta function, Proc. Amer. Math. Soc. **128** (2000), 1275–1283.

Soichi Ikeda
Department of Mathematics
Shibaura Institute of Technology
307 Fukasaku, Minuma-ku, Saitama 337-8570, Japan
e-mail: sikeda@sic.shibaura-it.ac.jp

Isao Kiuchi
Department of Mathematical Sciences
Faculty of Science, Yamaguchi University
Yoshida 1677-1, Yamaguchi 753-8512, Japan
e-mail: kiuchi@yamaguchi-u.ac.jp

Kaneaki Matsuoka
Graduate School of Mathematics, Nagoya University
Furocho, Chikusaku, Nagoya 464-8602, Japan
e-mail: m10041v@math.nagoya-u.ac.jp