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MAILLET TYPE THEOREMS FOR ALGEBRAIC DIFFERENCE EQUATIONS

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1. Introduction.

Near an irregular singular point (of pole type, say at ∞) of an ordinary, linear differential equation, there may exist formal power series solutions $\sum_{m\geq 0} y_m\ x^{-m}$ that diverge everywhere, but it is relatively easy to show that such formal solutions are in some *Geverey class*, i.e., the coefficients y_m grow at most like a fixed power of $m!$ as $m \to\infty$. While such formal solutions have no meaning in the ordinary sense (as $m \to\infty$), they do have meaning in the asymptotic sense of Poincaré (for fixed m, as $x \to\infty$ in certain sectorial regions). Recently, J.P. Ramis and Y. Sibuya [12] have shown that this phenomenon is a general property of formal solutions in a Gevrey class.

In the nonlinear case of algebraic differential equations, E. Maillet [9] showed that all formal solutions must still be in some Gevrey class (we will call such a result one of *Maillet's type*) and further went on to show that such series can be summed in the sense of Borel. Also see K. Mahler [8] for a proof of this result in a somewhat more general algebraic setting as well as other interesting number-theoretic consequences. The proofs of the above results follow by direct estimation of y_m from recurrence relations. Using a majorant argument, R. Gérard [3] has proven another result of Maillets type for certain more general "algebraic" differential equations (where the coefficients are allowed to be convergent series instead of just polynomials), but the equation must also satisfy a certain type of nondegeneracy condition.

In this paper, we wish to consider the corresponding situation for certain kinds of algebraic difference equations. Already in the linear case, the situation is somewhat more complicated than for differential equations; for example, there appears to be no appreciable class of singular difference equations having the so-called *singular regular property* with respect to power series, that is, that all formal power series solutions converge. Nörlund [10] has shown, however, that an analogue of the Fuchs-Frobenius theorem does carry over for linear difference equations when one works in the ring of *formal factorial series*, that is, series of the form

$$(1.1)\qquad y(x) = \sum_{m\geq 0} y_m\ m!/x(x+1)\cdots(x+m).$$

Here, the difference operators have a simpler structure, but the product formula for such series is more complicated. (Also see W. A. Harris [5] for an extension of Nörlund's result to systems.) For certain of the singular irregular cases for linear difference equations, one can conclude (implicitly) from the work of Birkhoff [1], Trjitzinsky [13], Turrittin [14], that formal factorial series solutions are also in some Gevrey class, which is defined analogously to the power series case. Also see Harris and Sibuya [6] for asymptotic power series solutions of certain nonlinear systems.

R. Gerard and D. A. Lutz [4] have recently obtained some nonlinear extensions of the singular regular phenomenon for factorial series solutions of certain types of singular operator equations (also including some differential - difference equations). Here, we wish to consider algebraic difference equations of the irregular singular type and show that they too must have all formal solutions in some Gevrey class. Moreover, as in the case of Maillet or Gérard, an explicit bound for the Gevrey exponent can be given a priori. To do this, we follow Gerard's approach and show that after a preliminary scaling transformation, the problem is mapped into one for which an analogue of our results on convergent factorial series solutions can be shown to apply. As a by product, we can also easily show by using well known expansions of factorial series as power series, that all formal power series solutions are likewise in some Gevrey class.

2. Preliminaries.

We will say that a formal factorial series (1.1) is in the Gevrey class α (where α is a nonegative real number) if

$$|y_m| = O((m!)^{\alpha}) \qquad \text{as } m \to \infty \tag{2.1}$$

Especially in the linear case, it also makes sense to consider finer types of Gevrey classes indexed by (α, β, γ) with the coefficients satisfying

$$|y_m| = O((m!)^{\alpha} \beta^m m^{\gamma}) \qquad \text{as } m \to \infty$$

In this latter situation, the convergence of (1.1) in some half plane is known from the results of Landau to correspond exactly to the cases when $\alpha = 0$ and $0 < \beta \leqq 1$. When $\alpha = 0$ and $\beta > 1$, the factorial series (1.1) diverges in every half plane, but sometimes it becomes convergent under a change of variable $x \to x/\omega$ for ω sufficiently large. However, for our purposes here, we will treat only the coarser Gevrey classes (2.1).

To see that formal solutions in the exact Gevrey class $\alpha = 1$ can occur for simple, algebraic, even linear difference equations, we construct the following example of an

irregular singular one (in the sense of Nörlund):
Let $y(x)$ be as in (1.1) and define the linear difference operator Δ by

(2.2) $$\Delta\, y(x) = (x-1)\{y(x)-y(x-1)\} = -\textstyle\sum_{m\geq 0} (m+1)\ m!y_m/x(x+1)\cdots(x+m).$$

(See [4] for details about this operator on factorial series.)

Now for any complex constants a,b,c consider the equation

(2.3) $$(1/x)\Delta^2 y(x) = a/x + by(x) + (c/x)y(x)$$

or equivalently

(2.3') $$\Delta^2 y(x) = a + bxy(x) + cy(x).$$

Using the identity $1/(x+m) = 1/x-m/x(x+m)$, it follows easily that

$$xy(x) = y_0+\textstyle\sum_{m\geq 0} \{(m+1)\ y_{m+1} - my_m\} m!/x(x+1)\cdots(x+m),$$

hence one sees that (1.1) formally satisfies (2.3') if and only if $a+by_0=0$ and for all $m\geqq 0$

$$(m+1)^2\ y_m = b\{(m+1)y_{m+1}-my_m\}+cy_m.$$

If $b\neq 0$ we see that the formal solutions are all in Gevrey class $1+\varepsilon$ for arbitrary $\varepsilon>0$. For the special choices $b=-1$ and $c=1$ ($a\neq 0$, but otherwise arbitrary and y_0 satisfying $a=y_0$) we even see that

$$y_m = (-1)^m\ (m-1)!\ y_0 \quad \text{for all } m\geqq 1,$$

hence the formal solution is in the exact Gevrey class $\alpha=1$. Comparing the equation (2.3) with the result of Nörlund on what he called "normal" difference equations or with the results of Gérard and Lutz on singular regular difference equations, one sees that the irregularity is caused by the term $(1/x)\Delta^2 y(x)$ and the fact that our sense the operator Δ^2 dominates the identity operator appearing on the right hand side. An important consideration is that the identity operator appears in a term of "degree" one, while the term $(1/x)\Delta^2 y(x)$ has degree two in the sense we now describe.

We consider a *"homogeneous polynomial" of degree* q in the variables $x, X_0, X_1, \ldots, X_N$ to be an expression of the form

$$F_q(x, X_0, X_1,\ldots, X_N)=\textstyle\sum_{|S|=q} b_S\, X^S+\sum_{r+|S|=q-1} a_{r,S}\ r!X^S/x(x+1)\cdots(x+r),$$

where $s = (s_0, s_1, \ldots, s_N)$ is an $(N+1)$ - tuple of nonnegative integers,

$|s| = s_0 + s_1 + \cdots + s_N$, $X = (X_0, X_1, \ldots, X_N)$, and $X^s = (X_0)^{s_0} (X_1)^{s_1} \cdots (X_N)^{s_N}$.

Thus, F_q is a polynomial of degree at most q in the variables $X_0, X_1, \ldots, X_N$ having coefficients that are inverse factorial expressions in the independent variable x. If each of the variables $X_0, X_1, \ldots, X_N$ would be replaced by a formal factorial series of the type (1.1), then F_q would have a factorial series beginning with a term of order

$$1/x(x+1)\cdots(x+q-1), \text{ that is, of order } O(1/x^q) \text{ as } x \to \infty.$$

The operator equations we treat here consist of expressions of the form $F_q(x, \theta_0 u, \theta_1 u, \ldots, \theta_N u)$, where for all i, $0 \leqq i \leqq N$, θ_i is a *diagonal linear operator* on factorial series in the sense that

$$\theta_i y(x) = \sum_{m \geq 0} m!\, \theta_{i,0}(m) u_m / x(x+1)\cdots(x+m)$$

and the quantities $\theta_{i,0}(m)$ are complex numbers. For example, in (2.3), the operators Δ^0 = identity, Δ and Δ^2 are all linear and diagonal and the term $(1/x)\Delta^2 y(x)$ can be considered as being generated by

$$(1/x)(X_0)^0\, (X_1)^0\, (X_2)^1 \quad \text{with } \theta_1 y = \Delta y \text{ and } \theta_2 y = \Delta^2 y,$$

hence in this sense it has degree 2. Observe that in the ordinary sense, the term $(1/x)\Delta^2 y(x)$ is a linear operator in y so it could also be considered as a first order term. However, as a linear operator in y it is not diagonal or even lower triangular (see[4]). Occasionally there are several such interpretations possible and it is convenient to allow for all consistent interpretations in our theory, although the reader should be aware that with one interpretation one of results might apply, while with another interpretation it might not.

If M is a given, fixed positive integer and if we also specify that the degree of F_q in the variables $X_0, X_1, \ldots, X_N$ does not exceed M, we write $F_{q,M}(x, X_0, X_1, \ldots, X_N)$. The "quasi-algebraic" operator equations we will consider have the form

$$\sum_{q \geq p} F_{q,M}(x, \theta_0 u, \theta_1 u, \ldots, \theta_N u) = 0, \tag{2.4}$$

where p and M are fixed positive integers, the operators $\theta_0, \theta_1, \ldots, \theta_N$ are all diagonal, and finally that there exist positive λ and ε such that for all $(x, X_0, X_1, \ldots, X_N)$ satisfying Re $x \geqq \lambda$ and $|X_0| + |X_1| + \cdots + |X_N| < \varepsilon$, the infinite series in (2.4) converges.

Between such diagonal operators, the order relationship we introduced in [4] takes on an especially simple form; we say that the operator θ_j well dominates the operator θ_i if for all $m \geqq 0$, $|\theta_{i,0}(m)| \leqq |\theta_{j,0}(m)|$. If, in addition, we have

$$\lim_{m\to\infty} \theta_{i,0}(m)/\theta_{j,0}(m) = 0,$$

then we say that θ_j *strictly dominates* the operator θ_i. For example, Δ^j strictly dominates Δ^i for all $i<j$ (including Δ^0 = identity).

Now assume that (2.4) is written in the form

$$F_q(x, \theta_0 u, \theta_1 u, \dots, \theta_N u) = R_{p+1,M}(x, \theta_0 u, \theta_1 u, \dots, \theta_N u), \tag{2.5}$$

where

$$R_{p+1,M}(x, \theta_0 u, \theta_1 u, \dots, \theta_N u) = \sum_{q\geq p+1} F_{q,M}(x, \theta_0 u, \theta_1 u, \dots, \theta_N u).$$

Here, F_p contains all terms of the lowest degree p in the equation.

For example, in (2.3), $p = 1$, $\theta_j = \Delta^j$,and $F_1(x, y) = a/x + by(x)$. The other terms, $(1/x)\Delta^2 y(x)$ and $c(1/x)y(x)$, have degree 2 as noted above.

If in (2.5), it would happen that $n = N$ and if θ_0 well dominates the identity, then under a certain mild nondegeneracy condition on F_p described below, it would follow from our previous result [4] that all formal factorial series solutions would converge. Note that in (2.3), $n = 0$ and $N = 2$.

Our purpose now is to show that when $n<N$ and $\theta_j = \Delta^j$, the formal solutions of (2.5) are nevertheless in some Gevrey class. To do this, we will first consider somewhat more general operator equations of the form (2.5) and make the following assumptions:

H$_1$: *All the operators θ_j are all diagonal for $0 \leqq j \leqq N$, θ_n strictly dominates all the operators θ_i with $i<n$, and also well dominates the identity.*

Letting

$$F_p(x, X_0, X_1, \dots, X_n) = \sum_{|s|=p} b_s X^s + \sum_{r+|s|=p-1} a_{r,s}\, r! X^s / x(x+1)\cdots(x+r),$$

we form the polynomial

$$C(T) = \sum_{|s|=p} b_s (T)^{s_0} (\theta_{1,0}(0)T)^{s_1} \cdots (\theta_{n,0}(0)T))^{s_n}$$

$$+\sum_{r+|s|=p-1} a_{0,s}\{1\}(T)^{s_0}(\theta_{1,0}(0)T)^{s_1}\cdots(\theta_{n,0}(0)T))^{s_n},$$

which can be interpreted as $F_p(\{1\}, T, \theta_{1,0}(0)T,\dots,\theta_{n,0}(0)T)$, with the symbol $\{1\}$ meaning that in the sum, the expression $r!/x(x+1)\cdots(x+r)$ is replaced by 1. In the same manner, we form the polynomial

$$C_n(T) = \partial F_p / \partial X_n(\{1\}, T, \theta_{1,0}(0)T,\dots,\theta_{n,0}(0)T).$$

The nondegeneracy assumption is given by

H₂ : *The two polynomials $C(T)$ and $C_n(T)$ have no common roots.*

Remark: If $p = 1$, and if the (constant) coefficient of X_n is different from zero, i.e., that the operator θ_n actually appears in the equation, then this assumption is automatically satisfied because $C_n(T)$ is just a non-zero constant. These two assumptions correspond with the ones we made in [4], where we also assumed that the operator θ_n well dominates all the other operators in the equation. This last assumption is too strong for our needs, so we now describe a rather technical, somewhat weaker, condition which will be substituted for the hypothesis that θ_n well dominates the operators θ_i for $i > n$:

H₃ : *For all $m \geqq 1$, all q $(p \leqq q \leqq M)$, all $N+2$-tuples $(r, s_0, s_1,\dots,s_N)$ satisfying $r+|s|=q$, and all sequences of nonnegative integers*

$$\{m_0^1, m_0^2,\dots,m_0^{s_0}; m_1^1, m_1^2,\dots,m_1^{s_1}; \dots; m_N^1, m_N^2,\dots,m_N^{s_N}\}$$

satisfying the relation $r+M_0+M_1+\cdots+M_N=m+p-1$, where $M_0=m_0^1+m_0^2+\cdots+m_1^{s_0}$; $M_1=m_1^1+m_1^2+\cdots+m_1^{s_1}$, etc., the set of all numbers $\{G(m)\}$ defined by

$$G(m) = 1/ | \theta_{n,0}(m) | \{\Pi_{1\le k\le s_0} | \theta_{0,0}(m_0^k) | \times \Pi_{1\le k\le s_1} | \theta_{1,0}(m_1^k) | \\ \times\cdots\times \Pi_{1\le k\le s_N} | \theta_{N,0}(m_N^k) | \}$$

is uniformly bounded by a constant.

(Note that if **H₃** is satisfied, then by a suitable normalization of θ_n we can assume that $G(m) \leqq 1$. Also, we allow in the above expressions some of the integers m_j^k to be zero and we make the convention that in such a case the term $\theta_{j,0}(m_j^k)$ does not occur in the product.)

With the above definitions and notation we now state:

THEOREM A. *Under the assumptions* H₁, **H₂**, *and* H₃, *every formal factorial series solution of the equation* (2.5) *is convergent in some half plane.*

3. Proof of Theorem A.

1°) *Existence of formal solutions.*

With the same kind of computations as in the proof of Theorem II of [4] (see §.6), the coefficients $(u_m)_{m\geq 0}$ of a formal inverse factorial series solution

$$u(x) = \sum_{m\geq 0} u_m \; m!/x(x+1)\cdots(x+m)$$

of the equation (2.5) are obtained by formal substitution and equating coefficients. For u_0 we obtain

$$\sum_{|S|=p} b_S \prod_{0\leq j\leq n} (\theta_{j,0}(0))^{s_j} (u_0)^{|S|} + \sum_{r+|S|=p-1} a_{r,S} \prod_{0\leq j\leq n} (\theta_{j,0}(0))^{s_j} (u_0)^{|S|} = 0,$$

and, for all $m\geqq 1$, by identification of the coefficients of $(m+p-1)!/x(x+1)\cdots(x+m+p-1)$, we have

$$\begin{aligned} &u_m F_{p,m}(u_0) \\ &= f_m(u_0, u_1,\ldots,u_{m-1}; \theta_{1,0}(0)u_0, \theta_{1,0}(1)u_1, \ldots, \theta_{1,0}(m-1)u_{m-1}; \ldots; \\ &\quad \theta_{j,0}(0)u_0, \theta_{j,0}(1)u_1, \ldots, \theta_{j,0}(m-1)u_{m-1}; \ldots; \theta_{n,0}(0)u_0, \theta_{n,0}(1)u_1,\ldots, \\ &\quad \theta_{n,0}(m-1)u_{m-1}; \ldots; \theta_{n+1,0}(0)u_0, \theta_{n+1,0}(1)u_1, \ldots, \theta_{n+1,0}(m-1)u_{m-1}; \ldots; \\ &\quad \theta_{N,0}(0)u_0, \theta_{N,0}(1)u_1,\ldots, \theta_{N,0}(m-1)u_{m-1}; (a)), \end{aligned}$$

where

$$\begin{aligned} F_{p,m}(u_0) &= \sum_{|S|=p} b_S \sum_{0\leq j\leq n} s_j\, \theta_{j,0}(m)(\theta_{j,0}(0)u_0)^{s_j-1} \prod_{k\neq j} (\theta_{k,0}(0)u_0)^{s_k} \\ &\quad + \sum_{r+|S|=p-1} a_{r,S} \sum_{0\leq j\leq n} s_j\, \theta_{j,0}(m)(\theta_{j,0}(0)u_0)^{s_j-1} \prod_{k\neq j} (\theta_{k,0}(0)u_0)^{s_k}. \end{aligned}$$

Here, f_m is a polynomial in its indicated arguments and the symbol (a) stands for the coefficients of $R_{p+1,M}$. For the argument in [4], this form sufficed for constructing a majorant equation. With our present weaker assumption on the dominance of the operators, we now require a more precise form showing the dependence on the quantities $\theta_{j,0}(k)$. This comes about because our equation is quasi-algebraic. To simplify these expressions, we introduce the following notation:

For each set of admissible indices $m_0, m_1,\ldots,m_n$ and $s_0, s_1,\ldots,s_n$ we denote certain universal positive constants coming from the product fromula for factorial series (see [4]) by the symbols

$$C_0(m_0, s_0) = C_{s_0}(m_0^1, m_0^2,\ldots, m_0^{s_0}),\; C_1(m_1, s_1), \ldots, C_n(m_n, s_n).$$

Then using these symbols we obtain

$$
\begin{aligned}
&F_{p,m}(u_0)\ u_m = \\
&-\{\textstyle\sum_{r+|s|=p-1} a_{r,s} \sum_{r+m_0+\cdots+m_n=m+p-1} \prod_{1\le k\le s_0,\ m_0^k\ne m} C_0(m_0, s_0)\,\theta_{0,0}(m_0^k)u_{m_0^k} \\
&\qquad \times \prod_{1\le k\le s_1,\ m_1^k\ne m} C_1(m_1, s_1)\,\theta_{1,0}(m_1^k)u_{m_1^k} \\
&\qquad \times\cdots\times \prod_{1\le k\le s_n,\ m_n^k\ne m} C_n(m_n, s_n)\,\theta_{n,0}(m_n^k)u_{m_n^k}\}
\end{aligned}
$$

$$
\begin{aligned}
&-\{\textstyle\sum_{|s|=p} b_s \sum_{m_0+\cdots+m_n=m+p-1} \prod_{1\le k\le s_0,\ m_0^k\ne m} C_0(m_0, s_0)\,\theta_{0,0}(m_0^k)u_{m_0^k} \\
&\qquad \times \prod_{1\le k\le s_1,\ m_1^k\ne m} C_1(m_1, s_1)\,\theta_{1,0}(m_1^k)u_{m_1^k} \\
&\qquad \times\cdots\times \prod_{1\le k\le s_n,\ m_n^k\ne m} C_n(m_n, s_n)\,\theta_{n,0}(m_n^k)u_{m_n^k}\}
\end{aligned}
$$

$$
\begin{aligned}
&+\{\textstyle\sum_{q>p} \sum_{r+|s|=q,\ |s|\le M} a_{r,s} \sum_{r+m_0+\cdots+m_N=m+p-1} \prod_{1\le k\le s_0} C_0(m_0, s_0)\,\theta_{0,0}(m_0^k)\,u_{m_0^k} \\
&\qquad \times \prod_{1\le k\le s_1} C_1(m_1, s_1)\,\theta_{1,0}(m_1^k)\,u_{m_1^k} \\
&\qquad \times\cdots\times \prod_{1\le k\le s_N} C_N(m_N, s_N)\,\theta_{N,0}(m_N^k)u_{m_N^k}\}
\end{aligned}
$$

$$
\begin{aligned}
&+\{\textstyle\sum_{p<|s|\le M} b_s \sum_{m_0+\cdots+m_N=m+p} \prod_{1\le k\le s_0} C_0(m_0, s_0)\,\theta_{0,0}(m_0^k)\,u_{m_0^k} \\
&\qquad \times \prod_{1\le k\le s_1} C_1(m_1, s_1)\,\theta_{1,0}(m_1^k)\,u_{m_1^k} \\
&\qquad \times\cdots\times \prod_{1\le k\le s_N} C_N(m_N, s_N)\,\theta_{N,0}(m_N^k)u_{m_N^k}\}\ .
\end{aligned}
$$

We have the following conclusions:
For each root u_0 of

$$
\begin{aligned}
C(T) &= \textstyle\sum_{|s|=p} b_s \prod_{0\le j\le n} (\theta_{j,0}(0))^{s_j}\,(T)^{|s|} \\
&\quad +\textstyle\sum_{r+|s|=p-1} a_{r,s} \prod_{0\le j\le n} (\theta_{j,0}(0))^{s_j}\,(T)^{|s|}
\end{aligned}
$$

satisfying $F_{q,m}(u_0) \ne 0$, the equation $Du = 0$ admits a formal factorial series as a solution whose first term is u_0/x ; but $F_{q,m}(u_0) = 0$ does not necessarily exclude the existence of a formal factorial series solution.

2°) *Convergence of a formal solution.*

Recall that from our assumptions, among the operators $\theta_0, \theta_1, \ldots, \theta_n$ the operator θ_n strictly dominates the others and also well dominates the identity. Hence we have $1 \le c_{0,0}\ |\theta_{n,0}(m)|$ for all $m = 0, 1, \ldots$, with $c_{0,0} \ne 0$. By a normalization of θ_n we can assume with no loss in generality that $c_{0,0} = 1$. Since

$$
\begin{aligned}
&\lim_{m\to\infty} F_{p,m}(u_0)/\theta_{n,0}(m) = \\
&\{\textstyle\sum_{|s|=p} b_s\, s_n\, (\theta_{n,0}(0))^{s_n-1} \prod_{k\ne j} (\theta_{k,0}(0))^{s_k}\, u_0^{\,|s|-1} \\
&+\textstyle\sum_{r+|s|=p-1} a_{r,s}\, s_n\, (\theta_{n,0}(0))^{s_n-1} \prod_{k\ne j} (\theta_{k,0}(0))^{s_k}\, u_0^{\,|s|-1}\} = C_n(u_0) \ne 0,
\end{aligned}
$$

there exists a number $\sigma > 0$ such that for all $m \ge N$,

$$
|\,F_{p,m}(u_0)\,| \ge \sigma\ |\ \theta_{n,0}(m)\,|\ .
$$

If moreover, $F_{p,m}(u_0) \neq 0$ for all $m<N$, there exists a (possibly different) number $\sigma>0$ such that for all m,

$$|F_{p,m}(u_0)| \geq \sigma\, |\theta_{n,0}(m)|.$$

We now also assume without loss in generality that this occurs for all m, since otherwise by changing a finite number of terms in the solution we could bring this about and not affect the convergence. It follows that

$$\begin{aligned}
|u_m| &\leq \{1/\sigma\, |\theta_{n,0}(m)|\} \\
&\times \{\textstyle\sum_{r+|s|=p-1} \sum_{r+m_0+\cdots+m_n=m+p-1} |a_{r,s}| \prod_{1\leq k\leq s_0,\, m_0^k\neq m} C_0(s_0, m_0)\, |\theta_{0,0}(m_0^k)|\,|u_{m_0^k}| \\
&\qquad \times \textstyle\prod_{1\leq k\leq s_1,\, m_1^k\neq m} C_1(s_1, m_1)\, |\theta_{1,0}(m_1^k)|\,|u_{m_1^k}| \\
&\qquad \times\cdots\times \textstyle\prod_{1\leq k\leq s_n,\, m_n^k\neq m} C_n(s_n, m_n)\, |\theta_{n,0}(m_n^k)|\,|u_{m_n^k}| \\
&+\textstyle\sum_{|s|=p} \sum_{m_0+\cdots+m_n=m+p} |b_s| \prod_{1\leq k\leq s_0,\, m_0^k\neq m} C_0(s_0, m_0)\, |\theta_{0,0}(m_0^k)|\,|u_{m_0^k}| \\
&\qquad \times \textstyle\prod_{1\leq k\leq s_1,\, m_1^k\neq m} C_1(s_1, m_1)\, |\theta_{1,0}(m_1^k)|\,|u_{m_1^k}| \\
&\qquad \times\cdots\times \textstyle\prod_{1\leq k\leq s_n,\, m_n^k\neq m} C_n(s_n, m_n)\, |\theta_{n,0}(m_n^k)|\,|u_{m_n^k}| \\
&+\textstyle\sum_{p<q\leq M} \sum_{r+|s|=q} |a_{r,s}| \sum_{r+m_0+\cdots+m_N=m+p-1} \prod_{1\leq k\leq s_0} C_0(s_0, m_0)\, |\theta_{0,0}(m_0^k)|\,|u_{m_0^k}| \\
&\qquad \times \textstyle\prod_{1\leq k\leq s_1} C_1(s_1, m_1)\, |\theta_{1,0}(m_1^k)|\,|u_{m_1^k}| \\
&\qquad \times\cdots\times \textstyle\prod_{1\leq k\leq s_N} C_N(s_N, m_N)\, |\theta_{N,0}(m_N^k)|\,|u_{m_N^k}|\} \\
&+\{\textstyle\sum_{p<|s|\leq M} b_s \sum_{m_0+\cdots+m_N=m+p} \prod_{1\leq k\leq s_0} C_0(s_0, m_0)\, |\theta_{0,0}(m_0^k)|\,|u_{m_0^k}| \\
&\qquad \times \textstyle\prod_{1\leq k\leq s_1} C_1(s_1, m_1)\, |\theta_{1,0}(m_1^k)|\,|u_{m_1^k}| \\
&\qquad \times\cdots\times \textstyle\prod_{1\leq k\leq s_N} C_N(s_N, m_N)\, |\theta_{N,0}(m_N^k)|\,|u_{m_N^k}|\}.
\end{aligned}$$

Now using assumption (**H**$_3$) we obtain,

$$\begin{aligned}
|u_m| \leq 1/\sigma\, &\{\textstyle\sum_{r+|s|=p-1} |a_{r,s}| \sum_{r+m_0+\cdots+m_n=m+p-1} \prod_{1\leq k\leq s_0,\, m_0^k\neq m} C_0(m_0, s_0)\, |u_{m_0^k}| \\
&\qquad \times \textstyle\prod_{1\leq k\leq s_1,\, m_1^k\neq m} C_1(m_1, s_1)\, |u_{m_1^k}| \\
&\qquad \times\cdots\times \textstyle\prod_{1\leq k\leq s_n,\, m_n^k\neq m} C_n(m_n, s_n)\, |u_{m_n^k}| \\
&+\textstyle\sum_{|s|=p} \sum_{m_0+\cdots+m_n=m+p} |b_s| \prod_{1\leq k\leq s_0,\, m_0^k\neq m} C_0(m_0, s_0)\, |u_{m_0^k}| \\
&\qquad \times \textstyle\prod_{1\leq k\leq s_1,\, m_1^k\neq m} C_1(m_1, s_1)\, |u_{m_1^k}| \\
&\qquad \times\cdots\times \textstyle\prod_{1\leq k\leq s_n,\, m_n^k\neq m} C_n(m_n, s_n)\, |u_{m_n^k}| \\
&+\textstyle\sum_{p<q\leq M} \sum_{r+|s|=q} |a_{r,s}| \sum_{r+m_0+\cdots+m_N=m+p-1} \prod_{1\leq k\leq s_0} C_0(m_0, s_0)\, |u_{m_0^k}| \\
&\qquad \times \textstyle\prod_{1\leq k\leq s_1} C_1(m_1, s_1)\, |u_{m_1^k}| \\
&\qquad \times\cdots\times \textstyle\prod_{1\leq k\leq s_N} C_N(m_N, s_N)\, |u_{m_N^k}|\}
\end{aligned}$$

$$+\sum_{p<|S|\le M} |b_S| \sum_{m_0+\cdots+m_n=m+p} \prod_{1\le k\le s_0,\ m_0^k\ne m} C_0(m_0, s_0)\,|u_{m_0^k}| \times \prod_{1\le k\le s_1,\ m_1^k\ne m} C_1(m_1, s_1)\,|u_{m_1^k}| \times\cdots\times \prod_{1\le k\le s_n,\ m_N^k\ne m} C_N(m_N, s_N)\,|u_{m_N^k}|\,.$$

Consider now the analytic (majorant) equation:

$$(*)\quad \sigma_1\,\{1/x(x+1)\cdots(x+q-2)\}\,Y = \{1/x(x+1)\cdots(x+q-1)\} + \sum_{|S|=p} |b_S| \prod_{0\le j\le n} (Y)^{s_j} + \sum_{r+|S|=p-1} |a_{r,s}|\, r!/x(x+1)\cdots(x+r) \prod_{0\le j\le n} (Y)^{s_j} + |R_{p+1,M}|\,,$$

where σ_1 and μ are real parameters and $|R_{p+1,M}|$ is a majorant series for the function $R_{p+1,M}$. By setting in this equation $Y=(1/x)\,Z$, it is easy to see that the implicit function theorem for factorial series [4] can be applied. This means that for each admissible Y_0 described below, the equation has a unique solution of the form

$$Y=\sum_{m\ge 0} Y_m\; m!/(x,\, x+m+1)$$

convergent in some half plane Re $x > \varrho$. Our objective is to show for a suitable choice of the constants σ_1 and μ that this series is a majorant series of each formal solution that we have computed for the equation (2.6). In the same way as above, we calculate explicit recursion formulas for the coefficients Y_m of the series solution of the given analytic equation $(*)$. This procedure yields for Y_0 the equation

$$\sigma_1 Y_0 = \mu + \sum_{|S|=p} |b_S| \prod_{0\le j\le n} (Y_0)^{s_j} + \sum_{r+|S|=p-1} |a_{r,S}| \prod_{0\le j\le n} (Y_0)^{s_j}.$$

(Note that there are, in general, p possible solutions Y_0 for each choice of the parameters.) For $m>0$, we have

$$\{\sigma_1 - \sum_{|S|=p} |b_S| \sum_{0\le j\le n} s_j\,(Y_0)^{s_j-1} \prod_{k\ne j} (Y_0)^{s_k} - \sum_{r+|S|=p-1} |a_{r,S}| \sum_{0\le j\le n} s_j\,(Y_0)^{s_j-1} \prod_{k\ne j} (Y_0)^{s_k}\}\,Y_m =$$

$$\{\sum_{r+|S|=p-1} \sum_{r+m_0+\cdots+m_n=m+p-1} |a_{r,S}| \prod_{1\le k\le s_0,\ m_0^k\ne m} C_0(m_0, s_0)\,Y_{m_0^k} \times \prod_{1\le k\le s_1,\ m_1^k\ne m} C_1(m_1, s_1)\,Y_{m_1^k} \times\cdots\times \prod_{1\le k\le s_N,\ m_N^k\ne m} C_N(m_N, s_N)\,Y_{m_N^k}\ \}$$

$$+\sum_{|S|=p} \sum_{r+m_0+\cdots+m_n=m+p-1} |b_S| \prod_{1\le k\le s_0,\ m_0^k\ne m} C_0(m_0, s_0)\,Y_{m_0^k} \times \prod_{1\le k\le s_1,\ m_1^k\ne m} C_1(m_1, s_1)\,Y_{m_1^k} \times\cdots\times \prod_{1\le k\le s_N,\ m_N^k\ne m} C_N(m_N, s_N)\,Y_{m_N^k}\ \}$$

$$+\sum_{p<q\le R} \sum_{r+|S|=q} \sum_{r+m_0+\cdots+m_N=m+p-1} |a_{r,S}| \prod_{1\le k\le s_0} C_0(m_0, s_0)\,Y_{m_0^k} \times \prod_{1\le k\le s_1} C_1(m_1, s_1)\,Y_{m_1^k} \times\cdots\times \prod_{1\le k\le s_N} C_N(m_N, s_N)\,Y_{m_N^k}\ \}$$

$$+\sum_{p<|s|\le M}\sum_{m_0+\cdots+m_n=m+p}|b_s|\prod_{1\le k\le s_0,\ m_0^k\ne m} C_0(m_0, s_0)Y_{m_0^k}$$
$$\times\prod_{1\le k\le s_1,\ m_1^k\ne m} C_1(m_1, s_1)Y_{m_1^k}$$
$$\times\cdots\times\prod_{1\le k\le s_N,\ m_N^k\ne m} C_N(m_N, s_N)Y_{m_N^k}\ \}.$$

Now make the following choices:
First select *any* positive real number Y_0 such that $|u_0|\le Y_0$.
Next, select a real number σ_1 such that

$$0<\sigma_1-\sum_{|s|=p}|b_s|\sum_{0\le j\le n} s_j\,(Y_0)^{s_j-1}\prod_{k\ne j}(Y_0)^{s_k}$$
$$-\sum_{r+|s|=K-1}|a_{r,s}|\sum_{0\le j\le n} s_j\,(Y_0)^{s_j-1}\prod_{k\ne j}(Y_0)^{s_k}<\sigma.$$

Finally, select the real number μ defined by

$$\sigma_1 Y_0=\mu+\sum_{|s|=p}|b_s|\prod_{0\le j\le n}(Y_0)^{s_j}+\Sigma_{r+|s|=p-1}|a_{r,s}|\prod_{0\le j\le n}(Y_0)^{s_j}$$

i.e., such that the above-selected Y_0 becomes a solution of the required equation for the first coefficient in the majoorant series. Now we will prove by induction that for all m, $|u_m|\le Y_m$. For $m=0$ this is already clear by our selection. Now assume that for all i, $1\le i\le m-1$, $|u_i|\le Y_i$. The formula majorizing u_m shows that we have,

$$|u_m|\le 1/\sigma\{\sum_{r+|s|=p-1}\sum_{r+m_0+\cdots+m_n=m+p-1}\{|a_{r,s}|\prod_{1\le k\le s_0,\ m_0^k\ne m} C_0(m_0, s_0)Y_{m_0^k}$$
$$\times\prod_{1\le k\le s_1,\ m_1^k\ne m} C_1(m_1, s_1)Y_{m_1^k}$$
$$\times\cdots\times\prod_{1\le k\le s_N,\ m_N^k\ne m} C_N(m_N, s_N)Y_{m_N^k}\ \}$$

$$+\sum_{|s|=p}\sum_{m_0+\cdots+m_n=m+p}|b_s|\prod_{1\le k\le s_0,\ m_0^k\ne m} C_0(m_0, s_0)Y_{m_0^k}$$
$$\times\prod_{1\le k\le s_1,\ m_1^k\ne m} C_1(m_1, s_1)\,Y_{m_1^k}$$
$$\times\cdots\times\prod_{1\le k\le s_n,\ m_n^k\ne m} C_n(m_n, s_n)\,Y_{m_n^k}$$

$$+\sum_{p<q\le M}\sum_{r+|s|=q}\sum_{r+m_0+\cdots+m_N=m+p-1}|a_{r,s}|\prod_{1\le k\le s_0} C_0(m_0, s_0)\,Y_{m_0^k}$$
$$\times\prod_{1\le k\le s_1} C_1(m_1, s_1)\,Y_{m_1^k}$$
$$\times\cdots\times\prod_{1\le k\le s_N} C_N(m_N, s_N)\,Y_{m_N^k}$$

$$+\sum_{p<|s|\le M}\sum_{m_0+\cdots+m_n=m+p}|b_s|\prod_{1\le k\le s_0,\ m_0^k\ne m} C_0(m_0, s_0)Y_{m_0^k}$$
$$\times\prod_{1\le k\le s_1,\ m_1^k\ne m} C_1(m_1, s_1)Y_{m_1^k}$$
$$\times\cdots\times\prod_{1\le k\le s_N,\ m_N^k\ne m} C_N(m_N, s_N)Y_{m_N^k}\ \}.$$

Using the positivity of the coefficients $C_k(s_k, m_k)$, it follows directly that

$$|u_m|\le Y_m,$$

and implies the convergence of the formal factorial series

$$\sum u_m\ m!/x(x+1)\cdots(x+m)$$

solution of (2.6).

4. A Maillet-type theorem for factorial series solutions of quasi-algebraic difference equations.

Consider first a quasi-algebraic difference equations that can be expressed in the form

$$F_p(x,\ y,\ \Delta y,\ \ldots,\ \Delta^n y) = R_{p+l,M}(x,\ y,\ \Delta y,\ \ldots,\ \Delta^N y) \tag{4.1}$$

and where $0 \leqq n < N$. We assume that the polynomials $C(T)$ and $C_n(T)$ constructed as in Section 2 satisfy $\mathbf{H}_2$. Then (4.1) has a formal factorial series solution

$$y(x) = \sum_{m\geq 0} y_m\ m!/x(x+1)\cdots(x+m).$$

Recall from (2.2) that for the operator Δ we have for each $k \geqq 0$,

$$\Delta^k\ y(x) = \sum_{m\geq 0}\ (-1)^k\ (m+1)^k\ y_m\ m!/x(x+1)\cdots(x+m),$$

so the assumption $\mathbf{H}_1$ is also satisfied, however, one can show that $\mathbf{H}_3$ is *not* satisfied. (Indirectly, we already know this by the example (2.3) and Theorem A.) To be able to apply Theorem A, we now introduce a kind of scaling operator as follows:

For each $\alpha \geqq 0$, consider the diagonal operator ϕ defined by

$$\phi u(x) = y_0/x + \sum_{m\geq 1}\ [(m-1)!]^\alpha\ u_m\ m!/x(x+1)\cdots(x+m).$$

The quantity α will be treated like an undetermined parameter that will be selected later. Let $y = \phi u$ and define the operators

$$\theta_j = \Delta^j \circ \phi$$

for all $j = 0,1,2,\ldots,N$. Then (4.1) can be expressed equivalently as

$$F_p(x, \theta_0 u, \theta_1 u,\ \ldots, \theta_n u) = R_{p+1,M}(x, \theta_0 u, \theta_1 u,\ \ldots, \theta_N u). \tag{4.2}$$

Observe that the operators θ_j satisfy $\mathbf{H}_1$. Although the identity operator does not explicitly appear in (4.2), note that $\theta_0 = \phi$ well dominates the identity. Clearly the assumption $\mathbf{H}_2$ in not changed by this identification. So it remains to check assumption $\mathbf{H}_3$. We now show that if α is selected appropriately, then the operators θ_j will satisfy $\mathbf{H}_3$, so that every formal factorial series solution u of (4.2) converges. Once we show this, it follows immediately from the definition of ϕ that each formal solution $y = \phi u$

of (4.1) is in the Gevrey class α. We now state this result as

THEOREM B. *Assume that F_p satisfies the nondegeneracy condition* H_2. *Then every formal solution of* (4.1) *is in a Gevrey class.*

PROOF: It remains to check H_3 and see how the parameter α could be selected. Note that for all nonnegative integers $p_1, p_2, \ldots, p_m$,

$$(p_1!)(p_2!)\cdots(p_m!) \leqq (p_1+p_2+\cdots+p_m)!,$$

hence it follows from the definition of M_j that

$$\Pi_{1\leq k\leq s_j} ((m_j^k-1)!)^\alpha (m_j^k+1) \leqq ((M_j-s_j)!)^\alpha \, \Pi_{1\leq k\leq s_j} (m_j^k+1)^j .$$

Also, from the definition of the operators θ_j we have

$$\Pi_{1\leq k\leq s_j} | \theta_{j,0}(m_j^k) | = \Pi_{1\leq k\leq s_j} ((m_j^k-1)!)^\alpha (m_j^k+1)^j$$

and

$$\theta_{n,0}(m) = ((m-1)!)^\alpha (m+1)^n.$$

Therefore we obtain the estimates

$$\begin{aligned} G(m) &\leqq (1/((m-1)!)^\alpha (m+1)^n) \Pi_{0\leq j\leq N} ((M_j-s_j)!)^\alpha \, \Pi_{0\leq j\leq N} \Pi_{1\leq k\leq s_j} (m_j^k+1)^j \\ &\leqq (1/((m-1)!)^\alpha (m+1)^n) ((\textstyle\sum_{0\leq j\leq N} ((M_j-s_j))!)^\alpha \, \Pi_{0\leq j\leq N} \Pi_{1\leq k\leq s_j} (m_j^k+1)^j \end{aligned}$$

Using

$$\textstyle\sum_{0\leq j\leq N} m_j = m-r \quad \text{and} \quad \sum_{0\leq j\leq N} s_j = q-r,$$

we have

$$\begin{aligned} G(m) &\leqq (1/\{(m-1)!)^\alpha (m+1)^n)\}((m-q)!)^\alpha \, \Pi_{0\leq j\leq N} \Pi_{1\leq k\leq s_j} (m_j^k+1)^j \\ &\leqq (1/\{(m-1)!)^\alpha (m+1)^n)\}((m-q)!)^\alpha \, \Pi_{0\leq j\leq N} (M_j+1)^{js_j} \\ &\leqq (1/\{(m-1)!)^\alpha (m+1)^n)\}((m-q)!)^\alpha (m+1)^{\Sigma_{0\leq j\leq N} js_j} \\ &\leqq (1/\{((m-1)!)^\alpha m^n)\}((m-q)!)^\alpha (m+p)^{\Sigma_{0\leq j\leq N} js_j} \\ &\leqq \{(m+p)^{\Sigma_{0\leq j\leq N} js_j}\}/\{m^n ((m+1-q)(m+1-q+1)\cdots(m-1))^\alpha \}. \end{aligned}$$

As m tends to $+\infty$ the right hand side member of this inequality is equivalent to

$$(m)^{(\Sigma_{0\leq j\leq N} js_j)-n-(q-1)\alpha}.$$

Hence if we select

(4.3) $\alpha \geqq \sup_{p<q\leq M} \{\sup_{|s|\leq q} \{((\sum_{0\leq j\leq N} js_j)-n)/(q-1))\}$,

then with this choice of α, we have for all $m \geqq 1$, $G(m) \leqq 1$.

Hence by Theorem A the formal factorial series $u=\sum_{m\geq 0} u_m\, m!/(x, x+m+1)$ solution of (4.2) is convergent. This completes the proof of Theorem B.

Remark 1: In the proof in [8], the existence of a formal power series solution of an algebraic differential equation is shown to be equivalent to the non-vanishing of a certain polynomial. This necessary condition is related to, but not exactly the same as our nondegeneracy assumption $\mathbf{H}_2$, which seems to be a "limiting situation" of that condition.

Remark 2: All quasi-algebraic difference equations of the form

(4.4) $$G(x, y, y(x-1), \ldots, y(x-N))=0,$$

where $G(x, X_0, X_1, \ldots, X_N)$ is a polynomial in the variables $X_0, X_1, \ldots, X_N$ with coefficients that are convergent factorial series in x, can be expressed in the form (4.1). To see this, first observe that from the definition (2.2) of the operator Δ, it follows by induction that for all positive integers k, $y(x-k)$ is a linear form in y, Δy, $\Delta^2 y, \ldots, \Delta^k y$ with coefficients that are rational in x. Then substituting such forms into (4.4), dividing bysuitable powers of $(x-m)$, ones sees that (4.4) is equivalent to a quasi-algebraic difference equation of the form (4.1). If in the expansion, the *terms of lowest order* F_p would satisfy the nondegeneracy assumption, then all formal solutions are in a Gevrey class. In particular, if $p=1$ and if $F_1(x, X_0, X_1, \ldots, X_n)$ non-trivially depends upon at least one of $X_0, X_1, \ldots, X_n$, i.e., $F_1 \neq a/x$, then as we have observed earlier, the assumption $\mathbf{H}_2$ is automatically satisfied. So if a quasi-algebraic difference equation has any nontrivial linear terms, all formal solutions are in a Gevrey class.

Remark 3: If in the example (2.3), we would apply the eatimate (4.3) on α we obtain $\alpha \geqq \sup_{|s|\leq 2} \{(\sum_{0\leq j\leq 2} js_j-0)/1\}=4$, hence one sees that the estimate is not sharp. In the case of many linear difference equations, the possible Gevrey classes can be determined from a Newton-type diagram. For nonlinear equations, this remains an open problem.

5. A Maillet-type result for formal power series solutions of quasi-algebraic difference equations.

Using the formal correspondence between power series in x^{-1} and factorial series, one can immediately see that any formal factorial series solution can be expressed as a formal power series. Applying Theorem B and some easy estimates on the coefficients, we thereby also obtain as a matter of course,

THEOREM C. *If $y(x)=\sum_{m\geq 1} a_m/x^m$ is a formal solution of (4.1), where F_p satisfies assumption* $\mathbf{H}_2$, *then $y(x)$ is in a Gevrey class.*

PROOF: If $y(x)$ is expressed as the factorial series

$$y(x)=\sum_{m\geq 0} y_m\ m!/x(x+1)\cdots(x+m),$$

then we know from Theorem B that there exist K and α such that for all $m\geqq 0$,

$$|y_m|\leqq k(m!)^{\alpha}\ .$$

Since for all $m\geqq 0$

$$m!/x(x+1)\cdots(x+m)=\sum_{n\geq m+1}(-1)^{n-m-1}\ m!\ S^m_{n-1}\ x^{-m}\ ,$$

where the quantities S^m_n are known as Stirling numbers of the second kind and can be expressed as

$$S^m_n=(n!/m!)\sum_{r_1+r_2+\cdots+r_m=n,\ r_i\geq 0}(r_1!\ r_2!\cdots r_m!)^{-1}.$$

(See [7], p.178 and p. 193 for details.) Then

$$\sum_{m\geq 0} y_m\ m!/x(x+1)\cdots(x+m)=\sum_{n\geq 1} a_n\ x^{-n},$$

where $a_n=\sum_{0\leq m\leq n-1} y_m\,(-1)^{n-m}\ m\,!\ S^m_{n-1}$. Using the above estimate for $|y_m|$ and estimating all terms in the above sum for S^m_n by 1, we see that

$$|a_m|\leqq K\sum_{0\leq m\leq n-1}(m!)^{\alpha}\ n!\ (n-1)^m=O((n!)^{\alpha+2+\varepsilon})$$

for any $\varepsilon>0$, hence the formal power series is also in a Gevrey class.

Remark: In the above argument, we have used some very crude estimates, but since the bound on α is also not sharp, we do not chose to make sharper ones at this time. It remains an open problem to determine a precise set of possible values for the Gevrey classes of formal power series solutions.

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