G-PRIMITIVE EXTENSIONS FOR LINEAR ORDINARY
DIFFERENTIAL EQUATIONS

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(Received October, 31, 1989)

The notion of primitive extensions is of theoretical importance in the algebraic
Galois theory. In differential algebra we have an analogous notion: G-primitive extensions,
where G stands for an algebraic group. Every Picard-Vessiot extension can be considered,
after some modification, as a G-primitive extension for G, the associated Picard-Vessiot
group. This interpretation furnishes a transformation of a linear ordinary differential
equation into, in some sense, a canonical form.

In this paper we explain the mechanism of the interpretation, and give instructive
examples.

We believe that a similar result holds for linear partial differential equations of
the first order, and then, by using it, we can reconstruct the work of Drach [1].

§1. G-extensions and G-primitive extensions

In this section we introduce several notions and results of differential algebra, which
entirely owe to Kolchin [2]; refer to it for details.

We denote the field of constants of a differential field K by Cx. Throughout this
section we fix a differential field K of characteristic 0 and its field of constants C=Cx.

For a strongly normal extension L of K, it is known that:
(i) L/K is a finitely generated extension,
(ii) Cx=Cx=C,
(iii) the set of all strong isomorphisms of L over K, which is denoted by Gal(L/K),
becomes an algebraic group over C.

When we are concerned with algebraic groups, we have the following definition.
Let G be an algebraic group over C.

Definition 1. We say that L/K is a G-extension if it is a strongly normal extension
and if there is an injective homomorphism

$$Gal(L/K) \rightarrow G_C$$

of algebraic groups over C, where C' is a field of constants of a differential extension
of L.

* Partially supported by the Inamori Foundation.
For example, a Picard-Vessiot extension defined by an \( n \)-th order linear ordinary differential equation is a \( GL(n) \)-extension. This is the subject of the following sections.

Next we introduce \( G \)-primitive extensions. We assume, for simplicity, that \( K \) is an ordinary differential field; we use \( \delta \) for the unique derivation.

For any connected algebraic group \( G \) over \( C \) and for a differential field extension \( L \) of \( K \), we can define canonically the logarithmic derivation \( \ell \delta \) of \( \delta \):

\[
\ell \delta : G_L \rightarrow \text{Lie}(G).
\]

We say that \( \alpha \in G_L \) is a \( G \)-primitive over \( K \) if

\[
\ell \delta (\alpha) \in \text{Lie}(G_K) = g \otimes c K,
\]

where \( g \) denotes the Lie algebra of \( G \).

**Definition 2.** Let \( G \) be a connected algebraic group over \( C \). We say that \( L/K \) is a \( G \)-primitive extension if there exists a \( G \)-primitive \( \alpha \) over \( K \) such that \( L = K\langle \alpha \rangle \).

We describe the relation between the two definitions.

**Theorem 1** (Kolchin [2], p.419). \( K \) and \( C \) being the fixed ones. Let \( G \) be a connected algebraic group over \( C \), and \( \alpha \) be a \( G \)-primitive over \( K \). Then \( L = K\langle \alpha \rangle \), which is by definition a \( G \)-primitive extension of \( K \), is a \( G \)-extension of \( K \).

**Remark 1.** In the situation of Theorem 1, we can see that \( \text{Gal}(L/K) \) acts on the group \( G_{C'} \) for some \( C' \). Then the injective homomorphism \( c : \text{Gal}(L/K) \rightarrow G_{C'} \) in Definition 1 is obtained by

\[
c(\sigma) = \alpha^{-1} \sigma(\alpha), \quad \sigma \in \text{Gal}(L/K).
\]

The converse of this theorem is substantial for our study; that is

**Theorem 2** (Kolchin [2], p.426). Let \( K \), \( C \) and \( G \) be as in Theorem 1. If the ordinary Galois cohomology \( H^1(K, G) = 1 \), then every \( G \)-extension is a \( G \)-primitive extension.

**Remark 2.** For a strongly normal extension \( L/K \), the differential Galois cohomology \( H^1((L, \delta)/(K, \delta), G) \) is defined. Then there exists a canonical injection

\[
H^1((L, \delta)/(K, \delta), G) \rightarrow H^1(K, G),
\]

which is essential for the proof of this theorem.
§ 2. Picard-Vessiot extensions

Let $K$ be an ordinary differential field of characteristic $0$ with a derivation $\delta$. We use the following notation:

$$\delta a = a', \quad \delta^m a = a^{(m)}$$

for every element $a$ of a differential field extension of $K$.

We say that a differential field extension $L/K$ is a Picard-Vessiot extension if $C_L = C_K$ (which we denote by $C$) and if $L$ is obtained from $K$ by differential adjunction of a fundamental system of solutions of a linear differential equation over $K$. Namely there is a linear ordinary differential equation

$$y^{(\omega)} + a_1 y^{(\omega-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

with $a_i \in K$ for $i = 1, \ldots, n$, and there are $\eta_1, \ldots, \eta_n \in L$ such that every $\eta_i$ is a solution of (E), $(\eta_1, \ldots, \eta_n)$ is linearly independent over $C$ and $L = K(\eta_1, \ldots, \eta_n)$. We call $(\eta_1, \ldots, \eta_n)$ a fundamental system of solutions of (E).

In this case $L/K$ is a strongly normal extension, and then $Gal(L/K)$ becomes an algebraic group over $C$. This group is injected into $GL(n)_C$ by using the fundamental system of solutions: Let

$$\eta = \begin{pmatrix} \eta_1 & \cdots & \eta_n \\ \eta'_1 & \cdots & \eta'_n \\ \vdots & & \vdots \\ \eta_1^{(\omega-1)} & \cdots & \eta_n^{(\omega-1)} \end{pmatrix},$$

then $\eta \in GL(n)_L$, because $(\eta_1, \ldots, \eta_n)$ is linearly independent over $C$. We call $\eta$ a fundamental matrix. Since $Gal(L/K)$ is known to be the group of differential automorphisms of $L$ over $K$, it naturally acts on $GL(n)_L$. Then we have the injective homomorphism

$$c: Gal(L/K) \rightarrow GL(n)_C$$

by defining $c(\sigma) = \eta^{-1} \sigma(\eta)$ for any $\sigma \in Gal(L/K)$. We denote the image of $c$ by $G$ and call it a Picard-Vessiot group relative to $\eta$; it becomes an algebraic subgroup of $GL(n)$ over $C$.

Recalling Definition 1, we see that the Picard-Vessiot extension $L/K$ is a $GL(n)$-extension, and moreover a $G$-extension.

The logarithmic derivation $\ell \delta$ for the algebraic group $GL(n)$ (and hence for any algebraic subgroup of $GL(n)$) is defined by
\ell (a) = a' a^{-1}, \ a \in GL(n)\ L.

Now we are interested in the mechanism how Picard-Vessiot extensions are considered as G-primitive extensions. A partial answer is given in the proof of the following theorem.

We denote the component of the identity of an algebraic group G by \( G^0 \).

**Theorem.** Let \( K \) be an ordinary differential field of characteristic 0, \( L \) be a Picard-Vessiot extension of \( K \) and \( G \) be a Picard-Vessiot group over \( K \) relative to a fundamental matrix. Then there exists a finite algebraic extension \( K^0 \) of \( K \) such that \( K^0 \) is algebraically closed in \( LK^0 \) and, whenever \( C_{K^0} = C_G \), \( LK^0 \) is a \( G^0 \)-primitive extension of \( K^0 \).

**Proof.** Let \( K, L \) and \( G \) be as in the theorem. We use \( C \) for \( C_K \). First we note the following facts: For any differential field extension \( M/K \), \( LM/M \) is also a Picard-Vessiot extension if \( C_{LM} = C \). Moreover \( \text{Gal}(LM/M) \) is isomorphic to \( \text{Gal}(L/L \cap M) \subset \text{Gal}(L/K) \). \( K^0 \) denoting the algebraic closure of \( K \) in \( L \), \( K^0/K \) is a finite algebraic extension and \( \text{Gal}(L/K^0) = \text{Gal}(L/K)^0 \), in particular it is connected. Now we are given the isomorphism

\[
\gamma : \text{Gal}(L/K) \rightarrow G_C.
\]

Then for an algebraic extension \( K' \) of \( K \) satisfying \( C_{K'} = C \), this isomorphism induces an injective homomorphism

\[
\text{Gal}(LK'/K') \rightarrow G_C.
\]

When \( K' \) is algebraically closed in \( LK' \), \( \text{Gal}(LK'/K') \) is connected, and hence it is injected into \( G_C^0 \).

Let \( \eta = (\eta_i^{(i-1)})_{ij} \) be a fundamental matrix which defines the isomorphism \( \gamma : \text{Gal}(L/K) \rightarrow G_C : \)

\[
c(\sigma) = \eta^{-1} \sigma(\eta), \ \sigma \in \text{Gal}(L/K).
\]

Let \( \mathfrak{p} \) be the prime ideal for \( \eta \) over \( K \):

\[
\mathfrak{p} = \{ \phi = K[Y_i^{(i-1)}]_{ij} \mid \phi(\eta) = 0 \},
\]

and we denote by \( W \) the locus of \( \mathfrak{p} \). Then \( W \) is an algebraic variety over \( K \).

The chase of the proof of Theorem 2 shows that, \( K \) denoting an algebraic closure of \( K \), we can take \( u \in W_K \) such that

\[
a \equiv u^{-1} \eta
\]
is contained in $G_{K^0}$. Take a finite algebraic extension $K'$ of $K$ so that $u \in W_{K'}$, and let $K^0$ be the algebraic closure of $K'$ in $L K'$. We assume that $C_{K'} = C$. Then as we noted above $K^0$ is a finite algebraic extension of $K$, and $Gal(L K^0 / K^0)$ is injected into $G_{K^0}$ by $c$. Obviously $\alpha \in G_{K^0}$. Now we show that $\alpha$ is a $G^0$-primitive over $K^0$, and $L K^0 = K^0 \langle \alpha \rangle$.

For any $\sigma \in Gal(L K^0 / K^0)$, $\sigma(u) = u$, then

$$
c(\sigma) = \eta^{-1} \sigma(\eta)$$

$$= (ua)^{-1} \sigma(ua)$$

$$= \alpha^{-1} u^{-1} \sigma(u) \sigma(\alpha)$$

$$= \alpha^{-1} \sigma(\alpha).$$

Therefore $\sigma(\alpha) = c(\sigma).$ Since $c(\sigma)$ is a $C$-valued point of $G^0$, we see

$$\sigma(\ell \delta(\alpha)) = \sigma(\alpha' \alpha^{-1})$$

$$= (\sigma(\alpha))'(\sigma(\alpha))^{-1}$$

$$= (\alpha c(\sigma))'(\alpha c(\sigma))^{-1}$$

$$= \alpha' c(\sigma) c(\sigma)^{-1} \alpha^{-1}$$

$$= \alpha' \alpha^{-1}$$

$$= \ell \delta(\alpha)$$

for any $\sigma \in Gal(L K^0 / K^0)$. This implies $\ell \delta(\alpha) \in Lie(G_{K^0})$, because $L K^0 / K^0$ is a strongly normal extension; namely $\alpha$ is a $G^0$-primitive over $K^0$. $L K^0 = K^0 \langle \alpha \rangle$ is clear, and hence $L K^0$ is a $G^0$-primitive extension of $K^0$.

In the viewpoint of transformations of differential equations, we can interpret the theorem as follows: A differential equation $(E)$ over $K$ with a fundamental matrix $\eta$ is always transformed, by a transformation $u^{-1} \eta$ for some $u \in GL(n, K^0)$ into an equation

$$(E') \quad z' = Az$$

such that the coefficient $A$ is a $K^0$-valued point of the Lie algebra of the Picard-Vessiot group for $(E)$ over $K$ relative to $\eta$, where $K^0$ is a finite algebraic extension of $K$, and $z$ denotes an $n \times n$ matrix of differential indeterminates. We call $u$ simply a transformation. In general, however, we have no way to obtain the transformation.

§ 3. Examples

We cite here several examples which illustrate how we can consider a Picard-Vessiot extension as a $G$-primitive extension. In this section we fix the differential field $(K, \delta) =$
(C(x), d/dx); then the field of constants C=C, the complex number field.

\textbf{Example 1.} 
\[ (E_1) \quad y'' + \frac{1}{x} y' = 0. \]

This equation has a fundamental system of solutions (1, log x). Then we have a fundamental matrix

\[ \eta = \begin{pmatrix} 1 & \log x \\ 0 & 1/x \end{pmatrix} \]

The Picard-Vessiot group relative to \( \eta \) is

\[ G = \left\{ \begin{pmatrix} 1 & c \log x \\ 0 & 1 \end{pmatrix} \mid c \text{ arbitrary} \right\}, \]

which is isomorphic to \( G_+ \) and hence is connected. The Lie algebra of \( G \) is

\[ \mathfrak{g} = \left\{ \begin{pmatrix} 0 & \ell \log x \\ 0 & 0 \end{pmatrix} \mid \ell \text{ arbitrary} \right\}. \]

Now we can take a transformation

\[ u = \begin{pmatrix} 1 & 0 \\ 0 & 1/x \end{pmatrix} \]

to obtain a \( G \)-primitive

\[ \alpha = u^{-1} \eta = \begin{pmatrix} 1 & \log x \\ 0 & 1 \end{pmatrix} \]

Thus \( L = K(\eta) \) is a \( G \)-primitive extension over \( K \), and the equation \((E_1)\) is transformed into

\[ (E_1') \quad z' = \begin{pmatrix} 0 & 1/x \\ 0 & 0 \end{pmatrix} z. \]

\textbf{Example 2.} 
\[ (E_2) \quad x(1-x)y'' + \left( \frac{1}{2} - x \right)y' + \frac{\nu^2}{4} y = 0, \quad \nu \in \mathbb{C}/\mathbb{Q}. \]
This is the Gauss hypergeometric equation with parameters \((-\nu/2, \, \nu/2, \, 1/2\)). It has a fundamental system of solutions \((\eta_1, \eta_2) = (\sqrt{x+1}, \, \sqrt{x-1})^*, \, (\sqrt{x-1}, \, \sqrt{x+1})^*)\), where we take branches of these functions (at a point in \(\mathbb{P}^1 \setminus \{0, \, 1, \infty\}\)) so that \(\eta_1 \eta_2 = 1\). Set

\[
\eta = \begin{pmatrix} \eta_1 \\ \eta_2^* \\ \eta_1^* \\ \eta_2 \end{pmatrix},
\]

a fundamental matrix. The Picard-Vessiot group relative to \(\eta\) is

\[
G = \left\{ \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix}, \begin{pmatrix} c_{12} \\ c_{21} \end{pmatrix} \middle| c_{11} c_{22} = 1, \, c_{12} c_{21} = 1 \right\}.
\]

\(G^s\) is not connected and has the component of the identity

\[
G^s = \left\{ \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix} \middle| c_{11} c_{22} = 1 \right\}.
\]

The Lie algebra of \(G^s\) is

\[
g = \left\{ \begin{pmatrix} \ell_{11} & 0 \\ 0 & \ell_{22} \end{pmatrix} \middle| \ell_{11} + \ell_{22} = 0 \right\}.
\]

Define \(K' = K(\sqrt{x(x-1)})\), then \(K' \subset L = K(\eta)\), and we have a \(K'\)-valued transformation

\[
u = \begin{pmatrix} 1 & 1 \\ a & -a \end{pmatrix}, \, a = \frac{\nu}{2} \cdot \frac{1}{\sqrt{x(x-1)}}
\]

to obtain a \(G^s\)-primitive

\[
a = u^{-1} \eta = \begin{pmatrix} \eta_1^* \\ \eta_2 \end{pmatrix}
\]

over \(K'\). Thus \(L\) is a \(G^s\)-primitive extension of \(K'\), and the equation \((E_1)\) is transformed into

\[
(E_1') \quad z' = \begin{pmatrix} a \\ -a \end{pmatrix} z.
\]

Example 3.

\[
(E_2) \quad y'' - \frac{1}{x} y' + (1 + \frac{3}{4x^2}) y = 0.
\]
This equation is regular singular at $x = 0$ and irregular singular at $x = \infty$. It has a fundamental system of solutions $(\eta_1, \eta_2) = (\sqrt{x} \cos x, \sqrt{x} \sin x)$. Set

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta'_1 \\ \eta'_2 \end{pmatrix}.$$  

The Picard-Vessiot group relative to $\eta$ is

$$G = \text{SO}(2),$$

which is connected, and hence, as we noted in the proof of the theorem, $K$ is algebraically closed in $L = K(\eta)$. The prime ideal for $\eta$ over $K$ is

$$= (Y_1^2 + Y_2^2 - x, Y'_1 - \frac{1}{2x} Y_1 + Y_2, Y'_2 - \frac{1}{2x} Y_2 - Y_1),$$

then it follows that, $W$ denoting the locus of $p$, $W_{L \cap K} = W_K = \phi$. Thus we need an algebraic extension of $K$ to obtain a transformation $u$. For example let $K' = K(\sqrt{x})$, then we have

$$u = \begin{pmatrix} \sqrt{x} \\ 0 \\ 1/\langle 2\sqrt{x} \rangle \\ \sqrt{x} \end{pmatrix} \in W_{K'}.$$  

We obtain a $G$-primitive

$$\alpha = u^{-1} \eta = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \in \text{SO}(2)(K').$$

Hence $LK'$ is a $G$-primitive extension of $K'$, and the equation $(E_2)$ is transformed into

$$(E_2')$$  

$$z' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z.$$  

References


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