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Twisted relative homology of the configuration spaces of n-points associated with the general hypergeometric integral

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Abstract. The twisted relative simplicial homology and the twisted relative singular homology of the configuration space with coefficients in a local system are investigated systematically. An exterior power structure of the relative homology group of the complement of hyperplanes in a projective space associated with the general hypergeometric integral is established.

1 Introduction

The generalized confluent hypergeometric function was introduced in the paper [14] as a "Radon transform" of characters of maximal abelian subgroup H_λ of $\mathrm{GL}(N)$ indexed by a partition λ of N. In the case $\lambda = (1, 1, ..., 1)$, the confluent hypergeometric function is called Aomoto-Gelfand hypergeometric function. The Aomoto-Gelfand hypergeometric functions are interpreted as the pairing between the twisted homology and the twisted cohomology of the complement of hyperplanes in a complex projective space. In this paper, we shall consider a general hypergeometric integral which includes the integral for the confluent hypergeometric function.

In the paper [9], the author considered the homology theory associated with the general hypergeometric integral, that is the homology group on the complement of hyperplanes in a complex projective space with coefficients in a local system and the family of supports. A fact is given in [9] that the homology group associated with general hypergeometric integral is isomorphic to a relative twisted homology group with coefficients in the local system.

In this paper, we shall investigate an exterior power structure associated with a general hypergeometric integral and make systematic investigation of the twisted relative simplicial homology and the twisted relative singular homology of the configuration space with coefficients in a local system. This kind of study will be

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important for further study, for example, the study of dimension of the homology group and of intersection theory associated with the general hypergeometric integral.

This paper consists of three parts, Part I(§2-3), Part II(§4-13) and Part III(§14-17). In Part I, we recall the twisted homology theory on the complement of hyperplanes in a complex projective spaces associated with the general hypergeometric integral. We shall establish the exterior power structure of the relative homology group of the complement of hyperplanes in a projective space associated with the general hypergeometric integral. This result will be stated in Theorem 3.1.7 - the main theorem of Part I. Theorem 3.2.1 gives a motivation to Part II and Part III.

In Part II and Part III, we give a discussion on a relative simplicial theory and a relative singular theory for the configuration spaces of n-points in detail, respectively. The main theorems of Part II and Part III are Theorem 12.2.2 and Theorem 17.2.3, respectively.

2 The general hypergeometric integrals

2.1 Definition of the integral

We recall briefly the definition of general hypergeometric functions (integrals) on the Grassmannian. Let N be a positive integer and $\lambda = (l_0, l_1, \dots, l_m)$ be a partition of N, namely, l_k are positive integers satisfying $l_0 \geq \cdots \geq l_m$ and $\sum_{k=0}^{m} l_k = N$. The partition λ is identified with the Young diagram which is obtained by arraying N boxes, l_0 boxes in the first row, l_1 boxes in the second row, and so on where the first boxes in each row are arrayed in the same column. The number of boxes N in the diagram is called the weight of λ and is denoted by $|\lambda|$. With the partition λ, we associate the maximal abelian subgroup H_λ of $\mathrm{GL}(N)$ of the form

$$H_\lambda = J(l_0) \times \cdots \times J(l_m),$$

where

$$J(l) := \left\{ h = \sum_{0 \leq i < l} h_i \Lambda^i \ ; \ h_i \in \mathbb{C}, \quad h_0 \neq 0 \right\} \subset \mathrm{GL}(l),$$

$\Lambda = (\delta_{i+1,j})_{1 \leq i,j \leq l}$ being the shift matrix of size l. The group $J(l)$ is a maximal abelian subgroup of $\mathrm{GL}(l)$ and is called the Jordan group since it is a centralizer of an element of the Jordan normal form $aI + \Lambda \in \mathrm{GL}(l)$. Note that $J(l)$ is isomorphic to the group of units of the quotient ring $\mathbb{C}[X]/(X^l)$ by an obvious correspondence

$$\sum_{0 \leq i < l} h_i \Lambda^i \mapsto \sum_{0 \leq i < l} h_i X^i$$

We describe the characters of unversal covering group $\tilde{H}_\lambda$ of H_λ. Let $x = (x_0, x_1, x_2, \dots)$ be a sequence of variables and let $\theta_k(x)$ $(k \geq 0)$ be the funciton defined by

$$\sum_{0\le k<\infty} \theta_k(x)T^k = \log(x_0 + x_1T + x_2T^2 + \cdots) \tag{2.1}$$

$$= \log x_0 + \log\left(1 + \frac{x_1}{x_0}T + \frac{x_2}{x_0}T^2 + \cdots\right). \tag{2.2}$$

Here $\theta_0(x) = \log x_0$, and $\theta_k(x)$ $(k \ge 1)$ is a quasihomogeneous polynomial of $x_1/x_0, \dots, x_k/x_0$ of weight k if the weight of x_i/x_0 is defined to be i which is written explicitly as

$$\theta_k(x) = \sum (-1)^{i_1+\cdots+i_k-1} \frac{(i_1+\cdots+i_k-1)!}{i_1!\cdots i_k!} \left(\frac{x_1}{x_0}\right)^{i_1} \cdots \left(\frac{x_k}{x_0}\right)^{i_k},$$

where the sum is taken over the indices $(i_1, \dots, i_k) \in \mathbb{Z}_{\ge 0}^k$ such that $i_1 + 2i_2 + \cdots + ki_k = k$.

Lemma 2.1.1. *[5] We have the isomorphism $J(l) \simeq \mathbb{C}^\times \times \mathbb{C}^{l-1}$ by the correspondence*

$$h = \sum_{0\le i<l} h_i\Lambda^i \mapsto (h_0, \theta_1(h), \dots, \theta_{l-1}(h)).$$

It follows that the character $\chi_l : \tilde{J}(l) \to \mathbb{C}^\times$is given by

$$\chi_l(h;\alpha) = \exp\left(\sum_{0\le i<l} \alpha_i\theta_i(h)\right) = h_0^{\alpha_0} \exp\left(\sum_{1\le i<l} \alpha_i\theta_i(h)\right),$$

where $\alpha = (\alpha_0, \dots, \alpha_{l-1})$ are arbitrary complex constants. Noting the fact that H_λ is a product of $J(l_k)$, we have the following.

Lemma 2.1.2. *A character $\chi : \tilde{H}_\lambda \to \mathbb{C}^\times$ is given, for some $\alpha = (\alpha^{(0)}, \dots, \alpha^{(m)}) \in \mathbb{C}^N$, $\alpha^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \dots, \alpha_{l_k-1}^{(k)}) \in \mathbb{C}^{l_k}$, by*

$$\chi(h;\alpha) = \prod_{0\le k\le m} \chi_{l_k}(h^{(k)};\alpha^{(k)}) = \prod_{0\le k\le m} (h_0^{(k)})^{\alpha_0^{(k)}} \exp\left(\sum_{1\le i<l_k} \alpha_i^{(k)}\theta_i(h^{(k)})\right), \tag{2.3}$$

where $h = (h^{(0)}, \cdots, h^{(m)}) \in \tilde{H}_\lambda, h^{(k)} \in \tilde{J}(l_k)$.

Next we consider the "Radon transform" of the character χ. Roughly speaking we substitute homogeneous polynomials of degree 1 in the homogeneous coordinates $t = (t_0, t_1, \dots, t_n)$ of $\mathbb{P}^n$ into the character and integrate. We first define the space of coefficients of these polynomials which is a Zariski open subset of the space $M(n+1, N)$ of $(n+1) \times N$ complex matrices.

For $\lambda = (l_0, l_1, \dots, l_m)$, a sequence $\mu = (i_0, \dots, i_m) \in \mathbb{Z}_{\ge 0}^{m+1}$ is called a subdiagram of λ of weight $|\mu| = \sum_k i_k$ if it satisfies $0 \le i_k \le m_k$ $(0 \le k \le m)$

and is denoted as $\mu \subset \lambda$. For $z = (z^{(0)}, \dots, z^{(m)}) \in \mathrm{Mat}(n+1, N)$ with $z^{(k)} = (z_0^{(k)}, \dots, z_{m_k-1}^{(k)})$ and for any subdiagram $\mu \subset \lambda, |\mu| = n+1$, we put

$$z_\mu = (z_0^{(0)}, \dots, z_{i_0-1}^{(0)}, \dots, z_0^{(m)}, \dots, z_{i_m-1}^{(m)}) \in \mathrm{Mat}(n+1).$$

Definition 2.1.3. *The generic stratum $Z_{n,\lambda} \subset \mathrm{Mat}(n+1, N)$ with respect to H_λ is defined by*

$$Z_{n,\lambda} = \{z \in \mathrm{Mat}(n+1, N)\ ;\ \det z_\mu \neq 0 \ \ \textit{for any} \ \ \mu \subset \lambda, |\mu| = n+1\}.$$

Define a biholomorphic map

$$\iota: \quad H_\lambda \to \prod_{0 \le k \le m} \left(\mathbb{C}^\times \times \mathbb{C}^{l_k - 1}\right) \subset \mathbb{C}^N$$

by

$$\iota(h) = (h_0^{(0)}, \dots, h_{l_0-1}^{(0)}, \dots, h_0^{(m)}, \dots, h_{l_m-1}^{(m)})$$

for $h = (h^{(0)}, \cdots, h^{(m)}) \in H_\lambda$. The map ι can be lifted to that from $\tilde{H}_l$ to $\prod_{0 \le k \le m} \left(\tilde{\mathbb{C}}^\times \times \mathbb{C}^{l_k-1}\right)$. This lift is also denoted by ι.

Definition 2.1.4. *For the character $\chi(\cdot\,; \alpha)$ given in (2.3), we assume*

$$\sum_{0 \le k \le m} \alpha_0^{(k)} = -n - 1, \tag{2.4}$$

$$\alpha_{l_k - 1}^{(k)} \neq 0 \ \textit{if} \ \ l_k \ge 2. \tag{2.5}$$

The general hypergeometric integral of type λ (GHI of type λ, for short) is defined, for $z \in Z_{n,l}$, by

$$\int_\Delta \chi(\iota^{-1}(tz), \alpha) \cdot \tau, \tag{2.6}$$

where

$$\tau = \sum_{i=0}^{n} (-1)^i dt_0 \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_n$$

and Δ_z is an n-dimensional cycle in $\mathbb{P}^n \setminus \cup_{0 \le k \le m} \{tz_0^{(k)} = 0\}$ of the homology group defined by the integrand $\chi(\iota^{-1}(tz), \alpha)$ (see also Section 2.2).

The integral (2.6) can be written in an affine coordinates of $\mathbb{P}^n$. For example, in the affine chart $\{t \in \mathbb{P}^n \mid t_0 \neq 0\}$, we take an affine coordinates $(s_1, \dots, s_n)$ by $s_i = t_i / t_0$. Noting $\tau = t_0^{n+1} d\left(\frac{t_1}{t_0}\right) \wedge \cdots \wedge d\left(\frac{t_n}{t_0}\right)$ and using the condition (2.4), we have

$$\int_\Delta \chi(\iota^{-1}(tz), \alpha) \cdot \tau = \int_\Delta \chi(\iota^{-1}(\mathbf{s}z), \alpha) ds_1 \wedge \cdots \wedge ds_n$$

where $\mathbf{s} = (1, s_1, \dots, s_n)$.

Let us write $\chi(\iota^{-1}(tz), \alpha) = P(t) \exp f(t)$ with

$$P(t) = \prod_{k=0}^{m} (t \cdot z_0^{(k)})^{\alpha_0^{(k)}}, \quad f(t) = \sum_{k=0}^{m} \sum_{i=1}^{l_k - 1} \alpha_i^{(k)} \theta_i(t \cdot z^{(k)}).$$

The multivalued n-form $P(t) \cdot \tau$ defines a local system $\mathfrak{L}$ of $\mathbb{C}$-vector space of rank 1 on $X := \mathbb{P}^n \setminus \bigcup_{k=0}^{m} D_k,\ D_k := \{t \in \mathbb{P}^n \mid t \cdot z_0^{(k)} = 0\}$ such that each branch of $P(t) \cdot \tau$ determines a horizontal local section of $\mathfrak{L}$, and $f(t)$ is the rational function on $\mathbb{P}^n$ with poles D_k of order $l_k - 1$. Here we used the assumption (2.5). In the following we fix a $z \in Z_{n,\lambda}$and consider the integral

$$\int_{\Delta} P(t) \exp f(t) \cdot \tau \tag{2.7}$$

2.2 Twisted homology with the family of supports

We recall the definition of the homology group associated with the general hypergeometric integral ([9] Section 2). For f defined as above, the family of supports $\mathbf{\Phi}$ is the family of closed subsets A of X, such that for any $\tau \in \mathbb{R}$, $A \cap f^{-1}(\Re w \geq \tau)$ is compact.

Let $\mathfrak{L}$ e the local system as in Section 2.1. For any singular q-simplex $\sigma : \Delta^q \to X$, let $\sigma^* \mathfrak{L}$ be the pull-back of $\mathfrak{L}$ by σ. We define a q-chain $S_q^{\mathbf{\Phi}}(X; \mathfrak{L})$ with the local system $\mathfrak{L}$ and with the family of supports $\mathbf{\Phi}$.

Definition 2.2.1. *A q-chain $c \in S_q^{\mathbf{\Phi}}(X; \mathfrak{L})$ is a formal infinite sum*

$$c = \sum_{\sigma} u_\sigma \cdot \sigma$$

where the sum is taken over all singular q-simplexes σ in X, such that

(1) *$u_\sigma \in \Gamma(\Delta_q, \sigma^* \mathfrak{L})$ is a global section of the local system $\sigma^* \mathfrak{L}$;*

(2) *The summation is locally finite, namely any compact subset in X intersects with only finite number of $\sigma(\Delta_q)$ with $u_\sigma \neq 0$;*

(3) *$supp(c) \in \mathbf{\Phi}$, where $supp(c) = \cup_{u_\sigma \neq 0} \sigma(\Delta_q)$.*

Let ∂ be the boundary map, then we get the chain complex $(S_p^{\mathbf{\Phi}}(X; \mathfrak{L}), \partial)$. The p^{th} homology group of the chain complex $(S_p^{\mathbf{\Phi}}(X; \mathfrak{L}), \partial)$ is denoted by $H_p^{\mathbf{\Phi}}(X; \mathfrak{L})$. For $\tau \in \mathbb{R}$, we put

$$A_\tau = \{t \in X; \Re f(t) < \tau\},$$

A_τ is a subspace of X. Consider the relative homology of the topological pair (X, A_τ) with coefficients in the local system $\mathfrak{L}$. Then the following result is known.

Theorem 2.2.2. *([9] Theorem 2.3) For any sufficiently small $\tau \in \mathbb{R}$, we have an isomorphism*

$$H_\bullet^{\mathbf{\Phi}}(X; \mathfrak{L}) \simeq H_\bullet(X, A_\tau; \mathfrak{L}).$$

2.3 P^1-case

We consider the integral (2.6) in the case $n = 1$. In this case the form in the integral (2.6) is multivalued on $X = \mathbb{P}^1 \setminus \{x_0, \ldots, x_m\}$, where x_k is the zero of $tz_0^{(k)}$. We may assume $x_0 = \infty$ without loss of generality and then the rational function f in (2.7) is written as

$$f = \sum_{i=1}^{l_0-1} c_{1,i} s^i + \sum_{k=1}^{m} \sum_{i=1}^{l_k-1} \frac{c_{k,i}}{(s-x_k)^i}.$$

in the affine coordinates $s = t_1/t_0$ with $c_{k,l_k-1} \neq 0$ for k satisfying $l_k \geq 2$ by virtue of the assumption $z \in Z_{1,\lambda}$ and (2.5).

Theorem 2.3.1. *([9],Theorem 3.1) Assume the condition (2.5) and $\alpha_0^{(k)} \notin \mathbb{Z}$ (for k s.t. $l_k = 1$) for α in the integral (2.6), then*

(1) $H_p^{\Phi}(X; \mathfrak{L}) = 0$ *if* $p \neq 1$,

(2) $dim_{\mathbb{C}} H_1^{\Phi}(X; \mathfrak{L}) = N - 2$,

where $N = \sum_{k=0}^{m} l_k$.

3 The exterior power structure

3.1 The exterior power structure associated with the hypergeometric integral

Let X, f and $\mathfrak{L}$ be the same as in Section 2.3. By virtue of Theorem 2.2.2, there exists a sufficiently small $\tau_1^0 \in \mathbb{R}$ such that, for any $\tau_1 \leq \tau_1^0$, the following isomorphism holds:

$$H_1^{\Phi}(X; \mathfrak{L}) \simeq H_1(X, A_{\tau_1}; \mathfrak{L}),$$

where $A_{\tau_1} = \{t \in X; \Re f(t) < \tau_1\}$.

Lemma 3.1.1. *For any sufficiently large $\tilde{\sigma} \in \mathbb{R}$ and any $\tau_1 \leq \tau_1^0$, we have an isomorphism*

$$H_1(X, A_{\tau_1}; \mathfrak{L}) \simeq H_1(\tilde{B}, A_{\tau_1}; \mathfrak{L})$$

where $\tilde{B} = \{t \in X; \Re f(t) < \tilde{\sigma}\}$.

To prove the Lemma 3.1.1, we need the following result.

Lemma 3.1.2. *([20], Corollary 5.1) Let X and Y be separated complex algebraic variety of finite type and let $f : X \to Y$ be a morphism. Then there exists a Zariski open set $U \subset Y$ such that $f : f^{-1}(U) \to U$ is a locally trivial topological fibration.*

Proof of Lemma 3.1.1 Let f be given in Section 2.3. We regard the rational function f a holomorphic map

$$f : X \to \mathbb{C}.$$

By virtue of Lemma 3.1.2, there is a finite subset Ω of the target space $\mathbb{C}$ such that

$$f : X \setminus f^{-1}(\Omega) \to \mathbb{C} \setminus \Omega$$

defines a locally trivial topological fibration. Let w be the coordinates of the target space $\mathbb{C}$. Take $\tilde{\sigma_0}$ sufficiently large so that $\{\Re w \geq \tilde{\sigma_0}\} \subset \mathbb{C}$ contains no point of Ω. Since $\{\Re w \geq \tilde{\sigma_0}\} \subset \mathbb{C}$ is contractible, then the fibration

$$f : X \setminus f^{-1}(\Omega) \to \mathbb{C} \setminus \Omega$$

is trivial over $\{\Re w \geq \tilde{\sigma_0}\} \subset \mathbb{C}$. It follows that, for any parameters $\tilde{\sigma_2} > \tilde{\sigma_1} > \tilde{\sigma_0}$, the inclusion

$$i_U^V : U = f^{-1}(\{w \in \mathbb{C}; \Re w < \tilde{\sigma_1}\}) \hookrightarrow V = f^{-1}(\{w \in \mathbb{C}; \Re w < \tilde{\sigma_2}\})$$

is a deformation retract. Put $\mathfrak{U} := \{U\}$, $\mathfrak{U}$ is a directed set for inclusion. Then i_U^V induces a chain map

$$i_{U\sharp}^V : S_\bullet(U; \mathfrak{L}) \to S_\bullet(V; \mathfrak{L}).$$

On the other hand, an inclusion

$$i_U : U \to X$$

induces a chain map

$$i_{U\sharp} : S_\bullet(U; \mathfrak{L}) \to S_\bullet(X : \mathfrak{L})$$

such that

$$i_{U\sharp} = i_{V\sharp} \circ i_{U\sharp}^V \qquad (U \subset V).$$

Then we have a chain map

$$\varinjlim i_{U\sharp} : \varinjlim S_\bullet(U; \mathfrak{L}) \to S_\bullet(X; \mathfrak{L}).$$

Note that for any compact subset W of X, W is contained in some $U \in \mathfrak{U}$, then we have a chain isomorphism

$$\varinjlim i_{U\sharp} : \varinjlim S_\bullet(U; \mathfrak{L}) \simeq S_\bullet(X; \mathfrak{L})$$

which induces a homology isomorphism

$$\varinjlim i_{U*} : \varinjlim H_1(U; \mathfrak{L}) \simeq H_1(X; \mathfrak{L}).$$

We deduce from this fact that, for sufficiently large $\tilde{\sigma}$,

$$H_1(\tilde{B}; \mathfrak{L}) \simeq H_1(X; \mathfrak{L})$$

where $\tilde{B} = \{t \in X; \Re f(t) < \tilde{\sigma}\}$. By the five-lemma, we have

$$H_1(X, A_{\tau_1}; \mathfrak{L}) \simeq H_1(\tilde{B}, A_{\tau_1}; \mathfrak{L}).$$

This proves the lemma. $\square$

Consider the n-copies of the pair (X, f), we use the following notation:

$$X^n = \overbrace{X \times X \times \cdots \times X}^{n \text{ times}}$$

$$\boxtimes^n \mathfrak{L} = \overbrace{\mathfrak{L} \boxtimes \mathfrak{L} \boxtimes \cdots \boxtimes \mathfrak{L}}^{n \text{ times}}$$

Let $t = (t_1, t_2, \ldots, t_n)$ be the coordinates of X^n, we define a rational function

$$F = F(t_1, t_2, \ldots, t_n) = f(t_1) + f(t_2) + \cdots + f(t_n).$$

Let $\boldsymbol{\Psi}$ be a family of supports defined by the function F, we consider the homology group of X^n with coefficients in the local system $\boxtimes^n \mathfrak{L}$ and with the family of supports $\boldsymbol{\Psi}$. We apply Theorem 2.2.2 to our case. Then, there exists a sufficiently small $\tau^0 \in \mathbb{R}$ such that, for any $\tau \leq \tau^0$, the following isomorphism holds:

$$H_n^{\boldsymbol{\Psi}}(X^n; \boxtimes^n \mathfrak{L}) \simeq H_n(X^n, A_\tau; \boxtimes^n \mathfrak{L}),$$

where

$$A_\tau = \{(t_1, t_2, \ldots, t_n) \in X; \Re F(t_1, t_2, ..., t_n) < \tau\}.$$

We may assume $\tau^0 < n\tau_1^0$ and for the fixed $\tau_1 < \tau_1^0$, we fix $\tau \leq n\tau_1$, where τ_1^0 is that in Lemma 3.1.1.

Take any basis vector $[c] \in H_n(X^n, A_\tau; \boxtimes^n \mathfrak{L})$,

$$c = \sum_{(X^n - A_\tau) \cap \mathrm{supp}\sigma \neq \emptyset} a_\sigma \cdot \sigma \in S_n(X^n, A_\tau; \boxtimes^n \mathfrak{L}).$$

Note that c is a finite sum, then $\mathrm{supp}(c)$ is compact. Moreover, the homology group $H_n(X^n, A_\tau; \boxtimes^n \mathfrak{L})$ has a finite dimension. It follows that, if we take $\sigma^0 \in \mathbb{R}$ sufficiently large and put $B_i = \{t_i \in X; \Re f(t_i) < \sigma^0\}$, for which we may assume Lemma 3.1.1 holds, then $B^n := B_1 \times B_2 \times \cdots \times B_n$ contains all the supports of the representative of the basis vectors of $H_n(X^n, A_\tau; \boxtimes^n \mathfrak{L})$. Hence the homomorphism

$$\rho : H_n(B^n, A_\tau \cap B^n; \boxtimes^n \mathfrak{L}) \to H_n(X^n, A_\tau; \boxtimes^n \mathfrak{L}),$$

induced from the natural chain map $S_n(B^n, A_\tau \cap B^n; \boxtimes^n \mathfrak{L}) \to S_n(X^n, A_\tau; \boxtimes^n \mathfrak{L})$, is surjective. Hence we obtain the following:

Lemma 3.1.3. *Let τ_1^0 be that in Lemma 3.1.1 and τ^0 be as $\tau^0 < n\tau_1^0$. Then for any $\tau \leq \tau^0$, there exists a sufficiently large $\sigma^0 \in \mathbb{R}$, the homology homomorphism*

$$\rho : H_n(B^n, A_\tau \cap B^n; \boxtimes^n \mathfrak{L}) \to H_n(X^n, A_\tau; \boxtimes^n \mathfrak{L})$$

is surjective, where $B^n := B_1 \times B_2 \times \cdots \times B_n$ and $B_i = \{t_i \in X; \Re f(t_i) < \sigma^0\}$ $(i = 1, 2, \ldots, n)$.

For a fixed $\tau_1 < \tau_1^0$, using Künneth formula, we have the isomorphisms:

$$\bigotimes^n H_1(X, A_{\tau_1}; \mathfrak{L}) \simeq H_n(X^n, A_{\tau_1}^{(n)}; \boxtimes^n \mathfrak{L}),$$

where $A_{\tau_1}^{(n)} = \bigcup_{i=1}^n X \times \cdots \times \overset{i}{\breve{A}_{\tau_1}} \times \cdots \times X$, and

$$\bigotimes^n H_1(B, A_{\tau_1}; \mathfrak{L}) \simeq H_n(B^n, \tilde{A}_{\tau_1}^{(n)}; \boxtimes^n \mathfrak{L}),$$

where $\tilde{A}_{\tau_1}^{(n)} := \bigcup_{i=1}^n B \times \cdots \times \overset{i}{\breve{A}_{\tau_1}} \times \cdots \times B$. On the other hand, we can easily obtain an isomorphism from Lemma 3.1.1:

$$\bigotimes^n H_1(B, A_{\tau_1}; \mathfrak{L}) \simeq \bigotimes^n H_1(X, A_{\tau_1}; \mathfrak{L}).$$

Hence these isomorphisms induce the following:

Lemma 3.1.4. *For any sufficiently small τ_1, we have an isomorphism*

$$H_n(B^n, \tilde{A}_{\tau_1}^{(n)}; \boxtimes^n \mathfrak{L}) \simeq H_n(X^n, A_{\tau_1}^{(n)}; \boxtimes^n \mathfrak{L}).$$

For the fixed τ and σ^0, where τ and σ^0 appeared in Lemma 3.1.3, we take $\tau_1' < \tau - n\sigma^0$ and fix it. Put

$$A_{\tau_1'} := \{t \in X; \Re f(t) < \tau_1'\},$$

$$\tilde{A}_{\tau_1'}^{(n)} := \bigcup_{i=1}^n B \times \cdots \times \overset{i}{\breve{A}_{\tau_1'}} \times \cdots \times B.$$

For the $\tau^0, \tau_1, \tau, \tau_1'$ taken as above, we have

$$\tilde{A}_{\tau_1'}^{(n)} \subset A_\tau \cap B^n \subset \tilde{A}_{\tau_1}^{(n)} \subset A_{\tau^0} \cap B^n,$$

where $\tilde{A}_{\tau_1'}^{(n)}$, A_τ, $\tilde{A}_{\tau_1}^{(n)}$ are defined as above and

$$A_{\tau^0} := \{(t_1, \ldots, t_n) \in X^n; \Re f(t_1) + \cdots + \Re f(t_n) < \tau^0\}.$$

Then we obtain the natural inclusion homomorphism diagram:

$$\begin{array}{ccc}
H_n(B^n, \tilde{A}_{\tau_1{}'}^{(n)}; \boxtimes^n \mathfrak{L}) & \xrightarrow{\zeta_1} & H_n(B^n, A_\tau \cap B^n; \boxtimes^n \mathfrak{L}) \\
{\scriptstyle \eta_1}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} \\
H_n(B^n, \tilde{A}_{\tau_1}^{(n)}; \boxtimes^n \mathfrak{L}) & = & H_n(B^n, \tilde{A}_{\tau_1}^{(n)}; \boxtimes^n \mathfrak{L})
\end{array}$$

$$\begin{array}{ccc}
H_n(B^n, A_\tau \cap B^n; \boxtimes^n \mathfrak{L}) & \xrightarrow{\zeta} & H_n(B^n, \tilde{A}_{\tau_1}^{(n)}; \boxtimes^n \mathfrak{L}) \\
{\scriptstyle \eta_2}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_2} \\
H_n(B^n, A_{\tau^0} \cap B^n; \boxtimes^n \mathfrak{L}) & = & H_n(B^n, A_{\tau^0} \cap B^n; \boxtimes^n \mathfrak{L})
\end{array}$$

such that $\eta_1 = \zeta \circ \zeta_1$, and $\eta_2 = \zeta_2 \circ \zeta$. By Künneth formula, the map η_1 is an isomorphism. On the other hand, we consider the homology exact sequence for $(B^n, A_{\tau'} \cap B^n, A_{\tau''} \cap B^n)$:

$$\begin{aligned} H_n(A_{\tau'} \cap B^n, A_{\tau''} \cap B^n; \boxtimes^n \mathfrak{L}) = 0 \to H_n(B^n, A_{\tau''} \cap B^n; \boxtimes^n \mathfrak{L}) \\ \to H_n(B^n, A_{\tau'} \cap B^n; \boxtimes^n \mathfrak{L}) \to H_{n-1}(A_{\tau'} \cap B^n, A_{\tau''} \cap B^n; \boxtimes^n \mathfrak{L}) = 0. \end{aligned}$$

where τ', τ'' are any sufficiently small complex numbers satisfying $\tau'' < \tau'$. This induces η_2 is an isomorphism. Hence we have the following:

Lemma 3.1.5. *There exists an isomorphism of* $\mathbb{C}$*-vector space*

$$H_n(B^n, A_\tau \cap B^n; \boxtimes^n \mathfrak{L}) \simeq H_n(B^n, \tilde{A}^{(n)}_{\tau_1}; \boxtimes^n \mathfrak{L}).$$

By Lemma 3.1.3, Lemma 3.1.4, Lemma 3.1.5, we obtain the homomorphism diagram:

$$\begin{array}{ccc} H_n(B^n, A_\tau \cap B^n; \boxtimes^n \mathfrak{L}) & \xrightarrow{\zeta} & H_n(B^n, \tilde{A}^{(n)}_{\tau_1}; \boxtimes^n \mathfrak{L}) \\ \rho \downarrow & & \downarrow \wr \\ H_n(X^n, A_\tau; \boxtimes^n \mathfrak{L}) & \xrightarrow[\eta]{} & H_n(X^n, A^{(n)}_{\tau_1}; \boxtimes^n \mathfrak{L}) \end{array}$$

where η is a natural homomorphism. We can easily check that ρ is injective homomorphism. By Lemma 3.1.3, ρ is bijective. We have proved the following theorem.

Theorem 3.1.6. *There exists an isomorphism of* $\mathbb{C}$*-vector space:*

$$H_n(X^n, A_\tau; \boxtimes^n \mathfrak{L}) \simeq H_n(X^n, A^{(n)}_{\tau_1}; \boxtimes^n \mathfrak{L}).$$

Let $\mathfrak{S}_n$ be the symmetric group. $\mathfrak{S}_n$ acts on X^n:

$$\sigma \cdot (t_1, t_2, ..., t_n) = (t_{\sigma(1)}, t_{\sigma(2)}, \dots, t_{\sigma(n)}).$$

for any $\sigma \in \mathfrak{S}_n$. We can easily see that the action of $\mathfrak{S}_n$ on X^n induces the action of $\mathfrak{S}_n$ on A_τ and $A^{(n)}_{\tau_1}$, respectively. Let $X_n := X^n/\mathfrak{S}_n$ be the configuration space of n-points (see Section 16), $\mathfrak{M}$ a local system on X_n. There exists a canonical topological projection

$$\pi : X^n \longrightarrow X^n/\mathfrak{S}_n.$$

We assume $\pi^*\mathfrak{M} = \boxtimes^n \mathfrak{L}$, then we have an isomorphism

$$H_n(X^n/\mathfrak{S}_n, A_\tau/\mathfrak{S}_n; \mathfrak{M}) \simeq H_n(\otimes^n S_\bullet(X, A_{\tau_1}; \mathfrak{L}))^{\mathfrak{S}_n}.$$

By Künneth formula, we have an isomorphism

$$H_n(\otimes^n S_\bullet(X, A_{\tau_1}; \mathfrak{L}))^{\mathfrak{S}_n} \simeq \{\otimes^n H_1(X, A_{\tau_1}; \mathfrak{L})\}^{\mathfrak{S}_n}.$$

Hence we have the main theorem as follows.

Theorem 3.1.7. *Let $\mathfrak{L}$, $\mathfrak{M}$ be the local systems of 1-dimension $\mathbb{C}$-vector space on X, $X^n/\mathfrak{S}_n$, respectively. Assume $\pi^*\mathfrak{M} = \boxtimes^n\mathfrak{L}$. Then for any sufficiently small τ_1, there exists an isomorphism of $\mathbb{C}$-vector space*

$$H_n(X^n/\mathfrak{S}_n, A_\tau/\mathfrak{S}_n; \mathfrak{M}) \simeq \wedge^n H_1(X, A_{\tau_1}; \mathfrak{L}),$$

where $\tau < n\tau_1$.

This result has previously been obtained by K.Iwasaki and M.Kita in the case of $\lambda = (1, 1, ..., 1)$, i.e., the case of Aomoto-Gel'fand hypergeometric functions, (see [7]). In their paper, the Wronskian determinant formula given by T.Terasoma (see [18]) was understood in the sense of homology theory. Theorem 3.1.7 is an extension of the result given by [7].

3.2 Application of the twisted relative homology theory

The subspace of A_{τ_1} of X can be decomposed into connected components

$$A_{\tau_1} = \cup_{k=0}^{m} \cup_{j=1}^{n_k-1} A_{jk},$$

where $A_{1,k}$,$A_{2,k}$,...,$A_{n_k-1,k}$ are components each of which contains the point x_k in its closure in $\mathbf{P}^1$. Note that each A_{jk} is contractible and we assume A_{jk} contract to a point a_{jk}. We take a simplicial pair (K, K_0), where K is a bouquet B_m with extra edges which is constructed in Section 13.1, K_0 is a subcomplex of B_m only contains 0-simplexes a_{ij}. Then the inclusion map $\nu : |K| \hookrightarrow X$ is a homotopy equivalence between $|K|$ and X so that the restriction of mapping $\nu|_{|K_0|} : |K_0| \to A_{\tau_1}$ is a homotopy equivalence between $|K_0|$ and A_{τ_1}. So (X, A_{τ_1}) is a polyhedral pair with underlying simplicial structure $((K, K_0), \nu)$. Let $\mathfrak{L}_K = \mathfrak{L}_e$, where $\mathfrak{L}_e$ is the simplicial local system defined in Section 13.2, then by Theorem 3.1.7 and Theorem 17.2.3, we have the following:

Theorem 3.2.1. *Let $\mathfrak{L}$, $\mathfrak{M}$ be the singular local systems of 1-dimension $\mathbb{C}$-vector space on X, $X^n/\mathfrak{S}_n$, respectively. Assume that $\pi^*\mathfrak{M} = \boxtimes^n\mathfrak{L}$, where $\pi : X^n \to X^n/\mathfrak{S}_n$ is the canonical projection. Then there exists a canonical isomorphism of $\mathbb{C}$-vector space:*

$$H_n(X^n/\mathfrak{S}_n, A_\tau/\mathfrak{S}_n; \mathfrak{M}) \simeq \wedge^n H_1(K, K_0; \mathfrak{L}_K),$$

where $\mathfrak{L}_K = \mathfrak{L}_e$ is the simplicial local system defined in Section 13.2 with $e_i = exp(2\pi\sqrt{-1}\alpha_i)$, $(i = 1, 2, ..., m)$.

4 Relative simplicial homology with local systems

4.1 Simplicial pair

Let us briefly recall some notions of simplicial local systems, and establish some notational conventions following those of [7] and [17]. By a simplicial complex K we mean an abstract simplicial complex. Namely, K is a collection of finite nonempty subsets of a set V such that the following conditions hold:

1. for any $a \in V$, $\{a\} \in K$,
2. if $\sigma \in K$, then any nonempty subset of σ belongs to K.

An element σ of K is called a simplex of K. A nonempty subset of a simplex is called a face. For $\sigma \in K$, its dimension $\sharp\sigma$ is one less than the number of its elements. The i-th skeleton of K is denoted by K^i, i.e.

$$K^i = \{\sigma \in K; \sharp\sigma \le i+1\} \qquad (i = 0, 1, ...)$$

The vertex set of K is denoted by V_K. A q-simplex ($q \ge 0$) with vertices $a_0, a_1, ..., a_q \in V_K$ is denoted by $\{a_0, a_1, ..., a_q\}$. A subcollection of K, which itself is a complex, is called a subcomplex of K. (K, K_0) is called simplicial pair, where K_0 is a subcomplex of K. We have $V_{K_0} \subset V_K$.

Let K, L be two simplicial complexes. A simplicial map is a map of the set of vertices f: $V_K \to V_L$ such that, for any simplex $\sigma \in K$, $f(\sigma) \in L$, where $f(\sigma) = \{f(a); a \in \sigma\}$. A simplicial map $f : (K, K_0) \to (L, L_0)$ is a simplicial map $K \to L$, such that $f(K_0) \subset L_0$.

4.2 Subdivision

There is a barycentric subdivision associated with a simplicial complex K, is denoted by SdK, whose vertices are the simplexes of K, i.e $V_{\mathrm{Sd}K} = K$, and whose simplexes are the sets $\{\sigma_0, \sigma_1, ..., \sigma_q\}$, where $\sigma_i \in V_{\mathrm{Sd}K}$, such that

$$\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_q.$$

The iterated barycentric subdivision SdK are defined for $n \ge 0$ inductively, so that

$$\mathrm{Sd}^0 K = K, \tag{4.1}$$

$$\mathrm{Sd}^n K = \mathrm{Sd}(\mathrm{Sd}^{n-1} K), \qquad n \ge 1. \tag{4.2}$$

If K_0 is a subcomplex of K, $\mathrm{Sd}K_0 := \mathrm{Sd}K|_{K_0}$ is a subcomplex of SdK. $(\mathrm{Sd}K, \mathrm{Sd}K_0)$ is called the barycentric subdivision of the simplicial pair (K, K_0).

Lemma 4.2.1. *If K_0 is a subcomplex of K, then* SdK_0 *is a full subcomplex of* SdK.

4.3 Local systems on simplicial complexes

We defined a local system $\mathfrak{L} = (\mathfrak{L}_a, \xi_{ba})$ of $\mathbb{C}$-vector spaces on K.

Definition 4.3.1. *A local system $\mathfrak{L} = (\mathfrak{L}_a, \xi_{ba})$ of $\mathbb{C}$-vector space on K is an assignment:*

(1) $V_K \ni a \longmapsto \mathfrak{L}_a : \mathbb{C}$ *vector space,*

(2) $K^{(1)} \ni \{a, b\} \longmapsto \xi_{ba} : \mathfrak{L}_a \longrightarrow \mathfrak{L}_b$ *: isomorphism,*

such that

(i) *for any $a \in V_K, \xi_{aa}$ is the identify map on $\mathfrak{L}_a$,*

(ii) *for any $\{a, b, c\} \in K^{(2)}$, $\xi_{cb} \circ \xi_{ba} = \xi_{ca}$.*

Remark 4.3.2. *For $K_0 \subset K$, the restriction of the local system $\mathfrak{L}$ on K to K_0 gives a local system $\mathfrak{L}|_{K_0}$ on K_0. The pair $(K_0, \mathfrak{L}|_{K_0})$ is also denoted by $(K_0, \mathfrak{L})$. So the simplicial pair (K, K_0) with local system $\mathfrak{L}$ is denoted by $(K, K_0; \mathfrak{L})$.*

Definition 4.3.3. *For any simplex σ of K, a section of $\mathfrak{L}$ on σ is a map*

$$u : \sigma \ni a \longmapsto u(a) \in \mathfrak{L}$$

such that

$$u(b) = \xi_{ba} u(a), \qquad \text{for any} \quad \{a, b\} \in \sigma.$$

The set of all sections of $\mathfrak{L}$ on σ is denoted by $\mathfrak{L}_\sigma$.

4.4 Pull-back of local systems

Let $(K, K_0), (L, L_0)$ be the simplicial pairs, $f : (K, K_0) \to (L, L_0)$ a simplicial map, i.e. f is a simplicial map $K \to L$ such chat $f(K_0) \subset L_0$. Let $\mathfrak{L}, \mathfrak{M}$ be the local systems on K and L, respectively.

Definition 4.4.1. *A local system map over f is a pair*

$$(f, \varphi) : (K, K_0; \mathfrak{L}) \longrightarrow (L, L_0; \mathfrak{M}),$$

where $\varphi = \{\varphi_a\}$ is a collection of homomorphism of $\mathbb{C}$-vector space

$$\varphi_a : \mathfrak{L}_a \longrightarrow \mathfrak{M}_{f(a)}, \qquad (a \in V_K)$$

such that, for each $\{a, b\} \in K^{(1)}$, the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{L}_a & \xrightarrow{\varphi_a} & \mathfrak{M}_{f(a)} \\ {\scriptstyle \xi_{ba}}\downarrow & & \downarrow{\scriptstyle \eta_{f(b)f(a)}} \\ \mathfrak{L}_b & \xrightarrow[\varphi_b]{} & \mathfrak{M}_{f(b)} \end{array}$$

There is a category of simplicial pairs (K, K_0) and local system maps (f, φ). This category is called the category of local systems and is denoted by $\mathbb{L}$. A simplicial map $f : K \to L$ induces a covariant functor $f^* : \mathbb{L}(L) \to \mathbb{L}(K)$ called pull back functor, where $\mathbb{L}(L), \mathbb{L}(K)$ are the categories of local systems on the simplicial complex L, K, respectively.

Definition 4.4.2. *Given a local system $\mathfrak{L} = \{\mathfrak{L}_a, \xi_{ba}\}$ on L, put*

$$(f^*\mathfrak{L})_a = \mathfrak{L}_{f(a)} \qquad (a \in V_K)$$
$$(f^*\xi)_{ba} = \xi_{f(b)f(a)} \qquad (\{a, b\} \in K^{(1)}).$$

Then $f^*\mathfrak{L} = \{(f^*\mathfrak{L})_a, (f^*\xi)_{ba}\}$ becomes a local system on K, called the pull-back of $\mathfrak{L}$ by f.

Remark 4.4.3. *A local system map*

$$(f, \varphi) : (K, K_0; f^*\mathfrak{L}) \longrightarrow (L, L_0; \mathfrak{L})$$

where $\varphi = \{\varphi_a\}, \varphi_a : (f^*\mathfrak{L})_a \to \mathfrak{L}_{f(a)} \quad (a \in V_K)$ is well-defined.

5 Polyhedra

5.1 The topological realization

For a simplicial complex K, let $|K|$ be the set of all functions $\alpha : V_K \longrightarrow [0,1]$ such that

(1) For any α, $\operatorname{supp}\alpha$ is a simplex σ of K,

(2) For any α , $\sum_{a\in\sigma}\alpha(a) = 1$,

where $\operatorname{supp}\alpha := \{a \in V_K; \alpha(a) \neq 0\}$.

We provide the set $|K|$ with the coherent topology ([17] Chapter 3 §2). Then $|K|$ is a topological space of K.

Remark 5.1.1. (1) *For any subcomplex K_0 of simplicial complex K, $|K_0|$ is a closed subset of $|K|$.*

(2)*If $\{K_{0j}\}_{j\in J}$ is a collection of subcomplexes of K, then $\cup|K_{0j}| = |\cup K_{0j}|$ and $\cap|K_{0j}| = |\cap K_{0j}|$.*

Remark 5.1.2. *For a simplicial map $f : K \to L$, let $|f| : |K| \to |L|$ be the map defined by*

$$|f|(\alpha) := \sum_{a\in V_K} \alpha(a)\langle f(a)\rangle,$$

where for $a, b \in V_K$,

$$\langle a\rangle(b) := \begin{cases} 1 & (b = a) \\ 0 & (b \neq a) \end{cases}$$

Then $|K| \to |L|$ becomes a continuous map.

5.2 The polyhedral pair

Let (K, K_0) be the simplicial pair, we call the topological realization associated with (K, K_0) a topological space pair $(|K|, |K_0|)$.

Definition 5.2.1. *Let (X, A) be a topological space pair. (X, A) is said to be an polyhedral pair if there exists a simplicial complex pair (K, K_0) and a continuous map $f : |K| \to X$ such that f is a homotopy equivalence between $|K|$ and X, and $f|_{|K_0|} : |K_0| \to A$ is homotopy equivalence between $|K_0|$ and A. We call $((K, K_0); f)$ an underlying simplicial structure of (X, A).*

Remark 5.2.2. *In the usual definition of polyhedral pair, the above condition on f is replaced by the one that $f : (|K|, |K_0|) \to (X, A)$ is a homeomorphism.*

6 Group action

6.1 Group action on simplicial complexes

Let K be a simplicial complex and G be a group. Let $\mathrm{Aut}K$ be the group of all simplicial automorphisms of K. A group action of G on K is a group homomorphism $\rho : G \to \mathrm{Aut}K$. For $g \in G$ and $a \in V_K$, we simply write $\rho(g)a = ga$. Then for any simplex $\sigma = \{a_0, a_1, ..., a_q\} \in K$, $g\sigma = \{ga_0, ga_1, ..., ga_q\}$ becomes another simplex of K.

If G acts on K, we define a simplicial complex of K/G as follows. The vertices of K/G are just the orbits $[a] = Ga$ of the action of G on the vertices of K, i.e. $V_{K/G} = \{Ga; a \in V_K\}$, and we take the simplexes of K/G to be those simplexes of the form $\{[a_0], [a_1], ..., [a_q]\}$, where $\{a_0, a_1, ..., a_q\}$ is a simplex of K. The simplex $\{a_0, a_1, ..., a_q\}$ is said to be over the simplex $\{[a_0], [a_1], ..., [a_q]\}$ of K/G. We have a natural projection

$$\pi : V_K \longrightarrow V_{K/G}, \qquad a \longmapsto [a],$$

then the projection

$$\pi : K \longrightarrow K/G, \qquad \{a_0, a_1, ..., a_q\} \longmapsto \{[a_0], [a_1], ..., [a_q]\}$$

is a simplicial map.

6.2 The regular action

Given a simplex σ of K/G, we put

$$O(\sigma) = \{\tilde{\sigma} \in K; \pi(\tilde{\sigma}) = \sigma\}.$$

$O(\sigma)$ is called the set of all simplexes of K over σ. The action of G on K leads to that on $O(\sigma)$.

Definition 6.2.1. *If G acts on $O(\sigma)$ transitively for any simplex σ of K/G, then the action of G on K is said to be regular.*

Remark 6.2.2. *If G acts on K regularly, then for any $\sigma \in K/G$, $O(\sigma)$ forms an orbit of the action of G on the simplexes of K.*

Let (K, K_0) be a simplicial pair, G a group. Then we consider the restriction of the action to K_0. If K_0 is invariant under the action of G, i.e. $G(K_0) = K_0$, where

$$G(K_0) := \{g\sigma; g \in G, \sigma \in K_0\},$$

then for any $\sigma \in K_0$ and $g \in G$, we have $g\sigma \in K_0$. Hence we obtain a subcomplex K_0/G of K/G, so that a simplex of K/G is a simplex of K_0/G if and only if there exists a simplex $\tilde{\sigma}$ of K_0 such that $\pi(\tilde{\sigma}) = \sigma$. For $\sigma \in K_0/G$, we put

$$O(\sigma) = \{\tilde{\sigma} \in K_0; \pi(\tilde{\sigma}) = \sigma\},$$

$O(\sigma)$ forms an orbit of K. Hence the regular action of G on K implies that on K_0 and $\pi : (K, K_0) \to (K/G, K_0/G)$ is a simplicial map.

6.3 Group action on the subdivision SdK

Let SdK be the subdivision of the simplicial complex K, G a group. The action of G on K induces an action of G on SdK in a natural manner:

$$G \times V_{\mathrm{Sd}K} \longrightarrow V_{\mathrm{Sd}K}, \qquad (g, \sigma) \longmapsto g\sigma.$$

We have the following:

Lemma 6.3.1. *Let K_0 be a subcomplex of K. If K_0 is invariant under the action of G, then $\mathrm{Sd}K_0$ is also invariant under the action of G.*

Proof. For any $g \in G$ and $\delta = \{\sigma_0, \sigma_1, ..., \sigma_q\} \in \mathrm{Sd}K_0$, we have

$$g\delta = \{g\sigma_0, g\sigma_1, ..., g\sigma_q\} \in \mathrm{Sd}K$$

and for $\sigma_i \in K_0 (i = 0, 1, \ldots, q)$, K_0 is G-invariant, then $g\sigma_0 \in K_0$. On the other hand, $\mathrm{Sd}K_0$ is full subcomplex of $\mathrm{Sd}K$ and $g\sigma_i \in V_{\mathrm{Sd}K_0} = K_0 \quad (i = 0, 1, ...q)$, then $g\delta \in \mathrm{Sd}K_0$. Hence $G(\mathrm{Sd}K_0) = \{g\delta; g \in G, \delta \in \mathrm{Sd}K_0\} = \mathrm{Sd}K_0$.

The following theorem is important:

Theorem 6.3.2. *([7] Theorem 8.3.2) If G acts on K, then G acts on Sd^2K regularly.*

7 External product

7.1 The external product of simplicial pair local system

Let $K_1, K_2, ..., K_n$ be ordered simplicial complexes, $K_{01}, K_{02}, ..., K_{0n}$ the subcomplex of $K_1, K_2, ..., K_n$, respectively. Let $\mathfrak{L}_i$, $(i = 1, 2, ..., n)$ be the local systems on K_i, $(i = 1, 2, ..., n)$. The direct product of $K_1, K_2, ..., K_n$ was described explicitly in [7]. We refer to the Definition 7.1.3 and Definition 7.2.1 in [7]. We can define the direct product of the simplicial pairs $(K_1, K_{01}), (K_2, K_{02}), ..., (K_n, K_{0n})$ in the same way.

We use the following notation.

- $K = K_1 \times K_2 \times \cdots \times K_n$: the direct product of $K_1, K_2, \ldots, K_n$,
- $K^{[i]} = K_1 \times \cdots \times K_{0i} \times \cdots \times K_n$: the direct product of $K_1, ..., K_{0i}, ..., K_n$,
- $(K, M) = (K_1, K_{01}) \times (K_2, K_{02}) \times \cdots \times (K_n, K_{0n})$
$= (K_1 \times K_2 \times \cdots \times K_n, K^{[1]} \cup K^{[2]} \cup \cdots \cup K^{[n]})$: the direct product of simplicial pairs $(K_1, K_{01}), ..., (K_n, K_{0n})$,
- $V_K = V_{K_1} \times V_{K_2} \times \cdots \times V_{K_n}$: the vertices of K,
- $\mathfrak{L} = (\mathfrak{L}_a, \xi_{ba})$: the external product of $\mathfrak{L}_1, ..., \mathfrak{L}_n$, where for each vertex $a = a_1 \times a_2 \times \cdots \times a_n$, $b = b_1 \times b_2 \times \cdots \times b_n \in V_K$, and $\{a, b\} \in K^{(1)}$,

$$\mathfrak{L}_a = (\mathfrak{L}_1)_{a_1} \otimes \cdots \otimes (\mathfrak{L}_n)_{a_n}$$

$$\xi_{ba} = (\xi_1)_{b_1,a_1} \otimes \cdots \otimes (\xi_n)_{b_n,a_n}.$$

$(K, M; \mathfrak{L})$ is denoted as $\boxtimes_{i=1}^n (K_i, K_{0i}; \mathfrak{L}_i)$ and is called the external product of $(K_1, K_{01}; \mathfrak{L}_1)$, ... , $(K_n, K_{0n}; \mathfrak{L}_n)$. We write $\mathfrak{L} = \mathfrak{L}_1 \boxtimes \cdots \boxtimes \mathfrak{L}_n$. The partial order on V_K is the lexicographic order, i.e., for $a = a_1 \times a_2 \times \cdots \times a_n$ and $b = b_1 \times b_2 \times \cdots \times b_n$, we put $a < b$ if and only if $a_j = b_j$ $(j < i))$ and $a_i < b_i$ for some $i \in \{1, 2, \ldots, n\}$.

- $q = (q_1, q_2, \dots, q_n)$: an n-tuple of nonnegative integers,
- $r_i = q_1 + q_2 + \cdots + q_i,\ (i = 1, 2, \dots, n)$,
- $j_i = (j_{i1}, j_{i2}, \dots, j_{iq_i})$: a q_i-tuple of integers such that

$$1 \le j_{i1} \le j_{i2} \le \cdots \le j_{iq_i} \le r,$$

- $j = (j_1, j_2, \dots, j_n)$: the map from $\{1, 2, \dots, r\}$ into itself defined by

$$j = \begin{pmatrix} 1 & \cdots & r_1 & r_1+1 & \cdots & r_2 & \cdots & r_{n-1}+1 & \cdots & r_n \\ j_{11} & \cdots & j_{1q_1} & j_{21} & \cdots & j_{2q_2} & \cdots & j_{n1} & \cdots & j_{nq_n} \end{pmatrix},$$

- $J(q)$: the set of all j's such that $j \in \mathfrak{S}_r$,
- $\sigma_i = \{a_{i0}, a_{i1}, \dots, a_{iq_i}\}$: a q_i-simplex of K_i such that $a_{i0} < a_{i1} < \cdots < a_{iq_i}$,
- $\Sigma(q) = \{\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) : \sigma_i \text{ is a } q_i\text{-simplex of } K_i\}$.
- $\langle \sigma; j \rangle$: the simplices of K, for any $\sigma \in \Sigma(q)$ and $j \in J(q)$, is defined as follows. For $i = 1, 2, \dots, n$, we put

$$(b_{i0}, b_{i1}, \dots, b_{ir}) = (\overset{0}{\breve{a}_{i0}}, \dots, a_{i0}, \overset{j_{i1}}{\breve{a}_{i1}}, \dots, a_{i1}, \dots, \overset{j_{iq_i}}{\breve{a}_{iq_i}}, \dots, \overset{r}{\breve{a}_{iq_i}}).$$

  For $k = 0, 1, \dots, r$, we define a vertex c_k of K by $c_k = b_{1k} \times b_{2k} \times \cdots \times b_{nk}$. Since $j \in J(q)$, we have $c_0 < c_1 < \cdots < c_r$ in the lexicographic order. Now we define $\langle \sigma; j \rangle = \langle \sigma_1, \dots, \sigma_n; j_1, \dots, j_n \rangle$ by $\langle \sigma; j \rangle = \{c_0, c_1, \dots, c_r\}$. A simplex of K is, by definition, a nonempty subset of $\langle \sigma; j \rangle$ for some $\sigma \in \Sigma(q)$ and $j \in J(q)$. Clearly, the partial order on V_K induces a total order on every simplices of K.

7.2 The symmetric group acts on the external product

Let (K_i, K_{0i}), $(i = 1, 2, ..., n)$ be the simplicial pairs, $\mathfrak{L}_i$, $(i = 1, 2, ..., n)$ the local systems on $K_i (i = 1, 2, ..., n)$. Let $\mathfrak{S}_n$ be the group of all permutation of $\{1, 2, ..., n\}$.

For $\tau \in \mathfrak{S}_n$, we use the following notation.

$$\begin{aligned} \tau K =& K_{\tau(1)} \times K_{\tau(2)} \times \cdots \times K_{\tau(n)}, \\ \tau K^{[i]} =& K_{\tau(1)} \times \cdots \times K_{0\tau(i)} \times \cdots \times K_{\tau(n)}, \\ \tau \mathfrak{L} =& \mathfrak{L}_{\tau(1)} \boxtimes \mathfrak{L}_{\tau(2)} \boxtimes \cdots \boxtimes \mathfrak{L}_{\tau(n)}, \\ \tau(K, M) =& (\tau K, \tau M) = (\tau K, \tau K^{[1]} \cup \tau K^{[2]} \cup \cdots \cup \tau K^{[n]}). \end{aligned}$$

For each vertex $a = a_1 \times a_2 \times \cdots \times a_n \in V_K$, we define

$$\tau : V_K \longrightarrow V_{\tau K},\ a_1 \times a_2 \times \cdots \times a_n \longmapsto a_{\tau(1)} \times a_{\tau(2)} \times \cdots \times a_{\tau(n)},$$

$$\tau : \mathfrak{L}_a \longrightarrow (\tau \mathfrak{L})_{\tau(a)},\ u_1 \otimes u_2 \otimes \cdots \otimes u_n \longmapsto u_{\tau(1)} \otimes u_{\tau(2)} \otimes \cdots \otimes u_{\tau(n)}.$$

Then $\tau : (K, M; \mathfrak{L}) \to (\tau K, \tau M; \tau\mathfrak{L})$ is an isomorphism of the category $\mathbb{L}$ of local systems.

We shall consider the following special case:

$$(K, K_0; \mathfrak{L}) := (K_1, K_{01}; \mathfrak{L}_1) = (K_2, K_{02}; \mathfrak{L}_2) = \cdots = (K_n, K_{0n}; \mathfrak{L}_n)$$

In this case, we write

$$\begin{aligned} K^n &= \overbrace{K \times K \times \cdots \times K}^{\text{n times}}, \\ K_0^{[i]} &= K \times \cdots \times \overset{i}{\breve{K_0}} \times \cdots \times K, \\ M &= \cup_{i=1}^n K_0^{[i]}, \\ \boxtimes^n \mathfrak{L} &= \overbrace{\mathfrak{L} \boxtimes \mathfrak{L} \boxtimes \cdots \boxtimes \mathfrak{L}}^{\text{n times}}. \end{aligned}$$

Then there is a natural action of $\mathfrak{S}_n$ on $(K^n; \boxtimes^n \mathfrak{L})$. We obtain the following:

Lemma 7.2.1. $\cup_{i=1}^n K_0^{[i]}$ *is an* $\mathfrak{S}_n$*-invariant subcomplex of* K^n*.*

8 Relative chain complexe with a local system

8.1 Chain complex with a local system

Let K be a simplicial complex, we let K_{ord} denote a set of all ordered simplexes of K (see [7] Definition 9.1.1) and K_{ori} a set of all oriented simplexes of K (see [7] Definition 9.1.3). If ϕ is an ordered simplex of σ, where σ is a q-simplex of K, then σ is said to be the simplex under ϕ. We put $\sigma = \langle\phi\rangle$. Let ϕ be an ordered simplex over a q-simplex σ and $[\phi]$ the equivalence class determinded by ϕ. We have a sequence of forgetting maps:

$$K_{ord} \to K_{ori} \to K, \qquad \phi \mapsto [\phi] \mapsto \langle\phi\rangle.$$

Let K be a simplicial complex. An ordering of K is a right-inverse $K \to K_{ord}, \sigma \mapsto \phi_\sigma$ of forgetting map $K_{ord} \to K$. Similarly, an orientation of K is a right-inverse $K \to K_{ori}, \sigma \mapsto \hat{\sigma}$ of the forgetting map $K_{ori} \to K$. An ordering $\sigma \mapsto \phi_\sigma$ induces an orientation $\sigma \mapsto \hat{\sigma} = [\phi_\sigma]$, called the associated orientization. If K is an ordered simplicial complex , then the associated orientization is called the natural orientization of K . In the case of K is an ordered simplicial complex, the natural orientization will be chosen unless otherwise is stated explicitly.

Let K be the simplicial complex, $\mathfrak{L}$ a local system on K. Given an orientization

$$K \to K_{ori}, \sigma := \{a_0, a_1, ..., a_q\} \mapsto [a_0, a_1, ..., a_q] =: \hat{\sigma},$$

then any oriented chain $c \in C_\bullet(K, \mathfrak{L})$ is uniquely expressed in the formal form

$$c = \sum_{\sigma \in K} u_\sigma \hat{\sigma}$$

such that

(1) for each $\hat{\sigma} \in K_{ori}, u_\sigma \in \mathfrak{L}_\sigma$,

(2) $\mathrm{supp}(c) := \{\sigma \in K; u_\sigma \neq 0\}$ is a finite set .

We shall often express an oriented chain c by

$$c = \sum_{\sigma \in K} u_\sigma \sigma$$

for simplicity of notation.

For $c_1 = \sum_\sigma u_\sigma \sigma, c_2 = \sum_\sigma v_\sigma \sigma \in C_\bullet(K, \mathfrak{L})$, we define $c_1 + c_2 = \sum_\sigma (u_\sigma + v_\sigma)\sigma \in C_\bullet(K, \mathfrak{L})$, then $C_\bullet(K, \mathfrak{L})$ becomes a $\mathbb{C}$-vector space. The boundary operator is a homomorphism of $\mathbb{C}$-vector space

$$\partial_p : C_p(K, \mathfrak{L}) \longrightarrow C_{q-1}(K, \mathfrak{L})$$

defined by

$$\partial_p c = \sum_{\sigma \in K} \sum_{\tau \in K} [\hat{\sigma} : \hat{\tau}](u_\sigma|_\tau)\hat{\tau}$$

where τ is a principal face of σ (see [7] Definition 9.2.1) and, $[\hat{\sigma} : \hat{\tau}]$ is the incidence number (see [7] Definition 9.7.1). We can easily check that $\partial^2 = 0$.

8.2 Relative chain complex with a local system

Let (K, K_0) be a simplicial pair, $\mathfrak{L}$ a local system on K. Then the chain complex $C_\bullet(K_0, \mathfrak{L})$ can be considered as a $\mathbb{C}$-vector subspace of the chain complex $C_\bullet(K, \mathfrak{L})$ in the natural way. We have the following:

Definition 8.2.1. *Let (K, K_0) be a complex pair, $\mathfrak{L}$ a local system on K. The quotient $\mathbb{C}$-vector space $C_\bullet(K, \mathfrak{L})/C_\bullet(K_0, \mathfrak{L})$ is called the relative chain complex of $(K, K_0; \mathfrak{L})$ and is denoted by $C_\bullet(K, K_0; \mathfrak{L})$.*

The element of $C_\bullet(K, K_0; \mathfrak{L}) = C_\bullet(K, \mathfrak{L})/C_\bullet(K_0, \mathfrak{L})$ is residual class $c + C_\bullet(K_0, \mathfrak{L})$, $(c \in C_\bullet(K, \mathfrak{L}))$. Then $C_\bullet(K, K_0; \mathfrak{L})$ with the boundary operator

$$\partial(c + C_q(K_0, \mathfrak{L})) = \partial c + C_{q-1}(K_0, \mathfrak{L}), \qquad (c \in C_q(K, \mathfrak{L}))$$

becomes a chain complex. Note that the boundary operator

$$\partial : C_q(K_0, \mathfrak{L}) \to C_{q-1}(K_0, \mathfrak{L})$$

is just the restriction of the boundary operator on $C_q(K, \mathfrak{L})$.

Remark 8.2.2. (1) *A relative q-chain $c + C_q(K_0, \mathfrak{L})$ is a relative cycle if and only if $\partial c \in C_{q-1}(K_0, \mathfrak{L})$.*

(2) *A relative q-chain $c + C_q(K_0, \mathfrak{L})$ is a relative boundary if and only if $c = \partial d + c' \quad (d \in C_{q+1}(K, \mathfrak{L}), c' \in C_q(K_0, \mathfrak{L}))$.*

8.3 The subdivision isomorphism

Let SdK be the subdivision of simplicial complex K, Sd$\mathfrak{L} = \{(\mathrm{Sd}\mathfrak{L})_\sigma, (\mathrm{Sd}\xi)_{\tau\sigma}\}$ the local system on SdK. There exists a natural chain map

$$(\mathrm{Sd} : C_\bullet(K, K_0; \mathfrak{L}) \to C_\bullet((\mathrm{Sd}K, (\mathrm{Sd}K_0; (\mathrm{Sd}\mathfrak{L}).$$

For any subcomplex K_0 of K, we let SdK_0 denote the induced subdivision of K_0. Then we have the following:

Lemma 8.3.1. *The subdivision*

$$\mathrm{Sd} : C_\bullet(K, K_0; \mathfrak{L}) \longrightarrow C_\bullet(\mathrm{Sd}K, \mathrm{Sd}K_0; \mathrm{Sd}\mathfrak{L})$$

is a chain homotopy equivalence.

As in the classical case where the local system $\mathfrak{L}$ is trivial, the method of acyclic models works out to prove this lemma. So the proof is omitted. Lemma 8.3.1 immediately imply the following:

Theorem 8.3.2. *There exists an isomorphism of* $\mathbb{C}$*-vector space:*

$$\mathrm{Sd} : H_\bullet(K, K_0; \mathfrak{L}) \longrightarrow H_\bullet(\mathrm{Sd}K, \mathrm{Sd}K_0; \mathrm{Sd}\mathfrak{L}).$$

9 The isomorphic relative chain complex

9.1 Group actions on relative chain complexes

Let K be a simplicial complex, G a finite group, G act on K regularly. Assume K_0 is a G-invariant subcomplex of K.

Let $\pi : K \to K/G$ be the canonical simplicial map, then

$$\pi : (K, K_0) \to (K/G, K_0/G)$$

is a canonical simplicial map.

Let $\mathfrak{L}$ be a local system on K. An action of G on $(K, K_0; \mathfrak{L})$ induces an action of G on the chain complex $C_\bullet(K, K_0; \mathfrak{L})$. For any $c \in C_\bullet(K; \mathfrak{L})$, $c = \sum_{\sigma \in K} u_\sigma \sigma$, the action of $g \in G$ on c is given explicitly by

$$gc = g \sum_{\sigma \in K} u_\sigma \sigma = \sum_{\sigma \in K} (gu_\sigma)(g\sigma),$$

where $u_\sigma \in \mathfrak{L}_\sigma$ is defined in Definition 4.3.3, and $g : \mathfrak{L}_\sigma \to \mathfrak{L}_{g\sigma},\ u \mapsto gu$ is defined by

$$(gu)(a) = g \cdot u(g^{-1}a), \quad a \in g\sigma.$$

Hence we have the following :

Lemma 9.1.1. *Let $\mathfrak{L}$ be a local system on K/G, then the canonical simplicial map $\pi : (K, K_0) \to (K/G, K_0/G)$ induces a chain map*

$$\pi : C_\bullet(K, K_0; \pi^*\mathfrak{L}) \to C_\bullet(K/G, K_0/G; \mathfrak{L}),$$
$$c + C_\bullet(K_0; \pi^*\mathfrak{L}) \mapsto \pi c + C_\bullet(K_0/G; \mathfrak{L})$$

where $\pi c = \sum_\sigma \pi_\sigma(u_\sigma)\pi(\sigma)$
and $\pi : (\pi^*\mathfrak{L})_\sigma \to \mathfrak{L}_{\pi(\sigma)}, \quad u_\sigma \mapsto \pi_\sigma(u_\sigma)$ *is defined by*

$$(\pi_\sigma(u_\sigma))(a) = (\pi_{(\pi|_\sigma)^{-1}(a)} \circ u_\sigma \circ (\pi|_\sigma)^{-1})(a), \qquad a \in \pi(\sigma),$$

$\pi|_\sigma : \sigma \to \pi(\sigma)$ *being a restriction of* π *to* $\sigma \in K$.

9.2 The transfer

Let $C_\bullet(K, K_0; \pi^*\mathfrak{L})^G$ be the G-invariant part of $C_\bullet(K, K_0; \pi^*\mathfrak{L})$, i.e.

$$C_\bullet(K, K_0; \pi^*\mathfrak{L})^G$$
$$= \{c + C_\bullet(K_0; \pi^*\mathfrak{L}) \in C_\bullet(K, K_0; \pi^*\mathfrak{L}); hc \equiv c \bmod C_\bullet(K_0; \pi^*\mathfrak{L}) \text{ for any } h \in G\}.$$

Then the inclusion map $C_\bullet(K, K_0; \pi^*\mathfrak{L})^G \hookrightarrow C_\bullet(K, K_0; \pi^*\mathfrak{L})$ induces a natural chain map

$$\pi : C_\bullet(K, K_0; \pi^*\mathfrak{L})^G \to C_\bullet(K/G, K_0/G; \mathfrak{L}).$$

We shall prove that the chain map π is an isomorphism. To see this, we construct its inverse chain map, called the transfer.

Definition 9.2.1. *Assume G is a finite group and G acts on K regularly. Let K_0 be a G-invariant subcomplex of K. The transfer*

$$\mathrm{tf} : C_\bullet(K/G, K_0/G; \mathfrak{L}) \longrightarrow C_\bullet(K, K_0; \pi^*\mathfrak{L})^G$$

is defined by

$$\mathrm{tf}(u\sigma + C_\bullet(K_0/G; \mathfrak{L})) \mapsto \frac{1}{\sharp G} \sum_{g \in G} (\pi^* u|_{g\tilde{\sigma}})(g\tilde{\sigma}) + C_\bullet(K_0; \pi^*\mathfrak{L}),$$

where $\sigma \in K/G$, $u \in \mathfrak{L}_\sigma$ *and* $\tilde{\sigma} \in O(\sigma) = \{\tilde{\sigma} \in K; \pi(\tilde{\sigma}) = \sigma\}$.

By assumption, G acts on $O(\sigma)$ transitively. So the sum is independent of the choice of $\tilde{\sigma} \in O(\sigma)$. Moreover, for any $\sigma' \in K_0/G$ and $\tilde{\sigma'} \in O(\sigma')$, we have $g\tilde{\sigma'} \in K_0$ since K_0 is G-invariant. This implies that $\frac{1}{\sharp G} \sum_{g \in G} (\pi^* u|_{g\tilde{\sigma'}})(g\tilde{\sigma'}) \in C_\bullet(K_0; \pi^*\mathfrak{L})$. So this definition is well-defined.

Lemma 9.2.2. $\mathrm{tf}(u\sigma + C_\bullet(K_0/G; \mathfrak{L})) \in C_\bullet(K, K_0; \pi^*\mathfrak{L})^G$.

Proof. By Definition 9.2.1, for $h \in G$ and $c = u\sigma + C_\bullet(K_0/G; \mathfrak{L})$,

$$\mathrm{tf}(c) = \frac{1}{\sharp G} \sum_{g \in G} (\pi^* u|_{g\tilde{\sigma}}) \cdot (g\tilde{\sigma}) + C_\bullet(K_0; \pi^*\mathfrak{L}).$$

For any $h \in G$,

$$h \cdot \frac{1}{\sharp G} \sum_{g \in G} (\pi^* u|_{g\tilde{\sigma}}) \cdot (g\tilde{\sigma}) = \frac{1}{\sharp G} \sum_{g \in G} h(\pi^* u|_{g\tilde{\sigma}}) \cdot (hg\tilde{\sigma}).$$

On the other hand, for any $\tilde{\sigma} \in O(\sigma)$ and any $a \in h\tilde{\sigma}$,

$$\begin{aligned}(h \cdot (\pi^* u|_{\tilde{\sigma}}))(a) &= (\pi^* u|_{\tilde{\sigma}})(h^{-1}(a)) \\ &= u(\pi(h^{-1}(a)) = u(\pi(a)) = (\pi^* u|_{h\tilde{\sigma}})(a),\end{aligned}$$

this implies that $h \cdot (\pi^* u|_{\tilde{\sigma}}) = \pi^* u|_{h\tilde{\sigma}}$. Then $h \cdot \mathrm{tf}(u\sigma) = \mathrm{tf}(u\sigma)$. Hence for any $h \in G$,

$$h \cdot \mathrm{tf}(u\sigma) \equiv \mathrm{tf}(u\sigma) \mod C_\bullet(K_0; \mathfrak{L}).$$

This establishes the lemma.

Lemma 9.2.3. *The transfer is the inverse of the natural chain map π, i.e.*

$$\pi \circ \mathrm{tf} = id|_{C_\bullet(K/G, K_0/G; \mathfrak{L})}$$

$$\mathrm{tf} \circ \pi = id|_{C_\bullet(K, K_0; \pi^* \mathfrak{L})^G}.$$

Proof. For any $g \in G$ and $\tilde{\sigma} \in O(\sigma)$, we have $\pi_{\tilde{\sigma}}(g\tilde{\sigma}) = \sigma$, where $\pi_{\tilde{\sigma}} : (\pi^* \mathfrak{L})_{\tilde{\sigma}} \to \mathfrak{L}_{\pi(\tilde{\sigma})} = \mathfrak{L}_\sigma$. Then $\pi_\sigma(\pi^* u|_{g\tilde{\sigma}}) = u|_{\pi_\sigma(g\tilde{\sigma})} = u|_\sigma = u_\sigma \in \mathfrak{L}_\sigma$. Hence

$$\begin{aligned}\pi_\sigma \circ \mathrm{tf}(c + C_\bullet(K_0/G; \mathfrak{L})) =& \pi \cdot \mathrm{tf}(\sum_{\sigma \in K/G} u_\sigma \cdot \sigma + C_\bullet(K_0/G; \mathfrak{L})) \\ =& \pi(\frac{1}{\sharp G} \sum_{g \in G} \sum_{\sigma \in K/G} (\pi^* u|_{g\tilde{\sigma}}) \cdot g\tilde{\sigma} + C_\bullet(K_0; \pi^* \mathfrak{L})) \\ =& \frac{1}{\sharp G} \sum_{g \in G} \sum_{\sigma \in K/G} \pi(\pi^* u|_{g\tilde{\sigma}}) \cdot \pi(g\tilde{\sigma}) + C_\bullet(K_0/G; \mathfrak{L}) \\ =& \frac{1}{\sharp G} \sum_{g \in G} \sum_{\sigma \in K/G} u_\sigma \cdot \sigma + C_\bullet(K_0/G; \mathfrak{L}) \\ =& \sum_{\sigma \in K/G} u_\sigma \cdot \sigma + C_\bullet(K_0/G; \mathfrak{L}).\end{aligned}$$

This shows $\pi \circ \mathrm{tf} = id|_{C_\bullet(K/G, K_0/G; \mathfrak{L})}$. On the other hand, let

$$c + C_\bullet(K_0; \pi^* \mathfrak{L}) = \sum_{\sigma \in K} u_\sigma \cdot \sigma + C_\bullet(K_0; \pi^* \mathfrak{L})$$

be any element of $C_\bullet(K, K_0; \pi^* \mathfrak{L})^G$, where $u_\sigma \in (\pi^* \mathfrak{L})_\sigma$. Similarly we have $\pi^* \pi_\sigma(u_\sigma)|_\sigma = u_\sigma$ for $\pi_\sigma : (\pi^* \mathfrak{L})_\sigma \to \mathfrak{L}_{\pi(\sigma)}$. Then

$$\pi(c + C_\bullet(K_0; \pi^* \mathfrak{L})) = \sum_{\sigma \in K} \pi_\sigma(u_\sigma) \cdot \pi(\sigma) + C_\bullet(K_0/G; \mathfrak{L})$$

and hence

$$
\begin{aligned}
(\mathrm{tf}\circ\pi)(c+C_\bullet(K_0;\pi^*\mathfrak{L})) =&\frac{1}{\sharp G}\sum_{g\in G}\sum_{\sigma\in K}(\pi^*\pi_\sigma(u_\sigma)|_{g\sigma})\cdot(g\sigma)+C_\bullet(K_0,\pi^*\mathfrak{L})\\
=&\frac{1}{\sharp G}\sum_{g\in G}\sum_{\sigma\in K}g(\pi^*\pi_\sigma(u_\sigma)|_{\sigma})\cdot(g\sigma)+C_\bullet(K_0;\pi^*\mathfrak{L})\\
=&\frac{1}{\sharp G}\sum_{g\in G}\sum_{\sigma\in K}(gu_\sigma)\cdot(g\sigma)+C_\bullet(K_0;\pi^*\mathfrak{L})\\
=&\frac{1}{\sharp G}g\sum_{g\in G}\sum_{\sigma\in K}u_\sigma\cdot\sigma)+C_\bullet(K_0;\pi^*\mathfrak{L})\\
=&\frac{1}{\sharp G}\sum_{g\in G}gc+C_\bullet(K_0;\pi^*\mathfrak{L})\\
=&\frac{1}{\sharp G}\sum_{g\in G}c+C_\bullet(K_0;\pi^*\mathfrak{L})\\
=&c+C_\bullet(K_0;\pi^*\mathfrak{L}).
\end{aligned}
$$

This shows $\mathrm{tf}\circ\pi = id|_{C_\bullet(K,K_0;\pi^*\mathfrak{L})^G}$. Hence the lemma is established.

Lemma 9.2.2 and Lemma 9.2.3 induce the following:

Theorem 9.2.4. *Let G be a finite group, G act on K regularly and K_0 be G-invariant. Then the chain map*

$$\pi : C_\bullet(K,K_0;\pi^*\mathfrak{L})^G \longrightarrow C_\bullet(K/G,K_0/G;\mathfrak{L})$$

is an isomorphism.

Corollary 9.2.5. *If G is a finite group, G acts on K regularly and K_0 is G-invariant, then the transfer induces an isomorphism*

$$\mathrm{tf} : H_\bullet(K/G,K_0/G;\mathfrak{L}) \longrightarrow H_\bullet(C_\bullet(K,K_0;\pi^*\mathfrak{L})^G).$$

Moreover, we have the following:

Theorem 9.2.6. *([7] Theorem 12.3.2) There exists a natural equivalence i_* induces an isomorphism*

$$i_* = (i_*)_{C_\bullet} : H_\bullet(C_\bullet^G) \longrightarrow H_\bullet(C_\bullet)^G.$$

Composing these isomorphisms in Corollary 9.2.5 and Theorem 9.2.6, we obtain the following:

Corollary 9.2.7. *Let G be a finite group, G act on K regularly and K_0 be G-invariant. Then there exists an isomorphism of $\mathbb{C}$-vector space:*

$$\mathrm{tf} : H_\bullet(K/G,K_0/G;\mathfrak{L}) \longrightarrow H_\bullet(K,K_0;\mathfrak{L})^G.$$

10 The relative chain complex of external products

10.1 The cross product

Let (K_i, K_{0i}) be the ordered simplicial pairs, $\mathfrak{L}_i$ the local systems on $K_i \quad (i = 1, 2, , ..., n)$. The external product of $(K_i, K_{0i}; \mathfrak{L}_i)$, $(j \in J(q))$ is defined in Section 7.1 and let $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma(q)$, and $j = (j_1, \dots, j_n) \in J(q)$. We denote by $\langle \sigma; j \rangle$ a simplex of K (see Section 7.1). We put $\mathfrak{L} = \mathfrak{L}_1 \boxtimes \mathfrak{L}_2 \boxtimes \cdots \boxtimes \mathfrak{L}_n$.

Definition 10.1.1. *The cross product*

$$C_\bullet(K_1, K_{01}; \mathfrak{L}_1) \otimes \cdots \otimes C_\bullet(K_n, K_{0n}; \mathfrak{L}_n) \longrightarrow C_\bullet(K, M; \mathfrak{L})$$

$$(u_1\sigma_1 + C_\bullet(K_{01}; \mathfrak{L}_1)) \otimes \cdots \otimes (u_n\sigma_n + C_\bullet(K_{0n}; \mathfrak{L}_n)) \mapsto u_1\sigma_1 \times \cdots \times u_n\sigma_n + C_\bullet(M; \mathfrak{L})$$

is a chain map defined by

$$u_1\sigma_1 \times u_2\sigma_2 \times \cdots \times u_n\sigma_n = \sum_{j \in J(q)} (sgn j) u_{\langle \sigma; j \rangle} \cdot \langle \sigma; j \rangle$$

where $u_i \in \mathfrak{L}_i$*,* $\sigma_i \in K_i \quad (i = 1, 2, ..., n)$ *and* $u_{\langle \sigma; j \rangle} \in \mathfrak{L}_{\langle \sigma; j \rangle}$ *is defined by*

$$u_{\langle \sigma; j \rangle}(a) = u_1(a_1) \otimes u_2(a_2) \otimes \cdots \otimes u_n(a_n)$$

for $a_1 \times a_2 \times \cdots \times a_n \in \langle \sigma; j \rangle$.

10.2 $\mathfrak{S}_n$-equivariance of the cross product

We shall give an $\mathfrak{S}_n$-equivariance of the cross product. To see this, we define a chain isomorphism for any $\tau \in \mathfrak{S}_n$ as follows.

(i) For a weight $(q_1, \dots, q_n)$, put

$$\Delta(x_1, \dots, x_n; q_1, \dots, q_n) = \prod_{i<j} (x_i - x_j)^{q_i q_j}.$$

(ii) For $\tau \in \mathfrak{S}_n$, the weight signature of τ with weight q is the number $sgn_q \tau \in \{\pm 1\}$ defined by

$$\Delta(x_{\tau(1)}, x_{\tau(2)}, \dots, x_{\tau(n)}; q_{\tau(1)}, q_{\tau(2)}, \dots, q_{\tau(n)}) = (sgn_q \tau) \Delta(x_1, \dots, x_n; q_1, \dots, q_n).$$

For any $\tau \in \mathfrak{S}_n$, define

$$\begin{aligned} \tau : & C_\bullet(K_1, K_{01}; \mathfrak{L}_1) \otimes \cdots \otimes C_\bullet(K_n, K_{0n}; \mathfrak{L}_n) \\ & \to C_\bullet(K_{\tau(1)}, K_{0\tau(1)}; \mathfrak{L}_{\tau(1)}) \otimes \cdots \otimes C_\bullet(K_{\tau(n)}, K_{0\tau(n)}; \mathfrak{L}_{\tau(n)}) \end{aligned}$$

by

$$\begin{aligned} & (\tau(u_1\sigma_1 + C_\bullet(K_{01}; \mathfrak{L}_1)) \otimes \cdots \otimes (u_n\sigma_n + C_\bullet(K_{0n}; \mathfrak{L}_n))) \\ & = sgn_q(\tau)((u_{\tau(1)}\sigma_{\tau(1)} + C_\bullet(K_{0\tau(1)}; \mathfrak{L}_{\tau(1)})) \otimes \cdots \otimes (u_{\tau(n)}\sigma_{\tau(n)} + C_\bullet(K_{0\tau(n)}; \mathfrak{L}_{\tau(n)})), \end{aligned}$$

where $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma(q)$ and $u_i \in \mathfrak{L}_{\sigma_i}$ $(i = 1, \ldots, n)$.

Note that

$$\tau : (K, M; \mathfrak{L}) \to (\tau K, \tau M; \tau\mathfrak{L})$$

is an isomorphism (see Section 7.2). By Definition 10.1.1 and the chain isomorphism defined above, we can easily show the following:

Lemma 10.2.1. *For any $\tau \in \mathfrak{S}_n$, there is a commutative diagram of chain complexes:*

$$\begin{array}{ccc} C_\bullet(K_1, K_{01}; \mathfrak{L}_1) \otimes \cdots \otimes C_\bullet(K_n, K_{0n}; \mathfrak{L}_n) & \longrightarrow & C_\bullet(K, M; \mathfrak{L}) \\ \tau \downarrow & & \downarrow \tau \\ C_\bullet(K_{\tau(1)}, K_{0\tau(1)}; \mathfrak{L}_{\tau(1)}) \otimes \cdots \otimes C_\bullet(K_{\tau(n)}, K_{0\tau(n)}; \mathfrak{L}_{\tau(n)}) & \longrightarrow & C_\bullet(\tau K, \tau M; \tau\mathfrak{L}) \end{array}$$

10.3 The special case

Let us consider the special case where

$$(K, K_0; \mathfrak{L}) = (K_1, K_{01}; \mathfrak{L}_1) = \cdots = (K_n, K_{0n}; \mathfrak{L}_n),$$

we put

$$\bigotimes^n C_\bullet(K, K_0; \mathfrak{L}) := C_\bullet(K, K_0; \mathfrak{L}) \otimes \cdots \otimes C_\bullet(K, K_0; \mathfrak{L}).$$

$$C_\bullet(K^n, M; \boxtimes^n \mathfrak{L}) := C_\bullet((K, K_0; \mathfrak{L}) \times \cdots \times (K, K_0; \mathfrak{L})).$$

Lemma 10.3.1. *The cross product*

$$\bigotimes^n C_\bullet(K, K_0; \mathfrak{L}) \to C_\bullet(K^n, M; \boxtimes^n \mathfrak{L})$$

is a chain homotopy equivalence.

We can use the method of acyclic models to prove this lemma as the classical case where the local system is trivial. So we omit the proof here.

The group $\mathfrak{S}_n$ acts on the chain complex $\otimes^n C_\bullet(K, K_0; \mathfrak{L}), C_\bullet(K^n, M; \boxtimes^n \mathfrak{L})$, and hence on the homology groups $H_\bullet(\otimes^n C_\bullet(K, K_0; \mathfrak{L}))$, $H_\bullet(C_\bullet(K^n, M; \boxtimes^n \mathfrak{L}))$. Using Lemma 10.2.1 and Lemma 10.3.1, we obtain the following:

Lemma 10.3.2. *The cross product*

$$H_\bullet(\otimes^n C_\bullet(K, K_0; \mathfrak{L})) \longrightarrow H_\bullet(C_\bullet(K^n, M; \boxtimes^n \mathfrak{L}))$$

is an $\mathfrak{S}_n$-equivariant isomorphism.

11 The Künneth formulae

In the paper, any local system is a local system of $\mathbb{C}$-vector space. Then the Künneth theorem simplifies considerably in this case:

Theorem 11.1. *(The Künneth formula) Suppose the chain complexes $C_\bullet$ and $D_\bullet$ ar those of $\mathbb{C}$-vector spaces, and the boundary operators are vector space homomorphisms. Then $H_p(C_\bullet)$ and $H_q(D_\bullet)$ are $\mathbb{C}$-vector spaces, and there is a natural isomorphism of $\mathbb{C}$-vector spaces*

$$\bigotimes_{p+q=m} H_p(C_\bullet) \otimes H_q(D_\bullet) \longrightarrow H_m(C_\bullet \otimes D_\bullet).$$

As an application of the Künneth formula, we obtain the following:

Theorem 11.2. *Let $C_\bullet$ be a chain complex of $\mathbb{C}$-vector space such that $H_q(C_\bullet) = 0$ if $q \neq r$. Then we have*
(1) *$H_q(\otimes^n C_\bullet) = 0$ if $q \neq nr$, and*
(2) *there exists a natural isomorphism of $\mathbb{C}$-vector space:*

$$\zeta : \bigotimes^n H_r(C_\bullet) \longrightarrow H_{nr}(\otimes^n C_\bullet)$$

where ζ is induced from the inclusion map

$$\bigotimes^n Z_r(C_\bullet) \longrightarrow Z_{nr}(\otimes^n C_\bullet).$$

Proof. We show this theorem by induction on n. If $n = 1$, there is nothing to show. Assume that the theorem holds for $n-1$ with $n \geq 2$. Put $D_\bullet = \otimes^{n-1} C_\bullet$, by virtue of theorem 11.1, there exists an isomorphism

$$\bigotimes_{p+q=m} H_p(C_\bullet) \otimes H_q(\otimes^{n-1} C_\bullet) \longrightarrow H_m(\otimes^m C_\bullet).$$

Since $H_p(C_\bullet) = 0 \quad (p \neq r)$, we have an isomorphism

$$\eta_{m,n} : H_r(C_\bullet) \otimes H_{m-r}(\otimes^{n-1} C_\bullet) \longrightarrow H_m(\otimes^n C_\bullet),$$

where $\eta_{m,n}$ is induced from the inclusion map

$$Z_r(C_\bullet) \otimes Z_{m-r}(\otimes^{n-1} C_\bullet) \longrightarrow Z_m(\otimes^n C_\bullet).$$

If $m \neq nr$, then by induction assumption, $H_{m-r}(\otimes^{n-1} C_\bullet) = 0$. So we have $H_m(\otimes^n C_\bullet) = 0$. This establishes (1) of Theorem.

To prove (2), we consider a commutative diagram of $\mathbb{C}$-vector space:

$$\begin{array}{ccc} \bigotimes^n H_r(C_\bullet) & \xrightarrow{1 \otimes \zeta_{n-1}} & H_r(C_\bullet) \otimes H_{(n-1)r}(\otimes^{n-1} C_\bullet) \\ \| & & \downarrow \eta \\ \bigotimes^n H_r(C_\bullet) & \xrightarrow[\zeta_n]{} & H_{nr}(\otimes^n C_\bullet) \end{array}$$

where ζ_n is induced from the inclusion map

$$\bigotimes^{n} Z_r(C_\bullet) \longrightarrow Z_{nr}(\otimes^n C_\bullet)$$

By induction assumption, ζ_{n-1} is $\mathbb{C}$-vector space isomorphism. Since $H_r(C_\bullet)$ is $\mathbb{C}$-vector space, $1 \otimes \zeta_{n-1}$ is also an isomorphism. Moreover, the isomorphism $\eta_{m,n}$ implies that $\eta_{nr,n}$ is an isomorphism. Hence ζ_n is an isomorphism.

12 Twisted relative homology associated with the configuration space

12.1 The configuration space of n-points

Let K be a order simplicial complex, $\mathfrak{L}$ a local system of $\mathbb{C}$-vector space on K. In the paper [7], the configuration space of n-points in K is defined by the quotient simplicial complex

$$K_n := \mathrm{Sd}^2 K^n / \mathfrak{S}_n.$$

Note that the natural action of $\mathfrak{S}_n$ on $\mathrm{Sd}^2 K^n$ is regular.

Let K_0 be a subcomplex of K. The external product of n-factors $(K, K_0; \mathfrak{L})$ is denoted by $(K^n, M; \boxtimes^n \mathfrak{L})$, see Section 7.2. By Lemma 6.3.1 and Lemma 7.2.1, we obtain the following:

Lemma 12.1.1. $\mathrm{Sd}^2 M$ *is a subcomplex of* M *which is invariant under the action of* $\mathfrak{S}_n$. *We denote* $\mathrm{Sd}^2 M/\mathfrak{S}_n$ *by* M_n.

12.2 Twisted homology of the configuration space

Let $\mathfrak{L}$, $\mathfrak{M}$ be the local systems of $\mathbb{C}$-vector spaces on K, K_n, respectively, $\mathrm{Sd}\mathfrak{L}$ the local system on $\mathrm{Sd}K$. We obtain a local system $\mathrm{Sd}^2 \boxtimes^n \mathfrak{L}$ on $\mathrm{Sd}^2 K^n$. We let $\pi : \mathrm{Sd}^2 K^n \to K_n$ denote a canonical projection.

Theorem 12.2.1. *Assume* $\pi^* \mathfrak{M} = \mathrm{Sd}^2 \boxtimes^n \mathfrak{L}$. *Then there exists a natural isomorphism of* $\mathbb{C}$*-vector spaces:*

$$H_\bullet(K_n, M_n; \mathfrak{M}) \longrightarrow H_\bullet(\otimes^n C_\bullet(K, K_0; \mathfrak{L}))^{\mathfrak{S}_n}.$$

Proof. We can apply Corollary 9.2.4 to obtain an isomorphism

$$\begin{aligned} H_\bullet(K_n, M_n; \mathfrak{M}) &= H_\bullet(\mathrm{Sd}^2 K^n, \mathrm{Sd}^2 M; \pi^* \mathfrak{M})^{\mathfrak{S}_n} \\ &= H_\bullet(\mathrm{Sd}^2 K^n, \mathrm{Sd}^2 M; \mathrm{Sd}^2 \boxtimes^n \mathfrak{L})^{\mathfrak{S}_n}. \end{aligned}$$

On the other hand, by Theorem 8.3.2 we have an $\mathfrak{S}_n$-equivariant isomorphism

$$H_\bullet(K^n, M; \boxtimes^n \mathfrak{L}) \longrightarrow H_\bullet(\mathrm{Sd}^2 K^n, \mathrm{Sd}^2 M; \mathrm{Sd}^2 \boxtimes^n \mathfrak{L})$$

which induces an isomorphism

$$H_\bullet(\mathrm{Sd}^2)^{-1} : H_\bullet(\mathrm{Sd}^2 K^n, \mathrm{Sd}^2 M; \mathrm{Sd}^2 \boxtimes^n \mathfrak{L})^{\mathfrak{S}_n} \longrightarrow H_\bullet(K^n, M; \boxtimes^n \mathfrak{L})^{\mathfrak{S}_n}.$$

Hence we have an isomorphism

$$H_\bullet(K_n, M_n; \mathfrak{M}) \longrightarrow H_\bullet(K^n, M; \boxtimes^n \mathfrak{L})^{\mathfrak{S}_n}.$$

Moreover, by Corollary 10.3.2 we have

$$H_\bullet(\otimes^n C_\bullet(K, K_0; \mathfrak{L}))^{\mathfrak{S}_n} \longrightarrow H_\bullet(K^n, M; \boxtimes^n \mathfrak{L})^{\mathfrak{S}_n}.$$

These isomorphism establish the desired isomorphism.

We have the following main theorem.

Theorem 12.2.2. *Assume* $\pi^*\mathfrak{M} = \mathrm{Sd}^2 \boxtimes^n \mathfrak{L}$ *and* $H_q(K, K_0; \mathfrak{L}) = 0$ $(q \neq r)$. *Then we have*

(1) $H_q(K_n, M_n; \mathfrak{M}) = 0 \quad (q \neq nr)$,

(2) $H_{nr}(K_n, M_n; \mathfrak{M}) \simeq \{\bigotimes^n H_r(K, K_0; \mathfrak{L})\}^{\mathfrak{S}_n}$ *is a canonical isomorphism of* $\mathbb{C}$-*vector space.*

When r *is odd,* (2) *implies that there is an isomorphism*

$$H_{nr}(K_n, M_n; \mathfrak{M}) \simeq \wedge^n H_r(K, K_0; \mathfrak{L}),$$

where $\wedge^n H_r(K, K_0; \mathfrak{L})$ *denotes* n^{th} *exterior power.*

Proof. If $q \neq nr$, Theorem 11.2 (1) implies that $H_q(\otimes^n C_\bullet(K, K_0; \mathfrak{L})) = 0$ and hence $H_q(\otimes^n C_\bullet(K, K_0; \mathfrak{L}))^{\mathfrak{S}_n} = 0$. Using Theorem 12.2.1 we obtain $H_q(K_n, M_n; \mathfrak{M}) = 0$. Next, we consider the case $q = nr$. By Theorem 11.2 (2), we have an isomorphism

$$\bigotimes^n H_r(K, K_0; \mathfrak{L})) \simeq H_{nr}(\otimes^n C_\bullet(K, K_0; \mathfrak{L})),$$

which induce an isomorphism

$$\{\bigotimes^n H_r(K, K_0; \mathfrak{L}))\}^{\mathfrak{S}_n} \simeq H_{nr}(\otimes^n C_\bullet(K, K_0; \mathfrak{L}))^{\mathfrak{S}_n}.$$

By Theorem 12.2.1, we obtain an isomorphism

$$H_{nr}(K_n, M_n; \mathfrak{M}) \simeq \{\bigotimes^n H_r(K, K_0; \mathfrak{L})\}^{\mathfrak{S}_n}.$$

We have obtained the desired isomorphism.

13 Local systems on a bouquet

13.1 Bouquets

We construct a bouquet B_m as follows. B_m is a 1-dimensional ordered simplicial complex whose vertices are a_{k1}, a_{k2},...,a_{k,l_k-1},c_{k1},c_{k2} $(k = 0, 1, ..., m)$, c and whose ordered 1-simplexes are (c, a_{k1}),...,(c, a_{k,l_k-1}), and (c, c_{k1}),(c_{k1}, c_{k2}), (c_{k2}, c), $(k = 0, 1, ..., m)$.

Given m-points $x_1, x_2, ..., x_m$ in $\mathbb{C}$, we take $\triangle_k = \triangle(c, c_{k1}, c_{k2})$ is the triangle in $\mathbb{C}$ with vertices c, c_{k1}, c_{k2} so that x_k is in the interior of $\triangle_k$ and $x_l (k \neq l)$ is in the outside of $\triangle_k$. The orientation of $\triangle_k$ is given by $\overrightarrow{cc_{k1}c_{k2}}$. We may assume that this orientation coincides with the anti-clockwise orientation of $\mathbb{C}$.

We take $K = B_m$ with vertices $a_{k1}, ..., a_{k,l_k-1}, c_{k1}, c_{k2}$ $(k = 0, 1, \ldots, m)$ and c, K_0 is a 0-dimensional simplicial complex with vertices $a_{k1}, ..., a_{k,l_k-1} (k = 0, 1, \ldots, m)$. The topological realization $|K|$ of K is the union of m-triangles $\triangle_k$ and n-edges $[c, a_{ik}]$ $(i = 1, 2, ..., l_k - 1)$

$$|K| = (\bigcup_{k=1}^{m} \triangle_k) \cup (\bigcup_{k=0}^{m} \bigcup_{i=1}^{l_k-1} [c, a_{ki}]).$$

13.2 Local systems on a bouquet

Let $e_1, e_2, ..., e_m \in \mathbb{C}^\times$, $K = B_m$ the m-bouquet. Put $e = (e_1, e_2, ..., e_m)$. We define the local system $\mathfrak{L} = \mathfrak{L}_e$ of $\mathbb{C}$-vector spaces on K by

$$\begin{aligned} &\mathfrak{L}_{c_{k1}} = \mathfrak{L}_{c_{k2}} = \mathfrak{L}_c = \mathbb{C} \quad (k = 1, 2, ..., m) \\ &\mathfrak{L}_{a_{k1}} = \mathfrak{L}_{a_{k2}} = \cdots = \mathfrak{L}_{a_{k,l_k-1}} = \mathbb{C} \quad (k = 0, 1, ..., m) \end{aligned}$$

and

$$\begin{aligned} \xi_{c_{k1},c} : \mathfrak{L}_c \to \mathfrak{L}_{c_{k1}},\ \xi_{c_{k1},c} &= \mathrm{id}_{\mathbb{C}} \\ \xi_{c_{k2},c_{k1}} : \mathfrak{L}_{c_{k1}} \to \mathfrak{L}_{c_{k2}},\ \xi_{c_{k2},c} &= \mathrm{id}_{\mathbb{C}} \\ \xi_{c,c_{k2}} : \mathfrak{L}_{c_{k2}} \to \mathfrak{L}_c,\ \xi_{c,c_{k2}} &= e_k \cdot \mathrm{id}_{\mathbb{C}} \\ \xi_{a_{ki},c} : \mathfrak{L}_c \to \mathfrak{L}_{a_{ki}},\ \xi_{a_{ki},c} &= \mathrm{id}_{\mathbb{C}}. \end{aligned}$$

The chain groups are

$$\begin{aligned} C_0(K, \mathfrak{L}) &= \bigoplus_{k=1}^{m} (\mathbb{C}c_{k1} \oplus \mathbb{C}c_{k2}) \oplus \mathbb{C}c \bigoplus_{k=0}^{m} \bigoplus_{i=1}^{l_k-1} \mathbb{C}a_{ki}, \\ C_1(K, \mathfrak{L}) &= \bigoplus_{k=1}^{m} \{\mathbb{C}(c, c_{k1}) \oplus \mathbb{C}(c_{k1}, c_{k2}) \oplus \mathbb{C}(c_{k2}, c)\} \bigoplus_{k=0}^{m} \bigoplus_{i=1}^{l_k-1} \mathbb{C}(c, a_{ki}). \end{aligned}$$

and the boundary map $\partial : C_1(K, \mathfrak{L}) \to C_0(K, \mathfrak{L})$ is defined by

$$\begin{aligned} &\partial((\sum_{k=1}^{m} u_k(c, c_{k1}) + v_k(c_{k1}, c_{k2}) + w_k(c_{k2}, c)) + \sum_{k=0}^{m} \sum_{i=1}^{l_k-1} s_{ki}(c, a_{ki})) \\ &= \sum_{k=1}^{m} \{(u_k - v_k)c_{k1} + (v_k - w_k)c_{k2}\} + \sum_{k=1}^{m} (e_k w_k - u_k)c \\ &+ \sum_{k=0}^{m} \sum_{i=1}^{l_k-1} s_{ki}(a_{ki} - c), \end{aligned}$$

where $u_k, v_k, w_k, s_{ki} \in \mathbb{C}$. Let

$$\Phi : \mathbb{C}^m \to \mathbb{C} \quad (u_1, ..., u_m) \mapsto \sum_{k=1}^{m} (e_k - 1) u_k.$$

The boundary group $B_0(K; \mathfrak{L})$ consists of elements of the form:

$$\sum_{k=1}^{m} (v_k c_{k1} + w_k c_{k2}) + \sum_{k=1}^{m} \{(e_k - 1)u_k - e_k(v_k + w_k)\}c$$
$$+ \sum_{k=0}^{m} \sum_{i=1}^{n_k - 1} s_{ki}(a_{ki} - c),$$

where $u_k, v_k, w_k, s_{ki} \in \mathbb{C}$. The cycle group $Z_1(K; \mathfrak{L})$ consists of elements of the form:

$$\sum_{k=1}^{m} u_k \sigma_k,$$

where $\sigma_k = (c, c_{k1}) + (c_{k1}, c_{k2}) + (c_{k2}, c)$, and $(u_1, u_2, ..., u_m) \in \ker\Phi$.

13.3 Homology of $(K, K_0; \mathfrak{L})$

Clearly, we have the following:

Lemma 13.3.1. (1) $H_0(K; \mathfrak{L}) = 0$ *if and only if* $\Phi : \mathbb{C}^m \to \mathbb{C}$ *is surjective.*
(2) $H_1(K; \mathfrak{L}) = Z_1(K; \mathfrak{L}) \simeq \ker\Phi$.

By the lemma, we can easily obtain the following:

Proposition 13.3.2. *If* $\Phi : \mathbb{C}^m \to \mathbb{C}$ *is surjective, then*

(1) $H_q(K; \mathfrak{L}) = 0$ *if* $q \neq 1$,

(2) $H_1(K; \mathfrak{L}) \simeq V_e$,

where $V_e = \{(u_1, u_2, ..., u_m) \in \mathbb{C}^m; \sum_{i=1}^{m} (1 - e_i) u_i\} = 0$.

Note that

$$C_0(K_0; \mathfrak{L}) = \bigoplus_{k=0}^{m} \bigoplus_{i=1}^{l_k - 1} \mathbb{C} a_{ki},$$

and we have $\partial c = 0$ for $c \in C_0(K_0; \mathfrak{L})$. Hence we have

$$H_0(K_0; \mathfrak{L}) \simeq \mathbb{C}^{n-m-1},$$

where $n = \sum_{k=0}^{m} l_k$.

From the Lemma 13.3.1 and the homology long exact sequence for the pair (K, K_0), we have

$$H_p(K, K_0; \mathfrak{L}) \simeq \begin{cases} 0 & p \neq 1 \\ \mathbb{C}^{n-2} & p = 1 \end{cases}$$

By Theorem 12.2.2, we obtain the following:

Theorem 13.3.3. *Let* $\pi : \mathrm{Sd}^2 K^n \to K_n$ *be the canonical projection,* $\mathfrak{M}$ *a local system on* K*. Assume that* $\pi^*\mathfrak{M} = \mathrm{Sd}^2 \boxtimes^n \mathfrak{L}$*, where* $\mathfrak{L} = \mathfrak{L}_e$*, and that* $\Phi : \mathbb{C}^m \to \mathbb{C}$ *is surjective. Then*

$$H_q(K_n, M_n; \mathfrak{M}) \simeq \begin{cases} 0 & q \neq n \\ \wedge^n H_1(K, K_0; \mathfrak{L}) & q = n \end{cases}$$

14 Relative Singular homology with local systems

14.1 The singular local systems

Let X be a topological space., $\mathfrak{L}$ a local system of $\mathbb{C}$-vector space on X. Let Δ^q be the standard q-simplex with vertices $v_0, v_1, ..., v_q$. For any singular q-simplex $\sigma : \Delta^q \to X$, let γ_σ be the curve in X defined by

$$\gamma_\sigma(t) := \sigma((1-t)v_0 + tv_1) \qquad (0 \leq t \leq 1).$$

There is an isomorphism

$$\xi(\gamma_\sigma) : \mathfrak{L}_{\sigma(v_0)} \to \mathfrak{L}_{\sigma(v_1)}.$$

We define the singular chain complex with coefficients in the local system $\mathfrak{L}$ as follows:

Definition 14.1.1. *A* q*-chain* $c \in S_q(X; \mathfrak{L})$ *is a formal sum:*

$$c = \sum_\sigma u_\sigma \cdot \sigma$$

where the sum is taken over all singular q*-simplex* σ *in* X*,* $u_\sigma \in \mathfrak{L}_{\sigma(v_0)}$ *and* $u_\sigma = 0$ *except for a finite number of* σ*'s. The boundary operator*

$$\partial : S_q(X; \mathfrak{L}) \longrightarrow S_{q-1}(X; \mathfrak{L})$$

is defined by

$$\partial c := \sum_\sigma \{\xi(\gamma_\sigma)(u_\sigma) \cdot \partial_0 \sigma + \sum_{i=1}^{q} (-1)^i u_\sigma \cdot \partial_i \sigma\},$$

where $\partial_i \sigma$ *is the ordered* $(q-1)$*-simplex defined by* $\partial_i \sigma = \sigma \circ \Delta_i^q$ *restricted to* $\{0, 1, \ldots, q-1\}$ *for* $i = 0, 1, \ldots, q$ *(see [7] Remark 9.1.4, Definition 9.2.6). For* $c' = \sum_\sigma u'_\sigma \cdot \sigma$*, we define*

$$c + c' := \sum_\sigma (u_\sigma + u'_\sigma)\sigma.$$

If X is a topological space and A is a subspace of X, the local system $\mathfrak{L}$ on X restricted to A induces a local system $\mathfrak{L}|_A$ on A, we also denote $\mathfrak{L}|_A$ by $\mathfrak{L}$. There is a natural inclusion map $S_\bullet(A; \mathfrak{L}) \to S_\bullet(A; \mathfrak{L})$. The quotient chain complex $S_\bullet(X, A; \mathfrak{L}) := S_\bullet(X; \mathfrak{L})/S_\bullet(A; \mathfrak{L})$ with boundary operator $\partial : S_p(X, A; \mathfrak{L}) \to S_{p-1}(X, A; \mathfrak{L})$ is called the singular chain complex of the pair (X, A) and its homology group $H_\bullet(X, A; \mathfrak{L})$ is called the singular homology of the pair (X, A) with coefficients in the local system $\mathfrak{L}$.

14.2 Homology invariance of the singular homology functor

Let (X, A), (Y, B) be the topological space pairs, $\mathfrak{I}$ a local system on Y. A continuous map $f : X \to Y$ induces a local system $f^*\mathfrak{I}$ on X, which is called the pull-back of $\mathfrak{I}$ by f.

Using the five-lemma, we obtain the following:

Lemma 14.2.1. *Let $f : (X, A) \to (Y, B)$ be a continuous map, $\mathfrak{I}$ a local system on Y. If $f : X \to Y$ and $f|_A : A \to B$ are homotopy equivalences, then*

$$f_* : H_\bullet(X, A; f^*\mathfrak{I}) \longrightarrow H_\bullet(Y, B; \mathfrak{I})$$

is an isomorphism.

15 Comparison theory

15.1 The comparison theorem

Let (K, K_0) be a simplicial pair, $(|K|, |K_0|)$ a topological pair of (K, K_0). Giving a singular local system $\mathfrak{L} = (\mathfrak{L}, \xi)$ on $|K|$, there exists an induced simplicial local system $\theta_K\mathfrak{L} = (\theta_K\mathfrak{L}, \theta_K\xi)$ on K defined by

(1) for any vertex $a \in V_K, (\theta_K\mathfrak{L})_a := \mathfrak{L}_{\langle a\rangle}$, and

(2) for any $\{a, b\} \in K^{(1)}, (\theta_K\xi)_{ba} := \xi(\gamma_{ba}) : \mathfrak{L}_{\langle a\rangle} \to \mathfrak{L}_{\langle b\rangle}$, here $\gamma_{ba}(t) := [0, 1] \to |K|$ is the curve in $|K|$ defined by $\gamma_{ba}(t) := t\langle a\rangle + (1 - t)\langle b\rangle \quad (0 \le t \le 1)$.

Definition 15.1.1. *The chain map $\theta_K : C_\bullet(K; \theta_K\mathfrak{L}) \to S_\bullet(|K|; \mathfrak{L})$ is defined by*

$$\theta_K(\sum_\sigma u_\sigma \cdot \sigma) := \sum_\sigma \tilde{u}_\sigma \cdot \tilde{\sigma},$$

where for any $\sigma = \{a_0, a_1, ...a_q\} \in K$, we define

$$\tilde{\sigma} : \Delta^q \to |K|, \quad \sum_{i=0}^{q} t_i v_i \mapsto \sum_{i=0}^{q} t_i\langle a_i\rangle \qquad (t_i \ge 0, \quad \sum_i t_i = 1)$$

and for any $u \in (\theta_K\mathfrak{L})_\sigma$, we define

$$\tilde{u} = u(a_0) \in \mathfrak{L}_{\tilde{\sigma}(v_0)} = \mathfrak{L}_{\langle a_0\rangle}.$$

For any q-simplex σ of the subcomplex K_0 of K, we can define the singular q-simplex $\tilde{\sigma} : \Delta^q \to |K_0|$, then θ_K maps $C_\bullet(K_0; \theta_K\mathfrak{L})$ to $S_\bullet(|K_0|; \mathfrak{L})$. Hence the chain map θ_K defined in Definition 15.1.1 induces a chain map

$$\theta_K : C_\bullet(K, K_0; \theta_K\mathfrak{L}) \longrightarrow S_\bullet(|K|, |K_0|; \mathfrak{L})$$

and a homomorphism of $\mathbb{C}$-vector space

$$\theta_K : H_\bullet(K, K_0; \theta_K\mathfrak{L}) \longrightarrow H_\bullet(|K|, |K_0|; \mathfrak{L}).$$

Theorem 15.1.2. *This homomorphism is an isomorphism of $\mathbb{C}$-vector space.*

Proof. Refer to the proof of the classical case where the local system is trivial. Since we can prove this theorem in an almost similar manner, we omit it.

15.2 Homology of the polyhedron

Let (X, A) be a polyhedral pair with underlying simplicial structure $((K, K_0), f)$, $\mathfrak{L}$ a local system on X. Then we have an isomorphism

$$f_* : H_\bullet(|K|, |K_0|; f^*\mathfrak{L}) \longrightarrow H_\bullet(X, A; \mathfrak{L}).$$

By Theorem 15.1.2, we have an isomorphism

$$\theta_K : H_\bullet(K, K_0; \theta_K f^*\mathfrak{L}) \longrightarrow H_\bullet(|K|, |K_0|; f^*\mathfrak{L}).$$

Composing these isomorphisms, we obtain the following:

Proposition 15.2.1. *There exists an isomorphism:*

$$H_\bullet(K, K_0; \theta_K f^*\mathfrak{L}) \simeq H_\bullet(X, A; \mathfrak{L}).$$

Remark 15.2.2. *Proposition 15.2.1 shows that if (X, A) is a polyhedral pair (X, A) with underlying simplicial structure $((K, K_0), f)$, then the singular homology of (X, A) is computed as the simplicial homology of (K, K_0).*

16 External product for singular local systems

Let $(X_i, A_i) \quad (i = 1, 2, ..., n)$ be the topological spaces with the local systems $\mathfrak{L}_i$ on X_i. Then the external product of $(X_1, A_1; \mathfrak{L}_1)$, $(X_2, A_2; \mathfrak{L}_2), \ldots, (X_n, A_n; \mathfrak{L}_n)$ is defined as follows:

- $X := X_1 \times X_2 \times \cdots \times X_n$,
- $A^{[i]} := X_1 \times \cdots \times A_i \times \cdots \times X_n$, $A := A^{[1]} \cup A^{[2]} \cup \cdots \cup A^{[n]}$,
- $\mathfrak{L} = (\mathfrak{L}, \xi) := \mathfrak{L}_1 \boxtimes \mathfrak{L}_2 \boxtimes \cdots \boxtimes \mathfrak{L}_n = (\mathfrak{L}_1, \xi_1) \boxtimes (\mathfrak{L}_2, \xi_2) \boxtimes \cdots \boxtimes (\mathfrak{L}_n, \xi_n)$,
- for each point $p = (p_1, p_2, ..., p_n) \in X$
$$\mathfrak{L}_p := \mathfrak{L}_{1,p_1} \otimes \mathfrak{L}_{2,p_2} \otimes \cdots \otimes \mathfrak{L}_{n,p_n},$$
- for each curve $\gamma : [0, 1] \to X$ with $\gamma(0) = p = (p_1, p_2, ..., p_n)$, $\gamma(1) = q = (q_1, q_2, ..., q_n)$, we define
$$\xi(\gamma) = \xi_1(\gamma_1) \otimes \xi_2(\gamma_2) \otimes \cdots \otimes \xi_n(\gamma_n) : \mathfrak{L}_p \to \mathfrak{L}_q.$$

The triple $(X, A; \mathfrak{L})$ is denoted by $(X_1, A_1; \mathfrak{L}_1) \boxtimes \cdots \boxtimes (X_n, A_n; \mathfrak{L}_n)$.

In particular, when

$$(X_1, A_1; \mathfrak{L}_1) = \cdots = (X_n, A_n; \mathfrak{L}_n),$$

we denote $X_i, A_i, \mathfrak{L}_i$ as $X, A, \mathfrak{L}$, respectively, and write

$$(X^n, N; \boxtimes^n \mathfrak{L}) = (X, A; \mathfrak{L}) \boxtimes \cdots \boxtimes (X, A; \mathfrak{L}),$$

where $N := \sum_{i=1}^n N^{[i]}$ for $N^{[i]} = X \times \cdots \times \overset{i}{\breve{A}} \times \cdots \times X \quad (i = 1, 2, ..., n)$.

The group $\mathfrak{S}_n$ acts on the product space X^n by permutation of n points which induces an action on N.

Definition 16.1. *The quotient space*

$$X_n := X^n/\mathfrak{S}_n$$

is called the (topological) configuration space of n points in X. $N_n := N/\mathfrak{S}_n$ is a subspace of X_n.

17 Twisted singular homology of the configuration space

17.1 Naturality of $|\cdot|$ with respect to a group action

Let (K, K_0) be a simplicial pair, G a finite group. G acts on K regularly. Let $\pi : K \to K/G$ be the canonical simplicial projection, $p : |K| \to |K|/G$ the canonical topological projection, respectively. Then we have the following:

Lemma 17.1.1. *If G acts on K regularly, then there exists a homeomorphism $\chi : |K/G| \to |K|/G$ such that the following diagram is commutative*

$$\begin{array}{ccc} |K| & = & |K| \\ {\scriptstyle |\pi|}\downarrow & & \downarrow{\scriptstyle p} \\ |K/G| & \xrightarrow[\chi]{} & |K|/G \end{array}$$

Proof. See [3] p.117.

Similarly, in the case of simplicial pair, we have the following:

Lemma 17.1.2. *Let K_0 be a G-invariant subcomplex of K and G act on K regularly. Then*

$$\chi|_{|K_0/G|} : |K_0/G| \to |K_0|/G$$

is a homeomorphism.

Proof. In fact, if K_0 is G-invariant, a regular action of G on K implies that on K_0. It follows that $\chi|_{|K_0/G|}$ is a homeomorphism.

Lemma 17.1.3. (1) *For any simplicial complex K, we have*

$$|K| \cong |\mathrm{Sd}K|.$$

If a group G acts on K, then this homeomorphism is G-equivariant.

(2) *For any ordered simplicial complex K,*

$$|K^n| \cong |K|^n$$

is $\mathfrak{S}_n$-equivariant.

Proof. See [7] Lemma 19.1.1 and Lemma 19.2.2.

By Lemma 17.1.3, we can easily obtain the homeomorphisms φ and ψ as follows.

$$\varphi : (|K|, |K_0|) \cong (|\mathrm{Sd}K|, |\mathrm{Sd}K_0|),$$
$$\psi : (|K^n|, |M|) \cong (|K|^n, ||M||)$$

where $||M|| := \cup_{i=1}^n ||K^{[i]}|| = \cup_{i=1}^n |K| \times \cdots \times |K_{0i}| \times \ldots |K|$ and ψ is $\mathfrak{S}_n$-equivariant homeomorphism. Note that the homeomorphism φ induces an $\mathfrak{S}_n$-equivarian homeomorphism

$$\varphi : (|\mathrm{Sd}^2 K^n|, |\mathrm{Sd}^2 M|) \cong (|K^n|, |M|).$$

17.2 Twisted relative singular homology of the configuration space

Let X, Y be the topological spaces. For any continuous map $f : X \to Y$, we define $f^n : X^n \to Y^n$ by

$$f^n(p_1, p_2, ..., p_n) = (f(p_1), f(p_2), ..., f(p_n)).$$

This map is $\mathfrak{S}_n$-equivariant. Hence it follows that we have a continuous map

$$f_n := f^n/\mathfrak{S}_n : X_n := X^n/\mathfrak{S}_n \to Y^n/\mathfrak{S}_n =: Y_n.$$

Let (X, A) be a polyhedral pair with underlying structure $((K, K_0), f)$, $(|K|, |K_0|)$ the topological space pair of (K, K_0). Then $f : |K| \to X$ and $f|_{|K_0|} : |K_0| \to A$ are homotopy equivalences. So the following lemma holds:

Lemma 17.2.1. *Let $f : (|K|, |K_0|) \to (X, A)$ be as above. The homotopy equivalences $f : |K| \to X$ and $f|_{|K_0|} : |K_0| \to A$ induce the $\mathfrak{S}_n$-equivariant homotopy equivalences*

$$f^n : |K|^n \to X^n$$

and

$$f|_{\cup_{i=1}^n ||K^{[i]}||} : \cup_{i=1}^n ||K^{[i]}|| \to \cup_{i=1}^n A^{[i]},$$

where the symbol $||\cdot||$ is same as in Section 17.1.

Here we use the following diagram of continuous maps:

$$\begin{array}{ccccccccc}
|\mathrm{Sd}^2 K^n| & = & |\mathrm{Sd}^2 K^n| & \xrightarrow{\varphi} & |K^n| & \xrightarrow{\psi} & |K|^n & \xrightarrow{f^n} & X^n \\
\downarrow{\scriptstyle |\pi_{\mathrm{Sd}^2 K^n}|} & & \downarrow{\scriptstyle \pi_{\mathrm{Sd}^2 K^n}} & & \downarrow{\scriptstyle \pi_{|K^n|}} & & \downarrow{\scriptstyle \pi_{|K|^n}} & & \downarrow{\scriptstyle \pi_{X^n}} \\
|K_n| & \xrightarrow[\chi]{} & |\mathrm{Sd}^2 K^n|/\mathfrak{S}_n & \xrightarrow[\varphi/\mathfrak{S}_n]{} & |K^n|/\mathfrak{S}_n & \xrightarrow[\psi/\mathfrak{S}_n]{} & |K|_n & \xrightarrow[f_n]{} & X_n
\end{array}$$

where χ is a homeomorphism obtained by Lemma 17.1.1. Hence, a continuous map $g : |K_n| \to X_n$ defined by

$$g := f_n \circ (\psi/\mathfrak{S}_n) \circ (\varphi/\mathfrak{S}_n) \circ \chi$$

is a homotopy equivalence between $|K_n|$ and X_n and $g|_{|M/\mathfrak{S}_n|} : |M/\mathfrak{S}_n| \to N_n$ is also a homotopy equivalence between $|M/\mathfrak{S}_n|$ and N_n. Hence we obtain the following:

Lemma 17.2.2. *(X_n, N_n) is a polyhedral pair with underlying simplicial structure $((K_n, M_n), g)$.*

Let $\mathfrak{L}_X$, $\mathfrak{L}_{X_n}$ be the local systems on X, X_n, respectively. We define the simplicial local systems $\mathfrak{L}_K$ and $\mathfrak{L}_{K_n}$ as follows:

$$\mathfrak{L}_K := \theta_K \circ f^* \mathfrak{L}_X, \quad \mathfrak{L}_{K_n} := \theta_{K_n} \circ g^* \mathfrak{L}_{X_n}.$$

Put $\pi_{X^n} : X^n \to X_n$. If $\pi_{X^n}^* \mathfrak{L}_{X_n} = \boxtimes^n \mathfrak{L}_X$, then

$$\pi_{\mathrm{Sd}^2 K^n}^* \mathfrak{L}_{K_n} = \mathrm{Sd}^2 \boxtimes^n \mathfrak{L}_K$$

where $\pi_{\mathrm{Sd}^2 K^n}^* \mathfrak{L}_{K_n} : \mathrm{Sd}^2 K^n \to K_n$ is the canonical projection (see [7] Lemma 21.3.2). We have the following main theorem.

Theorem 17.2.3. *Let (X, A) be a polyhedral pair with underlying simplicial structure $((K, K_0), f)$. Let $\mathfrak{L}_X$ and $\mathfrak{L}_{X_n}$ be singular local systems of $\mathbb{C}$-vector spaces on X, X_n, respectively. Assume that $\pi_{X^n}^* \mathfrak{L}_{X_n} = \boxtimes^n \mathfrak{L}_X$, where $\pi_{X^n} : X^n \to X_n$ is the canonical projection. Then*

(1) *there exists an isomorphism*

$$H_\bullet(X_n, N_n; \mathfrak{L}_{X_n}) \simeq H_\bullet(\otimes^n C_\bullet(K, K_0; \mathfrak{L}_K))^{\mathfrak{S}_n},$$

where $\mathfrak{L}_K$ is the simplicial local system of $\mathbb{C}$-vector space on K defined by $\mathfrak{L}_K = \theta_K \circ f^* \mathfrak{L}_K$.

(2) *Assume further that $H_q(K, K_0; \mathfrak{L}_K) = 0$ if $q \neq r$, then*

$$H_q(X_n, N_n; \mathfrak{L}_{X_n}) = 0 \quad q \neq nr$$

and there exists an isomorphism

$$H_{nr}(X_n, N_n; \mathfrak{L}_{X_n}) \simeq \begin{cases} \wedge^n H_r(K, K_0; \mathfrak{L}_K) & (r : odd) \\ \odot^n H_r(K, K_0; \mathfrak{L}_K) & (r : even) \end{cases}$$

where the symbol $\odot$ means the symmetric power and $\wedge$ means the exterior power.

Proof. By Lemma 17.2.2 and Proposition 15.2.2, we have

$$H_\bullet(X_n, N_n; \mathfrak{L}_{X_n}) \simeq H_\bullet(K_n, M_n; \mathfrak{L}_{K_n}),$$

where $\mathfrak{L}_{K_n} = \theta_{K_n} \circ g^* \mathfrak{L}_{X_n}$. By assumption we have $\pi_{\mathrm{Sd}^2 K^n}^* \mathfrak{L}_{K_n} = \mathrm{Sd}^2 \boxtimes^n \mathfrak{L}_K$. using Theorem 12.2.1, we obtain

$$H_\bullet(K_n, M_n; \mathfrak{L}_{K_n}) \simeq H_\bullet(\otimes^n C_\bullet(K, K_0; \mathfrak{L}_K))^{\mathfrak{S}_n}.$$

Combining the isomorphisms above, we obtain the first assertion (1).

By virtue of Theorem 12.2.2, we obtain the second assertion (2). This completes the proof of the theorem.

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References

[1] P. Appell; J. Kampé de Feriet: *Fonctions Hypergéometriques et Hyperspheriques-Polynomes d'Hermite*. Gauther-Villars. Paris, 1926.

[2] K. Aomoto: *On the structure of integral of integrals of power prower of linear function*. Sci. Papers Coll. Gen. Ed. Univ. Tokyo, **27** (1977), 49–61.

[3] G. E. Bredon: *Introduction to compact transformation group*. Academic Press, New York and London, 1972.

[4] I.M. Gel´fand: *General theory of hypergeometric functions*. Soviet. Math. Dokl., **33** (1986), 573–577.

[5] I.M. Gel´fand, V.S. Retahk and V.V. Serganova: *Generalized Airy funcitons, Schublrt cells, and Jordan groups*. Dokl. Akad. Nauk. SSSR , **298** (1988) 17–21. English transl.: Soviet Math. Dokl. **37** (1988) 8–12.

[6] Y. Haraoka: *Confluence of cycles for hypergeometric functions on* $Z_{2,n+1}$. Trans. Amer. Math. Soc., **349** no.2 (1997), 675–712.

[7] K. Iwasaki; M. Kita: *Twisted homology of the configuration spaces of* n*-points with application to the hypergeometric functions*. Kumamoto J. Math., **12** (1998), 9–72.

[8] K. Iwasaki; M. Kita: *Exterior power structure on the twisted de Rham cohomology of the complements of real Veronese arrangements*. J. Math. Pures Appl., **75** (1996), 69–84.

[9] H. Kimura: *On the homology group associated with the general hypergeometric integral on* $\mathbf{P}^r$. Preprint, (1998).

[10] H. Kimura: *On the homology group associated with the general Airy integral*. Kumamoto J. Math., **10** (1997), 11–29.

[11] H. Kimura: *On rational de Rham cohomology associated with the generalized confluent hypergeometric functions* I*,* $\mathbf{P}^1$ *case*. Royal Soc. Edinburgh Sec.A, **127** (1997), 145–155.

[12] H. Kimura: *On Wronskian determinant formulas of the general hypergeometric functions*. Tokyo J. Math., **34** (2011), 507–524.

[13] M. Kita: *On hypergeometric functions in several variables 1. New integral representations of Euler type*. Japan. J. Math., **18**(1) (1992), 25–74.

[14] H. Kimura; Y. Haraoka; K. Takano: *The generalized confluent hypergeometric functions*. Proc. Japan Acad., **68**, Ser. A (1992), 290–295.

[15] K. Okamoto; H. Kimura: *On particular solutions of Garnier systems and the hypergeometric functions of several variables*. Quarterly J. Math., **37** (1986), 61–80.

[16] F. Pham: *Vanishing homologies and the n-variable saddle point method.* Proc. Symposia in Pure and Applied Math., **40** (1983), 319–334.

[17] E. H. Spanier: *Algebraic topology.* Springer-Verlag, New York, Heidelberg and Berlin, (1966).

[18] T. Terasoma: *Exponential Kummer coverings and determinants of hypergeometric functions.* Tokyo J. Math., **6** (1993), 497–508.

[19] T. Terasoma: *Confluent hypergeometric functions and wild ramification.* Journal of Algebra, **85** (1996), 1–18.

[20] J.-L. Verdier: *Stratifications de whitney et théoréme de Bertini-Sard.* Invent. Math., **36** (1976), 295–312.

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