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G-PRIMITIVE EXTENSIONS FOR LINEAR ORDINARY DIFFERENTIAL EQUATIONS*

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The notion of primitive extensions is of theoretical importance in the algebraic Galois theory. In differential algebra we have an analogous notion: G-primitive extensions, where G stands for an algebraic group. Every Picard-Vessiot extention can be considered, after some modification, as a G-primitive extension for G, the associated Picard-Vessiot group. This interpretation furnishes a transformation of a linear ordinary differential equation into, in some sense, a canonical form.

In this paper we explain the mechanism of the interpretaion, and give instructive examples.

We believe that a similar result holds for linear partial differential equations of the first order, and then, by using it, we can reconstruct the work of Drach [1].

§1. G-extensions and G-primitive extensions

In this section we introduce several notions and results of differential algebra, which entirely owe to Kolchin [2]; refer to it for details.

We denote the field of constants of a differential field K by C_K. Throughout this section we fix a differential field K of characteristic 0 and its field of constants $C=C_K$.

For a strongly normal extension L of K, it is known that:

(i) L/K is a finitely generated extension,

(ii) $C_L=C_K=C$,

(iii) the set of all strong isomorphisms of L over K, which is denoted by $Gal(L/K)$, becomes an algebraic group over C.

When we are concerned with algebraic groups, we have the following definition. Let G be an algebraic group over C.

Definition 1. We say that L/K is a *G-extension* if it is a strongly normal extension and if there is an injective homomorphism

$$Gal(L/K) \to G_{C'}$$

of algebraic groups over C, where C' is a field of constants of a differential extension of L.

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For example, a Picard-Vessiot extension defined by an n-th order linear ordinary differential equation is a $GL(n)$-extension. This is the subject of the following sections.

Next we introduce G-primitive extensions. We assume, for simplicity, that K is an ordinary differential field; we use δ for the unique derivation.

For any connected algebraic group G over C and for a differential field extension L of K, we can define canonically the logarithmic derivation $\ell\delta$ of δ:

$$\ell\delta: G_L \to Lie(G).$$

We say that $\alpha \in G_L$ is a *G-primitive over K* if

$$\ell\delta(\alpha) \in Lie(G_K) = \mathfrak{g} \otimes_C K,$$

where $\mathfrak{g}$ denotes the Lie algebra of G.

Definition 2. Let G be a connected algebraic group over C. We say that L/K is a *G-primitive extension* if there exists a G-primitive α over K such that $L=K\langle\alpha\rangle$.

We describe the relation between the two definitions.

Theorem 1 (Kolchin [2], p.419). *K and C being the fixed ones. Let G be a connected algebraic group over C, and α be a G-primitive over K. Then $L=K\langle\alpha\rangle$, which is by definition a G-primitive extension of K, is a G-extension of K.*

Remark 1. In the situation of Theorem 1, we can see that $Gal(L/K)$ acts on the group $G_{LC'}$ for some C'. Then the injective homomorphism $c: Gal(L/K) \to G_{C'}$ in Definition 1 is obtained by

$$c(\sigma)=\alpha^{-1}\sigma(\alpha), \quad \sigma \in Gal(L/K).$$

The converse of this theorem is substantial for our study; that is

Theorem 2 (Kolchin [2], p.426). *Let K, C and G be as in Theorem 1. If the ordinary Galois cohomology $H^1(K, G) = 1$, then every G-extension is a G-primitive extension.*

Remark 2. For a strogly normal extension L/K, the differential Galois cohomology $H^1((L,\delta)/(K,\delta), G)$ is defined. Then there exists a canonical injection
$$H^1((L,\delta)/(K,\delta), G) \to H^1(K, G),$$
which is essential for the proof of this theorem.

§ 2. Picard-Vessiot extensions

Let K be an ordinary differential field of characteristic 0 with a derivation δ. We use the following notation:

$$\delta a=a',$$
$$\delta^m a=a^{(m)},\ m=0,\ 1,\ 2,\ \cdots$$

for every element a of a differential field extension of K.

We say that a differential field extension L/K is a *Picard-Vessiot extension* if $C_L=C_K$ (which we denote by C) and if L is obtained from K by differential adjunction of a fundamental system of solutions of a linear differential equation over K. Namely there is a linear ordinary differential equation

$$\text{(E)} \qquad y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny=0$$

with $a_i\in K$ for $i=1,\cdots,n$, and there are $\eta_1,\cdots,\eta_n\in L$ such that every η_i is a solution of (E), $(\eta_1,\ \cdots,\eta_n)$ is linearly independent over C and $L=K\langle\eta_1,\ \cdots,\eta_n\rangle$. We call $(\eta_1,\ \cdots,\ \eta_n)$ a fundamental system of solutions of (E).

In this case L/K is a strongly normal extension, and then $Gal(L/k)$ becomes an algebraic group over C. This group is injected into $GL(n)_C$ by using the fundamental system of solutions: Let

$$\eta=\begin{pmatrix} \eta_1 & \cdots & \eta_n \\ \eta'_1 & \cdots & \eta'_n \\ & \cdots & \\ \eta_1^{(n-1)} & \cdots & \eta_n^{(n-1)} \end{pmatrix},$$

then $\eta\in GL(n)_L$, because $(\eta_1,\ \cdots\eta_n)$ is linearly independent over C. We call η a *fundamental matrix*. Since $Gal(L/K)$ is known to be the group of differential automorphisms of L over K, it naturally acts on $GL(n)_L$. Then we have the injective homomorphism

$$c\colon Gal(L/K)\ \rightarrow\ GL(n)_C$$

by defining $c(\sigma)=\eta^{-1}\sigma(\eta)$ for any $\sigma\in Gal(L/K)$. We denote the image of c by G and call it a *Picard-Vessiot group relative to* η; it becomes an algebraic subgroup of $GL(n)$ over C.

Recalling Definition 1, we see that the Picard-Vessiot extension L/K is a $GL(n)$-extension, and moreover a G-extension.

The logarithmic derivation $\ell\delta$ for the algebraic group $GL(n)$ (and hence for any algebraic subgroup of $GL(n)$) is defined by

$$\ell\delta(a)=a'a^{-1},\quad a\in GL(n)_L.$$

Now we are interested in the mechanism how Picard-Vessiot extensions are considered as G-primitive extensions. A partial answer is given in the proof of the following theorem.

We denote the component of the identity of an algebraic group G by G^0.

Theorem. *Let K be an ordinary differential field of characteristic 0, L be a Picard-Vessiot extension of K and G be a Picard-Vessiot group over K relative to a fundamental matrix. Then there exists a finite algebraic extension K^0 of K such that K^0 is algebraically closed in LK^0 and, whenever $C_{LK}=C_K$, LK^0 is a G^0-primitive extension of K^0.*

Proof. Let K, L and G be as in the theorem. We use C for C_K. First we note the following facts : For any differential field extension M/K, LM/M is also a Picard-Vessiot extension if $C_{LM}=C$. Moreover $Gal(LM/M)$ is isomorphic to $Gal(L/L\cap M)\subset Gal(L/K)$. K^0 denoting the algebraic closure of K in L, K^0/K is a finite algebraic extension and $Gal(L/K^0)=Gal(L/K)^0$, in particular it is connected. Now we are given the isomorphism

$$c:\ Gal(L/K)\rightarrow G_C.$$

Then for an algebraic extension K' of K satisfying $C_{LK'}=C$, this isomorphism induces an injective homomorphism

$$Gal(LK'/K')\rightarrow G_C.$$

When K' is algebraically closed in LK', $Gal(LK'/K')$ is connected, and hence it is injected into $G_C{}^0$.

Let $\eta=(\eta_j{}^{(i-1)})_{i,j}$ be a fundamental matrix which defines the isomorphism c: $Gal(L/K)\rightarrow G_C$:

$$c(\sigma)=\eta^{-1}\sigma(\eta),\quad \sigma\in Gal(L/K).$$

Let $\mathfrak{p}$ be the prime ideal for η over K:

$$\mathfrak{p}=\{\phi=K[Y_j{}^{(i-1)}]_{i,j}\ \big|\ \phi(\eta)=0\},$$

and we denote by W the locus of $\mathfrak{p}$. Then W is an algebraic variety over K.

The chase of the proof of Theorem 2 shows that, K denoting an algebraic closure of K, we can take $u\in W_K$ such that

$$a\equiv u^{-1}\eta$$

is contained in $G_{LK}{}^0$. Take a finite algebraic extension K' of K so that $u \in W_{K'}$, and let K^0 be the algebraic closure of K' in LK'. We assume that $C_{LK'} = C$. Then as we noted above K^0 is a finite algebraic extension of K, and $Gal(LK^0/K^0)$ is injected into $G_C{}^0$ by c. Obviously $\alpha \in G_{LK^0}{}^0$. Now we show that α is a G^0-primitive over K^0, and $LK^0 = K^0\langle \alpha \rangle$.

For any $\sigma \in Gal(LK^0/K^0)$, $\sigma(u) = u$, then

$$\begin{aligned} c(\sigma) &= \eta^{-1}\sigma(\eta) \\ &= (u\alpha)^{-1}\sigma(u\alpha) \\ &= \alpha^{-1}u^{-1}\sigma(u)\sigma(\alpha) \\ &= \alpha^{-1}\sigma(\alpha). \end{aligned}$$

Therefore $\sigma(\alpha) = \alpha c(\sigma)$. Since $c(\sigma)$ is a C-valued point of G^0, we see

$$\begin{aligned} \sigma(\ell\delta(\alpha)) &= \sigma(\alpha'\alpha^{-1}) \\ &= (\sigma(\alpha))'(\sigma(\alpha))^{-1} \\ &= (\alpha c(\sigma))'(\alpha c(\sigma))^{-1} \\ &= \alpha' c(\sigma)c(\sigma)^{-1}\alpha^{-1} \\ &= \alpha'\alpha^{-1} \\ &= \ell\delta(\alpha) \end{aligned}$$

for any $\sigma \in Gal(LK^0/K^0)$. This implies $\ell\delta(\alpha) \in Lie(G_{K^0})$, because LK^0/K^0 is a strongly normal extension; namely α is a G^0-primitive over K^0. $LK^0 = K^0\langle\alpha\rangle$ is clear, and hence LK^0 is a G^0-primitive extension of K^0.

In the viewpoint of transformations of differential equations, we can interpret the theorem as follows: A differential equation (E) over K with a fundamental matrix η is always transformed, by a transformation $u^{-1}\eta$ for some $u \in GL(n)_{K^0}$ into an equation

$$(E^*) \qquad z' = Az$$

such that the coefficient A is a K^0-valued point of the Lie algebra of the Picard-Vessiot group for (E) over K relative to η, where K^0 is a finite algebraic extension of K, and z denotes an $n \times n$ matrix of differential indeterminates. We call u simply a *transformation*. In general, however, we have no way to obtain the transformation.

§ 3. Examples

We cite here several examples which illustrate how we can consider a Picard-Vessiot extension as a G-primitive extension. In this section we fix the differential field $(K, \delta) =$

$(\mathbf{C}(x), d/dx)$; then the field of constants $C=\mathbf{C}$, the complex number field.

Example 1.

$$(E_1) \qquad y'' + \frac{1}{x}y' = 0.$$

This equation has a fundamental system of solutions $(1, \log x)$. Then we have a fundamental matrix

$$\eta = \begin{pmatrix} 1 & \log x \\ 0 & 1/x \end{pmatrix}$$

The Picard-Vessiot group relative to η is

$$G=\left\{\begin{pmatrix} 1 & c_{12} \\ 0 & 1 \end{pmatrix} \middle| c_{12} \text{ arbitrary}\right\},$$

which is isomorphic to G_a and hence is connected. The Lie algebra of G is

$$\mathfrak{g} =\left\{\begin{pmatrix} 0 & \ell_{12} \\ 0 & 0 \end{pmatrix} \middle| \ell_{12} \text{ arbitrary}\right\}.$$

Now we can take a transformation

$$u=\begin{pmatrix} 1 & 0 \\ 0 & 1/x \end{pmatrix}$$

to obtain a G-primitive

$$\alpha =u^{-1}\eta =\begin{pmatrix} 1 & \log x \\ 0 & 1 \end{pmatrix}$$

Thus $L=K(\eta)$ is a G-primitive extension over K, and the equation (E_1) is transformed into

$$(E_1^*) \qquad z' = \begin{pmatrix} 0 & 1/x \\ 0 & 0 \end{pmatrix}z.$$

Example 2.

$$(E_2) \qquad x(1-x)y'' + (\frac{1}{2} -x)y' +\frac{\nu^2}{4}y =0, \quad \nu \in\mathbf{C}\backslash\mathbf{Q}.$$

This is the Gauss hypergeometric equation with parameters $(-\nu/2,\ \nu/2,\ 1/2)$. It has a fundamental system of solutions $(\eta_1, \eta_2)=((\sqrt{x}+\sqrt{x-1})^{\nu},\ (\sqrt{x}-\sqrt{x-1})^{\nu})$, where we take branches of these functions (at a point in $\mathbf{P}^1\backslash\{0,\ 1,\infty\}$) so that $\eta_1\eta_2= 1$. Set

$$\eta=\begin{pmatrix}\eta_1 & \eta_2\\ \eta_1' & \eta_2'\end{pmatrix},$$

a fundamental matrix. The Picard-Vessiot group relative to η is

$$G=\left\{\begin{pmatrix}c_{11} & 0\\ 0 & c_{22}\end{pmatrix},\begin{pmatrix}0 & c_{12}\\ c_{21} & 0\end{pmatrix}\ \middle|\ c_{11}c_{22}=1,\ c_{12}c_{21}=1\right\};$$

G is not connected and has the component of the identity

$$G^0=\left\{\begin{pmatrix}c_{11} & 0\\ 0 & c_{22}\end{pmatrix}\ \middle|\ c_{11}c_{22}=1\right\}.$$

The Lie algebra of G^0 is

$$\mathfrak{g}=\left\{\begin{pmatrix}\ell_{11} & 0\\ 0 & \ell_{22}\end{pmatrix}\ \middle|\ \ell_{11}+\ell_{22}=0\right\}.$$

Define $K'=K(\sqrt{x(x-1)})$, then $K'\subset L=K(\eta)$, and we have a K'-valued transformation

$$u=\begin{pmatrix}1 & 1\\ a & -a\end{pmatrix},\quad a=\frac{\nu}{2}\cdot\frac{1}{\sqrt{x(x-1)}}$$

to obtain a G^0-primitive

$$\alpha=u^{-1}\eta=\begin{pmatrix}\eta_1 & \\ & \eta_2\end{pmatrix}$$

over K'. Thus L is a G^0-primitive extension of K', and the equation (E_2) is transformed into

$$(E_2^*)\qquad z'=\begin{pmatrix}a & \\ & -a\end{pmatrix}z.$$

Example 3.

$$(E_3)\qquad y''-\frac{1}{x}y'+\left(1+\frac{3}{4x^2}\right)y=0.$$

This equation is regular singular at $x = 0$ and irregular singular at $x = \infty$. It has a fundamental system of solutions $(\eta_1, \eta_2) = (\sqrt{x}\cos x, \sqrt{x}\sin x)$. Set

$$\eta = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_1' & \eta_2' \end{pmatrix}.$$

The Picard-Vessiot group relative to η is

$$G = SO(2),$$

which is connected, and hence, as we noted in the proof of the theorem, K is algebraically closed in $L = K(\eta)$. The prime ideal for η over K is

$$= (Y_1{}^2 + Y_2{}^2 - x,\ Y_1' - \frac{1}{2x} Y_1 + Y_2,\ Y_2' - \frac{1}{2x} Y_2 - Y_1),$$

then it follows that, W denoting the locus of $\mathfrak{p}$, $W_{L \cap K} = W_K = \phi$. Thus we need an algebraic extension of K to obtain a transformation u. For example let $K' = K(\sqrt{x})$, then we have

$$u = \begin{pmatrix} \sqrt{x} & 0 \\ 1/(2\sqrt{x}) & \sqrt{x} \end{pmatrix} \in W_{K'}.$$

We obtain a G-primitive

$$\alpha = u^{-1}\eta = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \in SO(2)_{LK'}.$$

Hence LK' is a G-primitive extension of K', and the equation (E_3) is transformed into

$$(E_3^*) \qquad z' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z.$$

References

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