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REDUCTION OF CENERAL SINGLE LINEAR DIFFERENTIAL EQUATIONS TO SCHLESINGER'S CANONICAL SYSTEMS

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In [1], M.Kohno considered a reduction problem of a single linear differential equation with a regular singularity and an irregular singularity to a system of linear differential equations. He showed a method for the reduction of the single linear differential equation with one regular singularity at $t=0$ and an irregular singularity of rank q at infinity

$$(0.1) \qquad t^n \frac{d^n x}{dt^n} = \sum_{l=1}^{n} \left[\sum_{r=0}^{ql} a_{l,r} t^r \right] t^{n-l} \frac{d^{n-l} x}{dt^{n-l}}$$

to the Birkhoff canonical system

$$(0.2) \qquad t\frac{dY}{dt} = (B_0 + B_1 t + \cdots + B_q t^q) Y$$

by means of a linear transformation with polynomials in t^{-1} as its coefficients, where $B_i\,(i=0,1,..q)$ are $n\times n$ constant matrices. Furthermore, he considered the reduction problem of a single linear differential equation with p regular singularities at $t=t_i\,(i=1,2,..p)$ and an irregular singularity of rank q at infinity

$$(0.3) \qquad \phi^n \frac{d^n x}{dt^n} = \sum_{l=1}^{n} a_l(t)\, \phi^{n-l} \frac{d^{n-l} x}{dt^{n-l}}, \qquad \phi = \prod_{i=1}^{p} (t-t_i)$$

to the Schlesinger form

$$(0.4) \qquad \frac{dY}{dt} = \left(\sum_{i=1}^{p} \frac{C_i}{t-t_i} + \sum_{k=0}^{q-1} B_k t^k \right) Y$$

by means of a linear transformation with rational functions in t as its coefficients. In (0.3), each $a_l(t)$ is a polynomial of degree at most $(p+q-1)\,l$. In (0.4), C_i and B_k are $n\times n$ constant matricies.

He considered such a problem under the assumptions that no solutions near the regular singularity of the single linear differential equation included logarithmic terms and $p=2,\ q=1$. In this paper we shall consider the reduction problem of (0.3) to (0.4) without his assumptions.

1. We consider the reduction problem of the single linear differential equation

$$(1.1) \qquad \phi^n \frac{d^n x}{dt^n} = \sum_{l=1}^{n} a_l(t) \phi^{n-l} \frac{d^{n-l}x}{dt^{n-l}}$$

to the Schlesinger's canonical system

$$(1.2) \qquad \frac{dY}{dt} = \left(\sum_{i=1}^{p} \frac{C_i}{t-t_i} + \sum_{k=0}^{q-1} B_k t^k\right) Y.$$

where

$$\phi = \prod_{i=1}^{p} (t-t_i), \qquad t_i \neq t_j \ (i \neq j).$$

For simplisity, we can put $t_p=0$. So, equation (1.1) has p regular singularities at $t=t_i$ $(i=1,2,\ldots,p)$ and an irregular singularity of rank q at infinity. In (1.1), the coefficients $a_l(t)$ $(l=1,2,\ldots,n)$ are polynomials of degree at most $(p+q-1)l$ and are expressed in the from

$$a_l(t) = \sum_{r=0}^{l-1} \left\{\sum_{h=0}^{p-1} a_{l,r}^{h} \phi^{(p-h)} + \phi \cdot \sum_{v=0}^{q-2} a_{l,r}^{p+v} t^v \right\} \psi^r + a_{l,l}^{0} \psi^l$$

where

$$\psi = \phi t^{q-1}, \ \phi^{(p)} = 1.$$

The characteristic constants ρ_j^i $(j=1,2,\ldots,n)$ of the regular singularities at $t=t_i$ $(i=1,2,\ldots,p)$ are given by the roots of the equations

$$(1.3) \qquad \begin{aligned} [\rho]_n &= \sum_{l=1}^{n} \sum_{h=0}^{p-1} a_{l,0}^{h} \phi^{(p-h)}(t_i) \frac{1}{\{\phi'(t_i)\}^l} [\rho]_{n-l} \\ [\rho]_k &= \rho(\rho-1)\cdots(\rho-k+1). \end{aligned}$$

Here we put $\rho_j^i - \rho_k^i \geqq 0$ $(j<k)$ if $\rho_j^i = \rho_k^i$ (mod 1).

Furthemore, the characteristic constants λ_j $(j=1,2,\ldots,n)$ of the irregular singularity at infinity are given by the roots of the equation

$$(1.4) \qquad \lambda^n = \sum_{l=1}^{n} \alpha_l \lambda^{n-l} \qquad (l=1,2,\ldots,n),$$

where $\alpha_l = a_{l,l}^{0}$.

We use the following notation

$$(1.5) \qquad x_s(t) = t^{-(q-1)s} \frac{d^s x}{dt^s},$$

then we obtain

$$(1.6) \qquad t\frac{dx_s}{dt} + (q-1)sx_s = t^q x_{s+1} \quad (s=0,1,\dots,n-1).$$

Multiplying both sides of (1.1) by $t^{-(q-1)n}\phi^{-n+1}$, we can rewrite (1.1) in the form

$$(1.7) \qquad \phi x_n(t) = \sum_{l=1}^{n} a_l(t)\phi\,\phi^{-l}x_{n-l}(t) \;,$$

and we can obtain from (1.6-7)

$$\begin{cases} \phi\dfrac{dx_s}{dt} = \theta_{s+1}\phi t^{-1}x_s + \phi x_{s+1} \quad (s=0,1,\dots,n\text{-}2) \\ \phi\dfrac{dx_{n-1}}{dt} = \theta_n\phi t^{-1}x_{n-1} + \sum_{l=1}^{n} a_l(t)\phi^{-l+1}x_{n-l} \end{cases}$$

where $\theta_j = -(j-1)(q-1) \qquad (j=1,2,\dots,n)$.
Putting

$$(1.8) \qquad \phi t^{-1} = \sigma^0 + \sigma^1\phi^{(p-1)} + \cdots + \sigma^{p-1}\phi' \quad (\sigma^h;constants),$$

we can get a system of linear differential equations for the column vector $X(t)=(x_0(t), x_1(t),\dots,x_{n-1}(t))^*$ as follows:

$$\phi\frac{dX}{dt} = \begin{pmatrix} \theta_1\phi t^{-1} & \phi & & & \\ & \theta_2\phi t^{-1} & \phi & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & \phi \\ a_n(t)\phi^{1-n} & a_{n-1}(t)\phi^{2-n} & \cdots & & \theta_n f t^{-1} + a_1(t) \end{pmatrix} X$$

$$= \left\{ \begin{pmatrix} & 0 & & \\ A_n^0(\phi) & A_{n-1}^0(\phi) & \cdots\cdots & A_1^0(\phi) \end{pmatrix} + \begin{pmatrix} & 0 & \\ A_n^1(\phi) & \cdots & A_1^1(\phi) \end{pmatrix}\phi^{(p-1)} + \cdots \right.$$

$$+ \begin{pmatrix} & 0 & \\ A_n^{p-1}(\phi) & \cdots & A_1^{p-1}(\phi) \end{pmatrix}\phi' + \begin{pmatrix} & 0 & \\ A_n^p(\phi) & \cdots & A_1^p(\phi) \end{pmatrix}\phi + \cdots$$

$$+ \begin{bmatrix} & 0 & \\ & & \\ A_n^{p+q-2}(\phi) & \cdots & A_1^{p+q-2}(\phi) \end{bmatrix} \phi t^{q-2} + \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & \\ & & 0 & 1 \\ \alpha_n & \alpha_{n-1} & \cdots & \alpha_1 \end{bmatrix} \phi$$

$$+ \begin{bmatrix} \sigma^0\theta_1 & & \\ & \ddots & \\ & & \sigma^0\theta_n \end{bmatrix} + \begin{bmatrix} \sigma^1\theta_1 & & \\ & \ddots & \\ & & \sigma^1\theta_n \end{bmatrix} \phi^{(p-1)} + \cdots$$

$$+ \begin{bmatrix} \sigma^{p-1}\theta_1 & & \\ & \ddots & \\ & & \sigma^{p-1}\theta_n \end{bmatrix} \phi' \Bigg\} X$$

$$= [\{\sum_{h=0}^{p-1} A^h(\phi)\phi^{(p-h)} + \phi \sum_{\nu=0}^{q-2} A^{p+\nu}(\phi)\ t^\nu\} + A_\infty\phi + \sum_{h=0}^{p-1} \theta^h\phi^{(p-h)}]X$$

(1.9) $$= \{A(t) + \theta\}X ,$$

where

$$A_l^h(\phi) = \sum_{r=0}^{l-1} a_{l,r}^h \phi^{r-l+1}$$

$$(l=1,2,\dots,n; h=0,1,\dots,p,p+1,\dots,p+q-2)$$

$$\theta = \sum_{h=0}^{p-1} \theta^h\phi^{(p-h)} .$$

Now, we consider the following linear transformation with a triangular matrix as its coefficient

(1.10) $$Y = E(t)X = \{ \sum_{h=0}^{p-1} E^h(\phi)(\phi)^{(p-h)} + \phi \sum_{\nu=0}^{q-2} E^{p+\nu}(\phi)t^\nu \}X ,$$

where

$$E^h(\phi) = \begin{bmatrix} \delta^h & & & & \\ e_{2,1}^h(\phi) & \delta^h & & 0 & \\ \vdots & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ e_{n,1}^h(\phi) & e_{n,2}^h(\phi) & \cdots & e_{n,n-1}^h(\phi) & \delta^h \end{bmatrix}$$

$$(h=0,1,\dots,p,p+1,\dots,p+q-2;\ \delta^0=1,\ \ \delta^h=0\ \ (h\neq 0)).$$

Then our aim is to determine $E(t)$ such that $E(t)$ satisfies the system of linear differential equations

$$\phi\frac{d}{dt}E(t)\ +\ E(t)A(t)\ =\phi\{\sum_{i=1}^{p}\frac{C_i}{t-t_i}+\sum_{k=0}^{q-1}B_kt^k\}E(t)-E(t)\,\theta$$

(1.11)

$$=\ \{\sum_{m=0}^{p-1}G_m\phi^{(p-m)}+\phi\sum_{k=0}^{q-1}B_kt^k\}E(t)-E(t)\,\theta$$

together with an appropriate choice of the constant matrices C_i (or G_m) and B_k $(i=1,2,\dots,p;m=0,1,\dots,p-1;k=0,1,\dots,q-1)$. Here we have

$$C_i=\sum_{m=0}^{p-1}G_m\phi^{(p-m)}(t_i)\{\phi'(t_i)\}^{-1}.$$

Since the differential equation (1.1) includes n constants $\alpha_l\,(l=1,2,\dots,n)$ and $n(n+1)(p+q-1)/2$ constans $a^h_{l,r}$ $(l=1,2,\dots,n;r=0,1,\dots,l-1;\ h=0,1,\dots,p-1,p,\dots,p+q-1)$, we may put $B_{q-1}=A_\infty$ and we may assume that $G_m\,(m=0,1,\dots,n)$ and $B_k\,(k=0,1,\dots,q-2)$ are lower triangular matrices. We attempt to show that $n\,(n+1)(p+q-1)/2$ elements of G_m and B_k can be determined uniquely by the same number of the constants $a^h_{l,r}$ through the differential equation (1.11).

We shall substitute the expressions (1.9) and (1.10) for $A(t),\theta$ and $E(t)$ into (1.11) and compar the expressions attached to $\phi^{(p-h)}\,(h=0,1,\dots,p-1)$ and $\phi t^\nu\,(\nu=0,1,\dots q-2)$, respectively, in both side. We make some preparations for that purpose as follows:

Let $d^*_{m,h}$ $\varepsilon^*_{\omega,\nu}$ *and* $f^*_{\omega,\nu}$ *are constants and*

$$0\leq m,h\leq p-1.\quad 0\leq\omega,\nu\leq q-2\ .$$

[I] *We can set*

$$\phi^{(p-m)}\phi^{(p-h)}=\sum_{u=0}^{p-1}d^u_{m,h}\,\phi^{(p-\omega)}+\phi\sum_{\nu=0}^{q-2}d^{p+\nu}_{m,h}\ t^\nu$$

$$+\ \psi\{d^{p+q-1}_{m,h}\,\phi^{(p)}+d^{p+q}_{m,h}\phi^{(p-1)}+\ \cdots\cdots\ +d^{m+h}_{m,h}\phi^{(2p+q-1-\quad(m+h)}\}^2$$

(1.12)

$$=\{\phi^{(p-m)}\,\phi^{(p-h)}]^0+\phi[\phi^{(p-m)}\,\phi^{(p-h)}]^1+\psi[\phi^{(p-m)}\,\phi^{(p-h)}]^2.$$

If $p\leq m+h\leq p+q-1,$

(1.13)

$$\begin{cases}[\phi^{(p-m)}\,\phi^{(p-h)}]^1=d^p_{m,h}+d^{p+1}_{m,h}t+\ \cdots\cdots\ +d^{m+h}_{m,h}t^{(p-m-h)}\\ \\ [\phi^{(p-m)}\,\phi^{(p-h)}]^2=0\quad .\end{cases}$$

If $m+h<p,$

(1.14)
$$\begin{cases} [\phi^{(p-m)}\phi^{(p-h)}]^0 = d^0_{m,h}\phi^{(p)} + d^1_{m,h}\phi^{(p-1)} + \cdots\cdots + d^{m+h}_{m,h}\phi^{(p-m-h)} \\ [\phi^{(p-m)}\phi^{(p-h)}]^1 = [\phi^{(p-m)}\phi^{(p-h)}]^2 = 0 . \end{cases}$$

If $h=0$,

(1.15) $$[\phi^{(p-m)}\phi^{(p-h)}]^0 = \phi^{(p-m)} .$$

It is obiously that $d^u_{m,h} = d^u_{h,m}$.

[*II*] *We can set*

(1.16)
$$\phi t^\omega \phi^{(p-h)} = \phi \sum_{\nu=\omega}^{q-2} \varepsilon^\nu_{\omega,h} t^\nu + \phi\{\varepsilon^{q-1}_{\omega,h}\phi^{(p)} + \varepsilon^q_{\omega,h}\phi^{(p-1)} + \cdots\cdots + \varepsilon^{\omega+h}_{\omega,h}\phi^{(p+q-1-(\omega+h))}\}$$
$$= \phi[\phi t^\omega \phi^{(p-h)}]^1 + \phi[\phi t^\omega \phi^{(p-h)}]^2 .$$

If $\omega+h<q-1$,

(1.17)
$$\begin{cases} [\phi t^\omega \phi^{(p-1)}]^1 = \varepsilon^\omega_{\omega,h} t^\omega + \varepsilon^{\omega+1}_{\omega,h} t^{\omega+1} + \cdots\cdots + \varepsilon^{\omega+h}_{\omega,h} t^{\omega+h} \\ [\phi t^\omega \phi^{(p-h)}]^2 = 0 \end{cases}$$

If $h=0$,

(1.18) $$[\phi t^\omega \phi^{(p-h)}]^1 = t^\omega .$$

Generally, we have $\varepsilon^\nu_{\omega,h} \neq \varepsilon^\nu_{h,\omega}$.

[*III*] *If* $q-1<\omega+\nu$, *we have*

(1.19)
$$\phi t^\omega \phi t^\nu = \phi \sum_{u=0}^{p-1} f^{q-1+u}_{\omega,\nu}\phi^{(p-u)} + \{f^{p+q-1}_{\omega,\nu} + f^{p+q}_{\omega,\nu} t + \cdots\cdots + f^{p+\omega+\nu}_{\omega,\nu} t^{\omega+\nu-(q-1)}\}$$
$$= \phi\{[\phi t^\omega \phi t^\nu]^2 + \phi[\phi t^\omega \phi t^\nu]^3\} .$$

If $\omega+\nu<q-1$,

(1.20)
$$\phi t^\omega \phi t^\nu = \phi \sum_{u=\omega+\nu}^{q-2} f^u_{\omega,\nu} t^\nu + \phi\{f^{q-1}_{\omega,\nu}\phi^{(p)} + f^q_{\omega,\nu}\phi^{(p-1)} + \cdots\cdots + f^{p+\omega+\nu}_{\omega,\nu}\phi^{(q-1-(\omega+\nu))}\}$$
$$= \phi\{\phi t^\omega \phi t^\nu]^1 + \phi[\phi t^\omega \phi t^\nu]^2 .$$

Particularly, if $\omega+\nu<q-1-p$,

(1.21)
$$\begin{cases} [\phi t^\omega \phi t^\nu]^1 = f^{\omega+\nu}_{\omega,\nu} t^{\omega+\nu} + f^{\omega+\nu+1}_{\omega,\nu} t^{\omega+\nu+1} + \cdots\cdots + f^{\omega+\nu+p}_{\omega,\nu} t^{\omega+\nu+p} \\ [\phi t^\omega \phi t^\nu]^2 = 0 . \end{cases}$$

It is obviously that $f^{u}_{\omega,\nu} = f^{u}_{\nu,\omega}$.

As a result of (1.12-21), we can express these situation by the following figure.

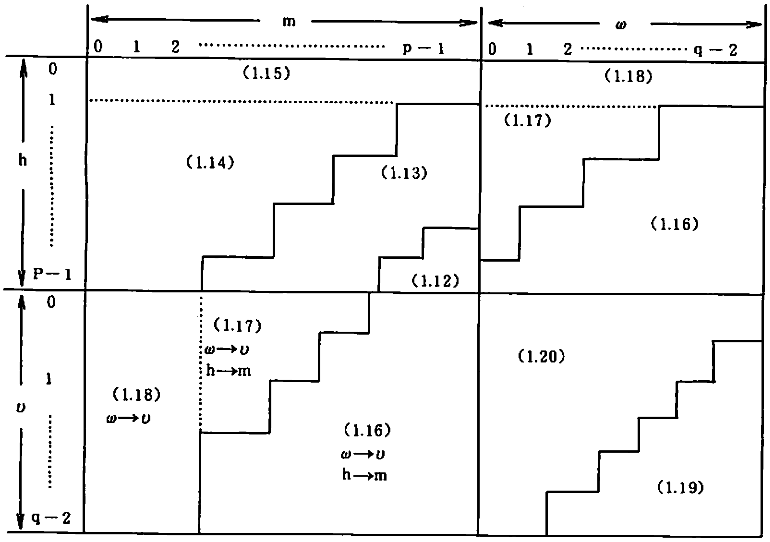

Fig.1.22

If $p<q$, *then (1.12) is not exist.*
If $p<q-1$, *then (1.21) is not exist.*

Substituting the expressions (1.3) and (1.4) for $A(t), \theta$ and $E(t)$ into (1.11) and using (1.12-21), we have as follows:

$$\underline{\sum_{h=0}^{p-1} \{ \{[\phi' \phi^{(p-h)}]^0 + (q-1)\cdot\sum_{m=0}^{p-1} \sigma^m [\phi^{(p-m)} \phi^{(p-h)}]^0 \}}$$

$$+ \{[\phi' \phi^{(p-h)}]^1 + (q-1)\cdot\sum_{m=0}^{p-1} \sigma^m [\phi^{(p-m)} \phi^{(p-h)}]^1 \}$$

$$\underline{+ \phi\{[\phi' \phi^{(p-h)}]^2 + (q-1)\cdot\sum_{m=0}^{p-1} \sigma^m [\phi^{(p-m)} \phi^{(p-h)}]^2 \}\} DE^h}$$

$$+ \sum_{\nu=0}^{q-2} \{ \phi\{[\phi t^\nu \phi']^1 + (q-1)\cdot\sum_{m=0}^{p-1} \sigma^m [\phi t^\nu \phi^{(p-m)}]^1 \}$$

$$\underline{+ \phi\{[\phi t^\nu \phi']^2 + (q-1)\cdot\sum_{m=0}^{p-1} \sigma^m [\phi t^\nu \phi^{(p-m)}]^2 \} DE^{p+\nu}}$$

$$+ \sum_{\nu=0}^{q-2} \{\phi\{[\phi t^\nu \phi']^1 + \nu\cdot\sum_{m=0}^{p-1} \sigma^m [\phi t^\nu \phi^{(p-m)}]^1 \}$$

$$\underline{+ \phi\{[\phi t^\nu \phi']^2 + \nu\cdot\sum_{m=0}^{p-1} \sigma^m [\phi t^\nu \phi^{(p-m)}]^2 \} E^{p+\nu}}$$

$$+ \phi\{p!E^1 + \sum_{h=2}^{p-1} [\phi\cdot\phi^{(p+1-h)}]^1 E^h \} + \underline{\phi\cdot\sum_{h=2}^{p-1} [\phi\cdot\phi^{p+1-h}]^2 E^h}$$

$$+ \underline{\sum_{m=0}^{p-1} \phi^{(p-m)} E^0 A^m} + \phi \cdot \sum_{\omega=0}^{q-2} t^{\omega} E^0 A^{p+\omega}$$

$$+ \underline{\phi \; \{\sum_{h=0}^{p-1} \phi^{(p-h)} E^h} + \sum_{\nu=0}^{q-2} \phi t^{\nu} E^{p+\nu} \} A_{\infty}$$

(1.23)

$$\begin{aligned}
= \quad & \sum_{h=0}^{p-1} \cdot \sum_{m=0}^{p-1} \{ \underline{[\phi^{(p-m)} \phi^{(p-h)}]^0} + \phi [\phi^{(p-m)} \phi^{(p-h)}]^1 \\
& \qquad + \underline{\phi [\phi^{(p-m)} \phi^{(p-h)}]^2 \} (G_m E^h - E^h \theta^m)} \\
& + \sum_{\nu=0}^{q-2} \sum_{m=0}^{p-1} \{ \phi [\phi t^{\nu} \phi^{(p-m)}]^1 + \underline{\phi [\phi t^{\nu} \phi^{(p-m)}]^2 \} (G_m E^{p+\nu} - E^{p+\nu} \theta^m)} \\
& + \sum_{\omega=0}^{q-2} \sum_{h=0}^{p-1} \{ \phi [\phi t^{\omega} \phi^{(p-h)}]^1 + \underline{\phi [\phi t^{\omega} \phi^{(p-h)}]^2 \} B_{\omega} E^h} \\
& + \sum_{\omega=0}^{q-2} \sum_{\nu=0}^{q-2-\omega} \{ \phi [\phi t^{\omega} \phi t^{\nu}]^1 + \underline{\phi [\phi t^{\omega} \phi t^{\nu}]^2 \} B_{\omega} E^{p+\nu}} \\
& + \underline{\sum_{\omega=0}^{q-2} \sum_{\nu=q-1-\omega}^{q-2}} \{ \phi [\phi t^{\omega} \phi t^{\nu}]^2 + \phi \phi [\phi t^{\omega} \phi t^{\nu}]^3 \} B_{\omega} E^{p+\nu} \\
& + \underline{\phi \cdot \sum_{h=0}^{p-1} \phi^{(p-h)} B_{q-1} E^h} + \phi \cdot \phi \cdot \sum_{\nu=0}^{q-2} t^{\nu} B_{q-1} E^{p+\nu} .
\end{aligned}$$

Here we used $E^h A^m = 0$ $(h \neq 0,\ 0 < m < p+q-2)$ and we denoted $D = \phi \dfrac{d}{dt}$.

(1.23) is formed by $\phi^{(p-h)}$ $(h=0,1,\ldots,p-1)$ *(under-line parts)* and ϕt^{ν} $(\nu = 0,1,\ldots,q-2)$. Taking account of the fact that $A^h(\phi)$ $(h=0,1,\ldots,p-1)$ are matrices of porinomials in ϕ^{-1} with no constant term, we can easily deduce from (1.23)'s under line parts that B_{q-1} must be equal to A_{∞} .

We denote

$$G_m = \begin{bmatrix} g^m_{1,1} & & & \\ g^m_{2,1} & g^m_{2,2} & & 0 \\ \vdots & \vdots & \ddots & \\ g^m_{n,1} & g^m_{n,2} & \cdots & g^m_{n,n} \end{bmatrix} \qquad (m=0,1,\ldots,p-1),$$

$$C_i = \begin{bmatrix} c^i_{1,1} & & & \\ c^i_{2,1} & c^i_{2,2} & & 0 \\ \vdots & \vdots & \ddots & \\ c^i_{n,1} & c^i_{n,2} & \cdots & c^i_{n,n} \end{bmatrix} \qquad (i=1,.2,\ldots,p),$$

and

$$B_\omega = \begin{pmatrix} b^\omega_{1,1} & & & \\ b^\omega_{2,1} & b^\omega_{2,2} & & 0 \\ \vdots & \vdots & \ddots & \\ b^\omega_{n,1} & b^\omega_{n,2} & \cdots & b^\omega_{n,n} \end{pmatrix} \quad (\omega=0,1,\dots,q-2),$$

where

$$C^i_{j,k} = \{ \sum_{m=0}^{p-1} g^m_{j,k}\,\phi^{(p-m)}(t_i)\} \frac{1}{\phi'(t_i)} \quad .$$

Comparing the expressions attached to $\phi^{(p-h)}(h=0,1,\dots,p-1)$ and ϕt^ν ($\nu=0,1,\dots,q-2$), respectively, in both sides of (1.23), we can rewrite (1.23) in the following elementwise form:

(1.24α) $$\phi e^h_{j,j-1}(\phi) = g^h_{j,j} - \sigma^h\theta_j + \phi e^h_{j+1,j}(\phi)$$

(1.24β) $$\phi\cdot\sum_{\nu=0}^{q-2} t^\nu e^{p+\nu}_{j,j-1}(\phi) = \phi\cdot\sum_{\omega=0}^{q-2} t^\omega b^\omega_{j,j} + \phi\cdot\phi\cdot\sum_{\nu=0}^{q-2} t^\nu e^{p+\nu}_{j+1,j}(\phi)$$

$$(j=1,2,\dots,n-1),$$

(1.25α) $$A^h_? + \phi e^h_{n,n-1}(\phi) = g^h_{n,n} - \sigma^h\theta_n$$
$$(h=0,1,\dots p-1) \quad ,$$

(1.25β) $$\phi\cdot\sum_{\omega=0}^{q-2} t^\omega A^{q+\omega}_? + \phi\cdot\phi\cdot\sum_{\nu=0}^{q-2} t^\nu e^{p+\nu}_{n,n-1}(\phi) = \phi\cdot\sum_{\omega=0}^{q-2} t^\omega b^\omega_{n,n}$$

(1.26α) $$\sum_{h=0}^{p-1}\{ [\phi'\phi^{(p-h)}]^0 + (q-1)\cdot\sum_{m=0}^{p-1}\sigma^m[\phi^{(p-m)}\phi^{(p-h)}]^0\} De^h_{j,j-k}$$
$$+\phi\ \{\sum_{h=0}^{p-1}\{[\phi'\phi^{(p-h)}]^2 + (q-1)\cdot\sum_{m=0}^{p-1}\sigma^m[\phi^{(p-m)}\phi^{(p-h)}]^2\} De^h_{j,j-k}$$
$$+\sum_{\nu=0}^{q-2}[\phi t^\nu\phi']^2 (De^{p+\nu}_{j,j-k} + e^{p+\nu}_{j,j-k}) + \sum_{h=2}^{p-1}[\phi\,\phi^{(p+1-h)}]^2 e^h_{j,j-k}$$
$$+\sum_{\nu=0}^{q-2}\sum_{m=0}^{p-1}\sigma^m[\phi t^\nu f^{(p-m)}]^2 ((q-1)De^{p+\nu}_{j,j-k} + \nu e^{p+\nu}_{j,j-k})\}$$
$$+\phi\cdot\sum_{h=0}^{p-1}\phi^{(p-h)}\ e^h_{j,j-k-1}$$
$$= \sum_{m=0}^{p-1}\phi^{(p-m)}\ (\sum_{l=j-k}^{j} g^m_{j,l}\,e^h_{l,j-k} - \sigma^m\theta_{j-k}\,e^h_{j,j-k})$$

$$+\sum_{h=1}^{p-1}\sum_{m=0}^{p-1}[\phi^{(p-m)}\phi^{(p-h)}]^{0}\,(\sum_{l=j-k}^{j} g^{m}_{j,l}\,e^{h}_{l,j-k}-\sigma^{m}\theta_{j-k}\,e^{h}_{j,j-k})$$

$$+\ \phi\{\sum_{h=0}^{p-1}\sum_{m=0}^{p-1}[\phi^{(p-m)}\phi^{(p-h)}]^{2}\,(\sum_{l=j-k+1}^{j} g^{m}_{j,l}\,e^{h}_{l,j-k}-\sigma^{m}\theta_{j-k}\,e^{h}_{j,j-k})$$

$$+\sum_{\nu=0}^{q-2}\sum_{m=0}^{p-1}[\phi t^{\nu}\phi^{(p-m)}]\,(\sum_{l=j-k+1}^{j}(g^{m}_{j,l}\,e^{p+\nu}_{l,j-k}+b^{\nu}_{j,l}\,e^{m}_{l,j-k})$$

$$-\sigma^{m}\theta_{j-k}\,e^{p+\nu}_{j,j-k})$$

$$+\sum_{\omega=0}^{q-2}\sum_{\nu=0}^{q-2}[\phi t^{\omega}\phi t^{\nu}]^{2}\,(\sum_{l=j-k+1}^{j} b^{\omega}_{j,l}\,e^{p+\nu}_{l,j-k})\}$$

$$+\ \phi\cdot\sum_{h=0}^{p-1}\phi^{(p-h)}\ e^{h}_{j+1,j-k}\ ,$$

$$(1.26\beta)\qquad \phi\cdot\sum_{h=0}^{p-1}\{[\phi'\phi^{(p-h)}]^{1}+(q-1)\cdot\sum_{m=0}^{p-1}\sigma^{m}\,[\phi^{(p-h)}\phi^{(p-m)}]^{1}\}De^{h}_{j,j-k}$$

$$+\ \phi\cdot\sum_{\nu=0}^{q-2}[\phi t^{\nu}\phi']^{1}\,(De^{p+\nu}_{j,j-k}+e^{p+\nu}_{j,j-k})$$

$$+\ \phi\cdot\sum_{\nu=0}^{q-2}\sum_{m=0}^{p-1}\sigma^{m}\,[\phi t^{\nu}\phi^{(p-m)}]\,((q-1)\ De^{p+\nu}_{j,j-k}+\nu\,e^{p+\nu}_{j,j-k})$$

$$+\ \phi\ \{p!e^{1}_{j,j-k}+\sum_{h=2}^{p-1}[\phi\,\phi^{(p+1-h)}]^{1}\,e^{h}_{j,j-k}\}+\phi\cdot\phi\cdot\sum_{\nu=0}^{q-2}t^{\nu}e^{p+\nu}_{j,j-k-1}$$

$$=\phi\cdot\sum_{h=0}^{p-1}\sum_{m=0}^{p-1}[\phi^{(p-m)}\phi^{(p-h)}]^{1}\,(\sum_{l=j-k+1}^{j} g^{m}_{j,l}\,e^{h}_{l,j-k}-\sigma^{m}\theta_{j-k}\,e^{h}_{j,j-k})$$

$$+\ \phi\cdot\sum_{\nu=0}^{q-2}\sum_{m=0}^{p-1}[\phi t^{\nu}\phi^{(p-m)}]^{1}\,(\sum_{l=j-k+1}^{j} g^{m}_{j,l}\,e^{p+\nu}_{l,j-k}-\sigma^{m}\theta_{j-k}\,e^{p+\nu}_{j,j-k})$$

$$+\ \phi\cdot\sum_{\omega=0}^{q-2}t^{\omega}\sum_{l=j-k}^{j} b^{\omega}_{j,l}\,e^{0}_{l,j-k}+\phi\cdot\sum_{\omega=0}^{q-2}\sum_{h=1}^{p-1}[\phi t^{\omega}\phi^{(p-h)}]^{1}\cdot\sum_{l=j-k+1}^{j} b^{\omega}_{j,l}\,e^{h}_{l,j-k}$$

$$+\ \phi\cdot\sum_{\omega=0}^{q-2}\{\sum_{\nu=0}^{q-2-\omega}[\phi t^{\omega}\phi t^{\nu}]^{1}+\phi\cdot\sum_{\nu=q-1-\omega}^{q-2}[\phi t^{\omega}\phi t^{\nu}]^{3}\}\,(\sum_{l=j-k+1}^{j} b^{\omega}_{j,l}\,e^{p+\nu}_{l,j-k})$$

$$+\ \phi\cdot\phi\cdot\sum_{\nu=0}^{q-2}t^{\nu}\,e^{p+\nu}_{j+1,j-k}$$

$$(j=2,3,\dots,n-1;k=1,2,\dots,j-1),$$

$$(1.27\alpha)\qquad \sum_{h=0}^{p-1}\{[\phi'\phi^{(p-h)}]^{0}+(q-1)\cdot\sum_{m=0}^{p-1}\sigma^{m}\,[\phi^{(p-m)}\phi^{(p-h)}]^{0}\}De^{h}_{n,n-k}$$

$$+\ \phi\ \{\sum_{h=0}^{p-1}\{[\phi'\phi^{(p-h)}]^{2}+(q-1)\cdot\sum_{m=0}^{p-1}\sigma^{m}\,[\phi^{(p-m)}\phi^{(p-h)}]^{2}\}De^{h}_{n,n-k}$$

$$+\sum_{\nu=0}^{q-2}[\phi t^{\nu}\phi']^{2}\,(De^{p+\nu}_{n,n-k}+e^{p+\nu}_{n,n-k})+\sum_{h=2}^{p-1}[\phi\,\phi^{(p+1-h)}]^{2}\,e^{h}_{n,n-k}$$

$$+\sum_{\nu=0}^{q-2}\sum_{m=0}^{p-1}\sigma^{m}\,[\phi t^{\nu}\phi^{(p-m)}]^{2}\,((q-1)De^{p+\nu}_{n,n-k}+\nu\,e^{p+\nu}_{n,n-k})\}$$

$$+\ \phi\cdot\sum_{h=0}^{p-1}\phi^{(p-h)}e^{h}_{n,\,n-k-1}+\sum_{m=0}^{p-1}\phi^{(p-m)}A^{m}_{k+1}$$

$$=\sum_{m=0}^{p-1}\phi^{(p-m)}\Big(\sum_{l=n-k}^{n}g^{m}_{n,\,l}\,e^{0}_{l,\,n-k}-\sigma^{m}\theta_{n-k}\,e^{0}_{n,\,n-k}\Big)$$

$$+\sum_{h=1}^{p-1}\sum_{m=0}^{p-1}[\phi^{(p-m)}\phi^{(p-h)}]^{0}\Big(\sum_{l=n-k+1}^{n}g^{m}_{n,\,l}\,e^{h}_{l,\,n-k}-\sigma^{m}\theta_{n-k}\,e^{h}_{n,\,n-k}\Big)$$

$$+\ \phi\ \Big\{\sum_{h=0}^{p-1}\sum_{m=0}^{p-1}[\phi^{(p-m)}\phi^{(p-h)}]^{2}\Big(\sum_{l=n-k+1}^{n}g^{m}_{n,\,l}\,e^{h}_{l,\,n-k}-\sigma^{m}\theta_{n-k}\,e^{h}_{n,\,n-k}\Big)$$

$$+\sum_{\nu=0}^{q-2}\sum_{m=0}^{p-1}[\phi t^{\nu}\phi^{(p-m)}]^{2}\Big(\sum_{l=n-k+1}^{n}(g^{m}_{n,\,l}\,e^{p+\nu}_{l,\,n-k}+b^{\nu}_{n,\,l}\,e^{m}_{l,\,n-k})$$

$$-\sigma^{m}\theta_{n-k}\,e^{p-\nu}_{n,\,n-k}\Big)$$

$$+\sum_{\omega=0}^{q-2}\sum_{\nu=0}^{q-2}[\phi t^{\omega}\phi t^{\nu}]^{2}\Big(\sum_{l=n-k+1}^{n}b^{\omega}_{n,\,l}\,e^{p+\nu}_{l,\,n-k}\Big)\Big\}$$

$$+\ \phi\cdot\sum_{h=0}^{p-1}\phi^{(p-h)}\cdot\sum_{l=n-k+1}^{n}\alpha_{n+1-l}\,e^{h}_{l,\,n-k}\ ,$$

$$(1.27\beta)\qquad \phi\cdot\sum_{h=0}^{p-1}\{[\phi'\phi^{(p-h)}]^{1}+(q-1)\cdot\sum_{m=0}^{p-1}\sigma^{m}\,[\phi^{(p-m)}\phi^{(p-h)}]^{1}\}De^{h}_{n,\,n-k}$$

$$+\ \phi\cdot\sum_{\nu=0}^{q-2}[\phi t^{\nu}\phi']^{1}\,(De^{p+\nu}_{n,\,n-k}+e^{p+\nu}_{n,\,n-k})$$

$$+\ \phi\cdot\sum_{\nu=0}^{q-2}\sum_{m=0}^{p-1}\sigma^{m}\,[\phi t^{\nu}\phi^{(p-m)}]^{1}\,((q-1)De^{p+\nu}_{n,\,n-k}+\nu\,e^{p+\nu}_{n,\,n-k})$$

$$+\ \phi\ \ \{p!e^{1}_{n,\,n-k}+\sum_{h=2}^{p-1}[\phi\,\phi^{(p+1-h)}]e^{h}_{n,\,n-k}\}$$

$$+\ \phi\cdot\phi\cdot\sum_{\nu=0}^{q-2}t^{\nu}e^{p+\nu}_{n,\,n-k-1}+\phi\cdot\sum_{\omega=0}^{q-2}t^{\omega}A^{p+\omega}_{k+1}$$

$$=\phi\cdot\sum_{h=0}^{p-1}\sum_{m=0}^{p-1}[\phi^{(p-m)}\phi^{(p-h)}]^{1}\Big(\sum_{l=n-k+1}^{n}g^{m}_{n,\,l}\,e^{h}_{l,\,n-k}-\sigma^{m}\theta_{n-k}\,e^{h}_{n,\,n-k}\Big)$$

$$+\ \phi\cdot\sum_{\nu=0}^{q-2}\sum_{m=0}^{p-1}[\phi t^{\nu}\phi^{(p-m)}]^{1}\Big(\sum_{l=n-k+1}^{n}g^{m}_{n,\,l}\,e^{p+\nu}_{l,\,n-k}-\sigma^{m}\theta_{n-k}\,e^{p+\nu}_{n,\,n-k}\Big)$$

$$+\ \phi\cdot\sum_{\omega=0}^{q-2}t^{\omega}\sum_{l=n-k}^{n}b^{\omega}_{n,\,l}\,e^{0}_{l,\,n-k}$$

$$+\ \phi\cdot\sum_{\omega=0}^{q-2}\sum_{h=1}^{p-1}[\phi t^{\omega}\ \phi^{(p-h)}]^{1}\cdot\sum_{l=n-k+1}b^{\omega}_{n,\,l}\,e^{h}_{l,\,n-k}$$

$$+\ \phi\cdot\sum_{\omega=0}^{q-2}\Big\{\sum_{\nu=0}^{q-2-\omega}[\phi t^{\omega}\phi t^{\nu}]^{1}+\phi\cdot\sum_{\nu=q-1-\omega}^{q-2}[\phi t^{\omega}\phi t^{\nu}]^{3}\Big\}\Big(\sum_{l=n-k+1}^{n}b^{\omega}_{n,\,l}\,e^{p+\nu}_{l,\,n-k}\Big)$$

$$+\ \phi\cdot\phi\cdot\sum_{\nu=0}^{q-2}t^{\nu}\cdot\sum_{l=n-k+1}^{n}\alpha_{n+1-l}\,e^{p+\nu}_{l,\,n-k}$$

$$(k=1,2,\dots,n-1).$$

We interprete $e^h_{j,k} = 0$ $(k \leq 0)$. From these relations we can easily observe that the k-th subdiagonal elements $\{e^h_{j,j-k}(\phi); h=0, 1,\ldots,p-1\}$, $\{e^{p+\nu}_{j,j-k}(\phi)\ ; \nu=0, 1,\ldots,q-2\}$ are taken as polynomials in ϕ^{-1} of degree k with no constant term.

2. We are now in a position to determine constants $c^i_{j,k}$, $b^\omega_{j,k}$ and all coefficients of polynomials $e^h_{j,k}(\phi)$ in ϕ^{-1} uniquely. To determine these, we use a lemma of M.Kohno [1]:

Lemma. *Let $\eta^1_0, \eta^2_0,\ldots,\eta^N_0$ be known constants such that*

$$[\rho]_N + \eta^1_0 [\rho]_{N-1} + \cdots\cdots + \eta^N_0 = \prod_{l=1}^{N} (\rho - \rho_l) \tag{2.1}$$

$$([\rho]_p = \rho(\rho-1)\cdots\cdots(\rho-p+1)).$$

Let μ be an unknown variable which satisfies the relations

$$\begin{cases} \xi^1_0 = (\mu-(n-1)) + \eta^1_0 \\ \xi^k_0 = (\mu-(n-1))\,\xi^{k-1}_0 + \eta^k_0 \qquad (k=2,3,\ldots,n-1) \\ 0 = \mu\,\xi^{N-1}_0 + \eta^N_0 \ . \end{cases} \tag{2.2}$$

Then μ is equal to one of ρ_l, i.e., for instance, $\mu = \rho_N$, and there holds

$$\begin{vmatrix} \xi^1_0 + \rho - (N-2) & -1 & & 0 \\ \xi^2_0 & \rho-(N-3) & -1 & \\ \vdots & & \ddots & -1 \\ \xi^{N-1}_0 & 0 & & \rho \end{vmatrix}$$

$$= [\rho]_{N-1} + \xi^1_0 [\rho]_{N-2} + \cdots\cdots + \xi^{N-1}_0 = \prod_{l=1}^{N-1} (\rho - \rho_l) \ .$$

Now, we shall begin with the determination of coefficients of $e^h_{n,n-k}(\phi)$ and $c^i_{n,n}$. Let us put

$$e^h_{n,n-k}(\phi) = \xi^h_k \phi^{-k} + \cdots\cdots \qquad (k=1,2,\ldots,n-1; h=0,1,\ldots,p-1). \tag{2.4}$$

Then, from (1.25α), we immediately obtain

$$a^h_{1,0} + \xi^h_1 = g^h_{n,n} - \sigma^h \theta_n \qquad (h=0,1,\ldots,p-1). \tag{2.5}$$

Substituting (2.4) into (1.27α) and equating coefficients of the power ϕ^{-k} attached to

$\phi^{(p-u)}$ $(u=0,1,\dots p-1)$ in both sides, we have

$$(2.6)\qquad -k\cdot\sum_{h=0}^{p-1}\{d^u_{p-1,h}+(q-1)\cdot\sum_{m=0}^{p-1}\sigma^m d^u_{m,h}\}\xi^h_k+\xi^u_{k+1}+a^u_{k+1,0}$$
$$=\sum_{h=0}^{p-1}\sum_{m=0}^{p-1}d^u_{m,h}(g^m_{n,n}-\sigma^m\theta_{n-k})\xi^h_k$$
$$(k=1,2,\dots,n-1),$$

where $\xi^h_n=0$ $(h=0,1,\dots p-1)$.

Here we use the following relations:

$$(2.7)\qquad \sum_{u=0}^{p-1}d^u_{m,h}\phi^{(p-m)}(t_i)=[\phi^{(p-m)}\phi^{(p-h)}]^0{}_{t=t_i}=\phi^{(p-m)}(t_i)\phi^{(p-h)}(t_i)$$

$$\sum_{m=0}^{p-1}\sigma^m\phi^{(p-m)}(t_i)=\{\phi t^{-1}\}_{t=t_i}=\begin{cases}0 & (i\neq p)\\ \phi'(t_p) & (i=p)\ .\end{cases}$$

Multiplying both sides of (2.6) by $\phi^{(p-u)}(t_i)$, adding to about u $(=0,1,\dots p-1)$, and dividing by $\{\phi'(t_i)\}^{k+1}$, we can rewrite (2.6) in the form

$$\eta^i_{k+1}=(\mu^i+k)\eta^i_k-\sum_{h=0}^{p-1}a^h_{k+1,0}\phi^{(p-h)}(t_i)\{\phi'(t_i)\}^{-(k+1)}\qquad(i\neq p).$$

$$\eta^p_{k+1}=(\mu^p+(n-1))q-(n-k-1))\eta^p_k-\sum_{h=0}^{p-1}a^h_{k+1,0}\phi^{(p-h)}(t_p)\{\phi'(t_p)\}^{-(k+1)}$$
$$(i=p)$$

where

$$(2.8)\qquad \mu^i=\{\sum_{m=0}^{p-1}g^m_{n,n}\phi^{(p-m)}(t_i)\}\{\phi'(t_i)\}^{-1}\qquad (i=1,2,\dots p)$$
$$\eta^i_k=\{\sum_{h=0}^{p-1}\xi^h_k\phi^{(p-h)}(t_i)\}\{\phi'(t_i)\}^{-k}\ .$$

Similarly, we can rewrite (2.5) as follows:

$$\eta^i_1=\mu^i-\{\sum_{h=0}^{p-1}a^h_{1,0}\phi^{(p-h)}(t_i)\}\{\phi'(t_i)\}^{-1}\qquad(i\neq p),$$

$$\eta^p_1=\mu^p+(n-1)q-(n-1))-\{\sum_{h=0}^{p-1}a^h_{1,0}\phi^{(p-h)}(t_p)\}\{\phi'(t_p)\}^{-1}$$
$$(i=p).$$

Hence we have obtained

$$(2.9)\qquad \eta^i_1=\mu^i-\{\sum_{h=0}^{p-1}a^h_{1,0}\phi^{(p-h)}(t_i)\}\{\phi'(t_i)\}^{-1}$$

$$\eta^i_{k+1}=(\mu^i+k)\eta^i_k-\sum_{h=0}^{p-1}a^h_{k+1,0}\phi^{(p-h)}(t_i)\{\phi'(t_i)\}^{-(k+1)}$$
$$(i=1,2,\dots,p-1;k=1,2,\dots n-1)$$

and

$$(2.10)\quad \begin{aligned} \eta_1^p &= (\mu^p+(n-1)q-(n-1)) - \{\sum_{h=0}^{p-1} a_{1,0}^h \phi^{(p-h)}(t_p)\}\{\phi'(t_p)\}^{-1} \\ \eta_{k+1}^p &= (\mu^p+(n-1))q-(n-k-1))\,\eta_k^p - \sum_{h=0}^{p-1} a_{k+1,0}^h \phi^{(p-h)}(t_p)\{\phi'(t_p)\}^{-(k+1)} \\ &\qquad (k=1,2,\dots,n-1), \end{aligned}$$

where

$$\eta_n^i = 0 \quad (i=1,2,\dots,p).$$

Applying the lemma in the beginning of section 2 to (2.9) and (2.10), we can see that $\mu^i+(n-1)$ $(i=1,1,\dots,p-1)$ and $\mu^p+(n-1)q$ are equal to the same of the characteristic constants ρ_j^i $(i=1,2,\dots p-1)$ and ρ_j^p $(j=1,2,\dots n)$, respectively. Therefore, putting $\mu^i=\rho_n^i-(n-1)$ $(i=1,2,\dots p-1)$ and $\mu^p=\rho_n^p-(n-1)q$, we can determine all values ξ_k^h in (2.4) uniquely by means of (2.8-10). Then we have $c_{n,n}^i=\mu^i$ $(i=1,2,\dots,p)$.

Next we set

$$(2.11)\quad e_{n-1,n-1-k}^h(\phi) = \zeta_k^h\,\phi^{-k} + \cdots\cdots \quad (k=1,2,\dots,n-2;\,h=0,1,\dots p-1)$$

and determine ζ_k^h and $c_{n-1,n-1}^i$ $(i=1,2,\dots,p)$. It follows from (1.24α) that

$$(2.12)\quad \zeta_1^h = g_{n-1,n-1}^h - \sigma^h\theta_{n-1} + \xi_1^h \quad (h=0,1,\dots,p-1).$$

Substituting (2.11) into (1.26α) and equating coefficients of the power ϕ^{-k} attached to $\phi^{(p-u)}$ $(u=0,1,\dots,p-1)$ in both sides, we have

$$(2.13)\quad \begin{aligned} &-k\cdot\sum_{h=0}^{p-1}\{d_{p-1,h}^u + (q-1)\cdot\sum_{m=0}^{p-1}\sigma^m d_{m,h}^u\}\cdot\zeta_k^h + \zeta_{k+1}^u \\ &= \sum_{h=0}^{p-1}\sum_{m=0}^{p-1} d_{m,h}^u\,(g_{n-1,n-1}^m - \sigma^m\theta_{n-1-k})\cdot\zeta_k^h + \xi_{k+1}^h \\ &\qquad (k=1,2,\dots,n-2), \end{aligned}$$

where $\zeta_{n-1}^h=0$ $(h=0,1,\dots,p-1)$.
Then, putting again

$$(2.14)\quad \begin{aligned} \widehat{\mu}^i &= \{\sum_{m=0}^{p-1} g_{n-1,h-1}^m\,\phi^{(p-m)}(t_i)\}\{\phi'(t_i)\}^{-1} \\ \gamma_k^i &= \{\sum_{h=0}^{p-1}\zeta_k^h\,\phi^{(p-h)}(t_i)\}\{\phi'(t_i)\}^{-k}. \end{aligned}$$

we can rewrite (2.12) and (2.13) as follows:

$$(2.15)\quad \begin{cases} \gamma_1^i = \widehat{\mu}^i + \eta_1^i \\ \gamma_{k+1}^i = (\widehat{\mu}^i + k)\,\gamma_k^i + \eta_{k+1}^i \end{cases}$$

$$(i=1,2,\dots,p-1;\,k=1,2,\dots,n-2),$$

$$(2.16)\qquad \begin{cases} \gamma_1^q = \widehat{\mu}^p + (n-2)q - (n-2)) + \eta_1^q \\ \gamma_{k+1}^q = (\widehat{\mu}^p + (n-2)q - (n-k-2))\gamma_k^q + \eta_{k+1}^q \qquad (k=1,2,\dots,n-2), \end{cases}$$

where $\gamma_{n-1}^i = 0$ $(i=1,2,\dots,p)$.
Since the lemma yields that

$$(2.17)\qquad [\rho]_{n-1} + \sum_{l=1}^{n-1} \eta_l^i [\rho]_{n-1-l} = \prod_{j=1}^{n-1} (\rho - \rho_j^i) \qquad (i=1,2,\dots,\rho),$$

we again apply the lemma to (2.15) and (2.16) and we can see that $\widehat{\mu}^i + (n-2)$ $(i=1, 2,\dots,\rho-1)$ and $\widehat{\mu}^p + (n-2)q$ are equal to some of ρ_j^i $(j=1,2,\dots,p-1)$ an ρ_j^p $(j=1,2,\dots, n-1)$, respectively. Furthermore, putting $\widehat{\mu}^i = \rho_{n-1}^i - (n-2)$ $(i=1,2,\dots,p-1)$ and $\widehat{\mu}^p = \rho_{n-1}^p - (n-2)q$, we can know that all values ζ_k^h are determined uniquely and then we have $c_{n-1,n-1}^i = \widehat{\mu}^i$ $(i=1,2,\dots,p)$. Moreover we can proceed to the determination of $c_{j,j}^i$ $(i= 1,2,\dots,p)$ and the coefficients of

$$e_{j,j-k}^h = \zeta_k^h \phi^{-k} + \cdots \qquad (k=1,2,\dots,j-1; h=0,1,\dots,p-1)$$

for $j=n-2,n-3,\dots,1$ in the same manner, and then we can put

$$(2.18)\qquad \begin{cases} c_{j,j}^i = \{\sum_{m=0}^{p-1} g_{j,j}^m \phi^{(p-m)}(t_i)\}\{\phi'(t_i)\}^{-1} = \rho_j^i - (j-1) \qquad (i=1,2,\dots,p-1) \\ c_{j,j}^p = \{\sum_{m=0}^{p-1} g_{j,j}^m \phi^{(p-m)}(t_i)\}\{\phi'(t_p)\}^{-1} = \rho_j^p - (j-1)q\ . \end{cases}$$

We shall begin with the determination of the coefficients $\xi_k^{p+\nu}$ $(e_{n,n-k}^{p+\nu}(\phi) = \xi_k^{p+\nu} \phi^{-k} + \cdots$ $(k=1,2,\dots,n-1;\ \nu=0,1,\dots,q-2))$ and $b_{n,n}^\nu$ $(\nu=0,1,\dots,q-2)$.

If we put

$$(2.19)\qquad e_{n,n-k}^{p+\nu}(\phi) = \xi_k^{p+\nu}\phi^{-k} + \cdots \qquad (k=1,2,\dots,n-1;\ \nu=0,1,\dots,q-2).$$

We immediately obtain from (1.25β) that

$$(2.20)\qquad a_{1,0}^{q+\nu} + \xi_1^{q+\nu} = b_{n,n}^\nu \qquad (\nu=0,1,\dots,q-2),$$

and then (1.27β) yields that

$$\frac{-k\cdot\phi\cdot\sum_{h=0}^{p-1}\{[\phi'\phi^{(p-h)}]' + (q-1)\cdot\sum_{m=0}^{p-1}\sigma^m[\phi^{(p-m)}\ \phi^{(p-h)}]'\}\xi_k^h}{-(k-1)\cdot\phi\cdot\sum_{\nu=0}^{q-2}[\phi t^\nu \phi']'\ \xi_k^{p+\nu}}$$

$$
\begin{aligned}
&+\ \phi\cdot\sum_{\nu=0}^{q-2}\{\nu-k(q-1)\}\cdot\sum_{m=0}^{p-1}\sigma^m\ [\phi t^\nu \phi^{(p-m)}]^1\ \xi_k^{p+\nu} \\
&+\ \phi\{p!\,\xi_k^1+\sum_{h=2}^{p-1}[\phi\,\phi^{(p+1-h)}]\,\xi_k^h\} \\
&+\ \phi\cdot\sum_{\nu=0}^{q-2} t^\nu\,\xi_{k+1}^{p+\nu}+\underline{\phi\cdot\sum_{\omega=0}^{q-2} t^\omega a_{k+1,0}^{p+\omega}} \\
=\ &\underline{\phi\cdot\sum_{h=0}^{p-1}\sum_{m=0}^{p-1}[\phi^{(p-m)}\,\phi^{(p-h)}]^1\ (g_{n,n}^m-\sigma^m\theta_{n-k})\,\xi_k^h} \\
&+\ \phi\cdot\sum_{\nu=0}^{p-1}\sum_{m=0}^{p-1}[\phi t^\nu\phi^{(p-m)}]^1\ (g_{n,n}^m-\sigma^m\theta_{n-k})\,\xi_k^{p+\nu} \\
&+\ \phi\cdot\sum_{\omega=0}^{q-2}\sum_{h=0}^{p-1}[\phi t^\omega\phi^{(p-h)}]^1\ b_{n,n}^\omega\,\xi_k^h \\
&+\ \phi\cdot\sum_{\omega=0}^{q-2}\sum_{\nu=0}^{q-2-\omega}[\phi t^\omega\phi t^\nu]^1\ b_{n,n}^\omega\,\xi_k^{p+\nu}\ ,
\end{aligned}
$$

where under-line parts are known. Comparing coefficients attached to ϕt^ν $(\nu=0,1,\ldots,q-2)$, we have

$$
\begin{aligned}
&-(k-1)\cdot\sum_{\omega=0}^{\nu}\varepsilon_{\omega,p-1}^{\nu}\,\xi_k^{p+\omega}+\sum_{m=0}^{p-1}\sigma^m\cdot\sum_{\omega=0}^{\nu}\{\omega-k(q-1)\}\,\varepsilon_{\omega,m}^{\nu}\,\xi_k^{p+\omega}+\xi_{k+1}^{p+\nu} \\
(2.21)\qquad &=\sum_{m=0}^{p-1}(g_{n,n}^m-\sigma^m\ \theta_{n-k})\cdot\sum_{\omega=0}^{\nu}\varepsilon_{\omega,m}^{\nu}\,\xi_k^{p+\omega}\ \sum_{h=0}^{p-1}\sum_{\omega=0}^{\nu}\varepsilon_{\omega,m}^{\omega}\ b_{n,n}^{\omega}\ \xi_k^h \\
&\quad+\sum_{\omega=0}^{\nu}\sum_{u=0}^{\nu-\omega}f_{\omega,u}^{\nu}\ b_{n,n}^{\omega}\ \xi_k^{p+u}+\ \textit{known values.}
\end{aligned}
$$

Let the set $\{\xi_k^{p+\omega},b_{n,n}^\omega\}$ $(k=1,2,\ldots,n-1)$ be known for $\omega=0,1,\ldots,\nu-1$. Since $f_{\nu,0}^{\nu}=0$ and $\varepsilon_{\nu,h}^{\nu}=\phi^{(p-h)}(t_p)$, we have

$$
\begin{pmatrix}
\phi'(t_p)\eta_1^p+\phi'(t_p)(\rho_n^p-(\nu+1)-(n-2)) & -1 & & \ddots & \\
\{\phi'(t_p)\}^n\eta_2^p & \phi'(t_p)(\rho_n^p-(\nu+1)-(n-3)) & \ddots & & 0 \\
\vdots & & \ddots & -1 & \\
\vdots & 0 & & \ddots & -1 \\
\{\phi'(t_p)\}^{n-1}\eta_{n-1}^p & & & & \phi'(t_p)(\rho_n^p-(\nu+1))
\end{pmatrix}
\begin{pmatrix}
\xi_1^{p+\nu} \\ \xi_2^{p+\nu} \\ \vdots \\ \vdots \\ \xi_{n-1}^{p+\nu}
\end{pmatrix}
\tag{2.22}
$$

$$= \textit{known values.}$$

Using the lemma, we see that the determinant of the matrix in the left hand side of the above formula is equal to

$$\{\phi'(t_p)\}^{n-1}\{[\rho_n^p-(\nu+1)]_{n-1}+\eta_1^p[\rho_n^p-(\nu+1)]_{n-2}+\cdots\cdots+\eta_{n-1}^p\}$$

$$=\{\phi'(t_p)\}^{n-1}\cdot\prod_{j=1}^{n-1}(\rho_n^p-(\nu+1)-\rho_j^p),$$

which is non-vanishing from (1.3). Thus we have determined $\xi_k^{p+\nu}$ and $b_{n,n}^{\nu}$ ($k=1,2,\dots,n-1$; $\nu=0,1,\dots,q-2$) uniquely.

Next we set

$$e_{n-1,n-1-k}^{p+\nu}(\phi)=\zeta_k^{p+\nu}\phi^{-k}+\cdots\cdots \qquad (k=1,2,\dots,n-2;\ \nu=0,1,\dots,q-2) \tag{2.23}$$

and determine $\zeta_k^{p+\nu}$ ($k=1,2,\dots,n-2$) and $b_{n-1,n-1}^{\nu}$ ($\nu=0,1,\dots,q-2$). It follows from (1.24 β) that

$$\zeta_1^{p+\nu}=b_{n-1,n-1}^{\nu}+\xi_1^{p+\nu} \qquad (\nu=0,1,\dots,q-2). \tag{2.24}$$

Since $\xi_k^{p+\nu}$ ($\nu=0,1,\dots,q-2$; $k=1,2,\dots,n-1$) and ζ_k^h ($h=0,1,\dots,p-1$; $k=1,2,\dots,n-2$) are known, (1.26β) yields that

$$(-k-1)\cdot\phi\cdot\sum_{\nu=0}^{q-2}[\phi t^{\nu}\phi']\zeta_k^{p+\nu}+\phi\cdot\sum_{\nu=0}^{q-2}\{\nu-k(q-1)\}\cdot\sum_{m=0}^{p-1}\sigma^m[\phi t^{\nu}\phi^{(p-m)}]^1\zeta_k^{p+\nu}$$
$$+\ \phi\cdot\sum_{\nu=0}^{q-2}t^{\nu}\zeta_{k+1}^{p+\nu}$$
$$=\phi\cdot\sum_{\nu=0}^{q-2}\sum_{m=0}^{p-1}[\phi t^{\nu}\phi^{(p-m)}]^1(g_{n-1,n-1}^m-\sigma^m\theta_{n-1-k})\zeta_k^{p+\nu}$$
$$+\ \phi\cdot\sum_{\omega=0}^{q-2}\sum_{h=0}^{p-1}[\phi t^{\omega}\phi^{(p-h)}]^1 b_{n-1,n-1}^{\omega}\zeta_k^h$$
$$+\ \phi\cdot\sum_{\omega=0}^{q-2}\sum_{\nu=0}^{q-2-\omega}[\phi t^{\omega}\phi t^{\nu}]^1 b_{n-1,n-1}^{\omega}\zeta_k^{p+\nu}+\textit{known values}.$$

Furthermore, comparing coefficients attached to ϕt^{ν} ($\nu=0,1,\dots,q-2$), we have

$$-(k-1)\cdot\sum_{\omega=0}^{\nu}\varepsilon_{\omega,p-1}^{\nu}\zeta_k^{p+\omega}+\sum_{m=0}^{p-1}\sigma^m\cdot\sum_{\omega=0}^{\nu}\{\omega-k(q-1)\}\varepsilon_{\omega,m}^{\nu}\zeta_k^{p+\omega}+\zeta_{k+1}^{p+\nu}$$
$$=\sum_{m=0}^{p-1}(g_{n-1,n-1}^m-\sigma^m\theta_{n-1-k})\cdot\sum_{\omega=0}^{\nu}\varepsilon_{\omega,m}^{\nu}\zeta_k^{p+\omega}$$
$$+\sum_{h=0}^{p-1}\cdot\sum_{\omega=0}^{\nu}\varepsilon_{\omega,h}^{\nu}b_{n-1,n-1}^{\omega}\zeta_k^h+\sum_{\omega=0}^{\nu}\cdot\sum_{u=0}^{\nu-\omega}f_{\omega,u}^{\nu}b_{n-1,n-1}^{\omega}\zeta_k^{p+u}$$
$$+\ \textit{known values}.$$

Let the set $\{\zeta_k^{p+\omega},\ b_{n-1,n-1}^{\omega}\}$ ($k=1,2,\dots,n-2$) be known for $\omega=0,1,\dots,\nu-1$. Then we have

$$\begin{bmatrix} \phi'(t_p)\gamma_1^p + \phi'(t_p)\,(\rho_{n-1}^p-(\nu+1)-(n-3)) & -1 & & \\ \{\phi'(t_p)\}^2\,\gamma_2^p & \phi'(t_p)(\rho_{n-1}^p-(\nu+1)-(n-4)) & -1 & 0 \\ \vdots & 0 & \ddots & -1 \\ \{\phi'(t\ )\}^{n-1}\,\gamma_{n-2}^p & & & \phi'(t_p)(p_{n-1}^p-\ (\nu+1)) \end{bmatrix} \begin{bmatrix} \zeta_1^{p+\nu} \\ \zeta_2^{p+\nu} \\ \vdots \\ \zeta_{n-2}^{p+\nu} \end{bmatrix}$$

(2.25)

$$= \textit{known values}.$$

Again, from the lemma, we can see that the determinant of the matrix in the left hand side is equal to

$$\{\phi'(t_p)\}^{n-2}\,\{[\rho_{n-1}^p-(\nu+1)]_{n-2}+\gamma_1^p\,[\rho_{n-1}^p-(\nu+1)]_{n-3}+\cdots\cdots+\gamma_{n-2}^p\}$$

$$=\{\phi'(t_p)\}^{n-2}\cdot\prod_{j=1}^{n-2}(\rho_{n-1}^p-(\nu+1)-\rho_j^p)\neq 0.$$

Thus we have determined $\zeta_k^{p+\nu}$ and $b_{n-1,n-1}^{\nu}$ $(k=1,2,\ldots,n-2;\ \nu=0,1,\ldots,q-2)$ uniquely.

Moreover we can proceed to the determination of $b_{j,j}^{\nu}$ $(\nu=0,1,\ldots,q-2)$ and the coefficients

$$e_{j,j-k}^{p+\nu}=\zeta_k^{p+\nu}\phi^{-k}+\cdots\cdots \qquad (k=1,2,\ldots,j-1;\ \nu=0,1,\ldots,q-2)$$

for $j=n-2,\ n-3,\ldots,1$ in the same manner. From now on, we regard these constants as known values.

We now return to (1.27α) and determine $c_{n,n-1}^{i}$ $(i=1,2,\ldots,p)$ and the coefficients x_k^h. Here we denote

$$(2.26)\qquad e_{n,n-k}^{h}(\phi)=\xi_k^h\phi^{-k}+x_{k-1}^h\phi^{-k+1}+\cdots\cdots$$
$$(k=2,3,\ldots,n-1;h=0,1,\ldots,p-1).$$

Substituting $k=1$ into (1.27α) and comparing the expression included $\phi^{(p-h)}$ $(h=0,1,\ldots,p-1)$ in both sides, we have

$$-\{\sum_{h=0}^{p-1}[\phi'\phi^{(p-h)}]+(q-1)\cdot\sum_{m=0}^{p-1}\sigma^m\,[\phi^{(p-m)}\ \ \phi^{(p-h)}]^2\}\,\xi_1^h$$

$$+\sum_{\nu=0}^{q-2}(\nu-q+1)\cdot\sum_{m=0}^{p-1}\sigma^m\,[\phi t^\nu\phi^{(p-m)}]^2\xi_1^{p+\nu}+\sum_{h=2}^{p-1}[\phi\,\phi^{(p+1-h)}]^2\xi_1^h$$

$$
(2.27)\quad
\begin{aligned}
&+\underline{\sum_{h=0}^{p-1}\phi^{(p-h)}x_1^h+\sum_{m=0}^{p-1}\phi^{(p-m)}a_{2,1}^m}\\
&=\sum_{m=0}^{p-1}\phi^{(p-m)}g_{n,n-1}^m+\underline{\sum_{h=0}^{p-1}\sum_{m=0}^{p-1}[\phi^{(p-m)}\phi^{(p-h)}]^2(g_{n,n}^m-\sigma^m\theta_{n-1})\xi_1^h}\\
&+\underline{\sum_{\nu=0}^{q-2}\sum_{m=0}^{p-1}[\phi t^\nu\phi^{(p-m)}]^2\{g_{n,n}^m-\sigma^m\theta_{n-1})\xi_1^{p+\nu}+b_{n,n}^\nu\xi_1^m\}}\\
&+\underline{\sum_{\omega=0}^{q-2}\sum_{\nu=0}^{q-2}[\phi t^\omega\phi t^\nu]^2 b_{n,n}^\omega\xi_1^{p+\nu}}
\end{aligned}
$$

Here under-line parts are known values. From above relation, equating coefficients attached to $\phi^{(p-h)}$ in both sides, we have

$$(2.28)\qquad x_1^h=g_{n,n-1}^h+\textit{known values}\quad(h=0,1,\ldots p-1).$$

We here put

$$(2.29)\qquad \tau_k^i=\{\sum_{h=0}^{p-1}x_k^h\,\phi^{(p-h)}(t_i)\}\{\phi'(t_i)\}^{-k}\qquad(i=1,2,\ldots,p),$$

and rewrite (2.28) as follows:

$$(2.30)\qquad \tau_1^i=\{\sum_{h=0}^{p-1}g_{n,n-1}^h\,\phi^{(p-h)}(t_i)\}\{\phi'(t_i)\}^{-1}+\textit{known values}\quad(i=1,2,\ldots,p).$$

Substituting (2.26) into (1.27α) and equating coefficients of the power ϕ^{-k+1} attached to $\phi^{(p-u)}$ $(u=0,1,\ldots p-1)$ in both sides, we have

$$
(2.31)\quad
\begin{aligned}
&-(k-1)\cdot\sum_{h=0}^{p-1}\{d_{p-1,h}^u+(q-1)\cdot\sum_{m=0}^{p-1}\sigma^m d_{m,h}^u\}x_{k-1}^h+x_k^u\\
&=\sum_{h=0}^{p-1}\sum_{m=0}^{p-1}d_{m,h}^u\{(g_{n,n}^m-\sigma^m\theta_{n-k})x_{k-1}^h+g_{n,n-1}^m\zeta_{k-1}^h\}\\
&\quad+\textit{known values}\qquad(k=2,3,\ldots,n-1).
\end{aligned}
$$

Using (2.29), we rewrite (2.3) as follows:

$$(2.32)\qquad \tau_j^i=\{\rho_n^i-1-(n-1-k)\}\tau_{k-1}^i+\tau_1^i\gamma_{k-1}^i+\textit{known values}\quad(i\neq p),$$

$$(2.33)\qquad \tau_k^p=\{\rho_n^p-q-(n-1-k)\}\tau_{k-1}^p+\tau_1^p\gamma_{k-1}^p+\textit{known values},$$

where

$$\tau_{n-1}^i=0\quad(i=1,2,\ldots,p).$$

Then, it follows from (2.32) and (2.33) that

$$\begin{bmatrix} \gamma_1^i+\rho_n^i-1-(n-3) & -1 & & & 0 \\ \gamma_2^i & \rho_n^i-1-(n-4) & -1 & & \\ \vdots & & \ddots & \ddots & \\ \vdots & 0 & & \ddots & -1 \\ \gamma_{n-2}^i & & & & \rho_n^i-1 \end{bmatrix} \begin{bmatrix} \tau_1^i \\ \tau_2^i \\ \vdots \\ \tau_{n-2}^i \end{bmatrix}$$

(2.34)

$$= \textit{known values} \qquad (i=1,2,\dots p-1),$$

and that

$$\begin{bmatrix} \gamma_1^p+\rho_n^p-q-(n-3) & -1 & & & 0 \\ \gamma_2^p & \rho_n^p-q-(n-4) & -1 & & \\ \vdots & & \ddots & \ddots & \\ \vdots & 0 & & \ddots & -1 \\ \gamma_{n-2}^p & & & & \rho_n^p-q \end{bmatrix} \begin{bmatrix} \tau_1^p \\ \tau_2^p \\ \vdots \\ \tau_{n-2}^p \end{bmatrix}$$

(2.35)

$$= \textit{known values} \quad .$$

We see the determinants of the coefficient matrices in (2.34) and (2.35) are equal to

$$[\rho_n^i-1]_{n-2}+\sum_{l=1}^{n-2}\gamma_l^i\,[\rho_n^i-1]_{n-2-l} = \prod_{j=1}^{n-2}(\rho_n^i-1-\rho_j^i) \qquad (i=1,2,\dots,p-1),$$

and

$$[\rho_n^p-q]_{n-2}+\sum_{l=1}^{n-2}\gamma_l^p\,[\rho_n^p-q]_{n-2-l} = \prod_{j=1}^{n-2}(\rho_n^p-q-\rho_j^p) \quad ,$$

respectively, which are non-vanishing. Hence $c_{n,n-1}^i$ $(i=1,2,\dots,p)$ and the coefficients x_k^h are determined uniquely.

Lastly we shall determine $b_{n,n-1}^\nu$ $(\nu=0,1,\dots,q-2)$ and the coefficients $x_k^{p+\nu}$. Here we denote

(2.36) $$e_{n,n-k}^{p+\nu}(\phi)=\xi_k^{p+\nu}\phi^{-k}+x_{k-1}^{p+\nu}\phi^{-k+1}+\cdots\cdots$$
$$(k=2,3,\dots,n-1;\ \nu=0,1,\dots,q-2).$$

Substituting $k=1$ into (1.27β) and equating coefficients attached to ϕt^ν in both sides, we have

(2.37) $$b_{n,n-1}^\nu = x_1^{p+\nu}+\alpha_{2,1}^{p+\nu}-f_{\omega,u}^{p+q+\nu-1}\,b_{n,n}^\omega\,\xi_1^{p+u} \qquad (\nu=0,1,\dots,q-2).$$

Furthermore, substituting (2.36) into (1.27β) and equating coefficients of the power ϕ^{-k+1} attached to ϕt^ν $(\nu=0,1,\dots,q-2)$ in both sides, we obtain

(2.38) $$-(k-2)\cdot\sum_{\omega=0}^{\nu}\varepsilon_{\omega,p-1}^{\nu}\,x_{k-1}^{p+\omega}+\sum_{m=0}^{p-1}\sigma^m\sum_{\omega=0}^{\nu}\{\omega-(k-1)(q-1)\}\,\varepsilon_{\omega,m}^{\nu}\,x_{k-1}^{p+\omega}+x_k^{p+\nu}$$

$$= \sum_{m=0}^{p-1} (g_{n,n}^{m} - \sigma^{m}\theta_{n-k}) \cdot \sum_{\omega=0}^{\nu} \varepsilon_{\omega,m}^{\nu}\, x_{k-1}^{p+\omega} + \sum_{h=0}^{n-1}\sum_{\omega=0}^{\nu} \varepsilon_{\omega,h}^{\nu}\, b_{n,n-1}^{\omega}\zeta_{k-1}^{h}$$

$$+ \sum_{\omega=0}^{\nu}\sum_{u=0}^{\nu-\omega} f_{\omega,u}^{\nu}\, (b_{n,n-1}^{\omega}\, \zeta_{k-1}^{p+u} + b_{n,n}^{\omega}\, x_{k-1}^{p+u}) + \textit{known values.}$$

Let the set $\{x_k^{p+\omega}, b_{n,n-1}^{\omega}\}$ $(k=1,2,\ldots,n-2)$ be known for $\omega=0,1,\ldots,\nu-1$. Nothing that $f_{\nu,0}^{\nu}=0$ and $\varepsilon_{\nu,h}^{\nu}=\phi^{(p-h)}(t_p)$, we get from (2.37) and (2.38) that

$$\begin{bmatrix}
\phi'(t_p)\gamma + \phi_1^{p}(t_p)(\rho_n^p - q-(\nu+1)-(n-3)) & -1 & & 0 \\
\{\phi'(t_p)\}^2\gamma_2^p & \phi'(t_p)(\rho_n^p-q-(\nu+1)-(n-4)) & \ddots & \\
\vdots & & \ddots & -1 \\
\{\phi'(t_p)\}^{n-2}\gamma_{n-2}^{p} & 0 & & \phi'(t_p)(\rho_n^p-q-(\nu+1))
\end{bmatrix}
\begin{bmatrix} x_1^{p+\nu} \\ x_2^{p+\nu} \\ \vdots \\ x_{n-2}^{p+\nu} \end{bmatrix}$$

(2.39) $\qquad = \textit{known values}\ .$

Similarly, it follows from the lemma that the determinant of the above matrix is equal to

$$\{\phi'(t_p)\}^{n-2}\, \{[\rho_n^p - q-(\nu+1)]_{n-2} + \gamma_1^p[\rho_n^p-q-(\nu+1)]_{n-3} + \cdots\cdots + \gamma_{n-2}^{p}\}$$

$$= \{\phi'(t_p)\}^{n-2} \cdot \prod_{j=1}^{n-2} (\rho_n^p - q-(\nu+1)-\rho_j^p)\ ,$$

which is non-vanishing. Therefore, we have determined $x_k^{p+\nu}$ and $b_{n,n-1}^{\nu}$ $(k=1,2,\ldots,n-1;\ \nu=0,1,\ldots q-2)$ uniquely.

Continuing the above procedure in succession, we can determine all $c_{j,k}^{i}$, $b_{j,k}^{\omega}$ and all the coefficients of polynomials $e_{j,k}^{h}(\phi)$ and $e_{j,k}^{p+\nu}(\phi)$. The complete proof of the validity will be done by mathematical inductin. We here omit the details.

We consequently obtain the following theorem:

Theorem. *Under the assumption that* $\rho_j^i - \rho_k^i \geq 0$ $(i=1,2,\ldots,\rho)$ *in the case* $\rho_j^i = \rho_k^i$ (mod *1*) $(j<k)$ *the single linear differential equation (1.1) can be reduced to the Schlesinger system (1.2) by the following linear transformation with rational functions*

as its coefficients,

$$Y=E(t)X=\{\sum_{h=0}^{p-1} E^{h}(\phi)\phi^{(p-h)}+\phi\cdot\sum_{\nu=0}^{q-2} E^{p+\nu}(\phi)t^{\nu}\}X .$$

And that, in (1.2)

$$B_{q-1}=\begin{bmatrix} 0 & 1 & & & \\ \vdots & & \ddots & \ddots & \\ 0 & & & 0 & 1 \\ a^{0}_{n,n} & a^{0}_{n-1,n-1} & \cdots\cdots & a^{0}_{2,2} & a^{0}_{1,1} \end{bmatrix} ,$$

whose eigenvalues are the characteristic constants λ_j ($j=1,2,\ldots,n$), and C_i ($i=1,2,\ldots p$) are triangular matrices, whose diagonal elements are given by $(\rho^{i}_{1}, \rho^{i}_{2}-1, \ldots, \rho^{i}_{n}-(n-1))$ $(i=1,2,\ldots p-1)$, $(\rho^{p}_{1}, \rho^{p}_{2}-q, \ldots, \rho^{p}_{n}-(n-1)q)$, *respectively.*

Appendix. In [1], M. Kohno reffered to the possibility of algeblaic manipulation of the reduction problem. We state here a program of an algebraic manipulation for a reduction of single linear differential equation to Birkhoff canonical systems by REDUCE on PC (Ver.3.2).

```
%%%%%%%        Birkhoff . RED          %%%%%%%%%%%%%%%%%%
%
%   A Simple reduction of Single Linear Differential
%       Equation to Birkhoff and Schlesinger's
%              Canonical Systems
%
%
%
%
   %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
ARRAY AA(N,Q*N);
 FOR I:=1:N DO
  BEGIN
    FOR K:=0:Q*N DO
      AA(I,K):=SUB(S=0,DF(A(I,S),S,K))
                /(FOR J:=1:K PRODUCT J):
  END;

OPERATOR P;
ARRAY KSI(N,N,Q,N);

%%  -JYO-   %%

 KSI(N,N-1,0,1):=P(N)-AA(1,0);
 FOR K:=1:N-1 DO
    KSI(N,N-(K+1),0,K+1)
      :=(P(N)-(N-K-1))*KSI(N,N-K,0,K)
             -AA(K+1,0);

 FOR I:=1:Q-1 DO
   BEGIN
     KSI(N,N-1,I,1):=U;
     FOR K:=1:N-1 DO
      KSI(N,N-(K+1),I,K+1)
        :=(P(N)-I-(N-K-1))*KSI(N,N-K,I,K)
            +(FOR L:=0:I-1 SUM
                KSI(N,N-K,L,K)*
                 (KSI(N,N-1,I-L,1)+AA(1,I-L)))
            -AA(K+1,I);
      KSI(N,N-1,I,1):=-SUB(U=0,KSI(N,0,I,N))
                       /(DF(KSI(N,0,I,N),U));
```

```
      CLEAR U;
    END;

%% -HA- %%

FOR J:=N-1 STEP -1 UNTIL 2 DO
    KSI(J,J-1,0,1):=P(J)-(J-1)+KSI(J+1,J,0,1);
FOR J:=N-1 STEP -1 UNTIL 2 DO
   BEGIN
     FOR K:=1:J-1 DO
        KSI(J,J-(K+1),0,K+1):=(P(J)-(J-K-1))*
                    KSI(J,J-K,0,K)
                       +KSI(J+1,J+1-(K+1),0,K+1);
   END;

FOR J:=N-1 STEP -1 UNTIL 2 DO
 BEGIN
   FOR I:=Q-1 DO
    BEGIN
      KSI(J,J-1,I,1):=U;
      FOR K:=1:J-1 DO
       KSI(J,J-(K+1),I,K+1)
          :=(P(J)-I-(J-K-1))*KSI(J,J-K,I,K)
            +(FOR L:=0:I-1 SUM
                  KSI(J,J-K,L,K)*(KSI(J,J-1,I-L,1)
                               -KSI(J+1,J,I-L,1)))
            +KSI(J+1,J-K,I,K+1);
       KSI(J,J-1,I,1):=-SUB(U=0,KSI(J,0,I,J))
                 /(DF(KSI(J,0,I,J),U));
     END;
 END;

%% -KYUU-  %%

ARRAY B(N,N);
    B(N,N):=AA(1,Q)*(S**Q)
            +(FOR L:=1:Q-1 SUM (KSI(N,N-1,L,1)
               +AA(1,L))*(S**L))
            +P(N)-(N-1)*Q $

    FOR J:=N-1:2 DO
       B(J,J):=P(J)-(J-1)*Q
```

```
            +(FOR L:=1:Q-1 SUM (KSI(J,J-1,L,1)
                +KSI(J+1,J,L,1))*S**L);

   FOR I:=1:N-1 DO
     B(I,I+1):=S**Q;

  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    PROCEDURE EE(I,J,K,S);
       EE(I,J,K,S):=(FOR L:=0:Q-1 SUM
            KSI(I,J,L,K)*(S**L));
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    PROCEDURE  BEI(I,J,K,M);
      BEGIN
        SCALAR PROC;
        LET S**Q=0 $
        PROC:=B(I,J)*EE(J,K,M,S);
        CLEAR S**Q;
        RETURN PROC;
      END;
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
     PROCEDURE BEO(I,J,K,M);
       BEGIN
        SCALAR PROCE;
        PROCE:=(B(I,J)*EE(J,K,M,S)
            -BEI(I,J,K,M)/(S**Q);
        RETURN PROCE;
      END;
  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

ARRAY BB(N,N-1,Q-1);
  FOR K:=1:N-1 DO
   BEGIN
     FOR R:=0:Q-1 DO
       BB(N,N-K,R)
         :=SUB(S=0,DF(EE(N,N-K,1,S),S,R))
           *(FOR I:=1:R PRODUCT I)
          +AA(K+1,Q*K+R)
          -SUB(S=0,DF((FOR L:=0:K-1 SUM
              BEO(N,N-L,N-K,1)),S,R));
```

```
    FOR L:=1:Q-1 DO
      BEGIN
        FOR M:=1:N-K-1 DO
          BEGIN
            KSI(N,N-K-1,L,1):=U;
            F:=(FOR I:=1:K-1 SUM
                   (BEI(N,N-I,N-K-M,M)
                     +BEO(N,N-I,N-K-M,M+1)))
                 +BEO(N,N,N-K-M,M+1):
            KSI(N,N-K-(M+1),L,M+1)
              :=(P(N)-K*Q-L-(N-K-M-1))
                   *KSI(N,N-K-M,L,M)
                 +BB(N,N-K,L)*KSI(N-K,N-K-M,0,M)
                 -AA(K+M+1,M*Q+L)
                 +(FOR I:=1:L SUM
                     KSI(N,N-K-M,L-I,M)
                        *(KSI(N,N-1,I,1)
                      +AA(1,I)))
                 +(FOR I:=1:L SUM
                     KSI(N-K,N-K-M,I,M)
                       *BB(N,N-K,L-I))
                 +SUB(S=0,DF(F,S,L))*
                      (FOR I:=1:L PRODUCT I);
            CLEAR F;
            KSI(N,N-K-1,L,1)
               :=-SUB(U=0,KSI(N,0,L,N-K))
                    /(DF(KSI(N,0,L,N-K),U));
          END;
        END;
    B(N,N-K):=(FOR R:=0:Q-1 SUM
               BB(N,N-K,R)*S**R)
                 +AA(K+1,Q*K+Q)*S**Q

  END;

%%  -KYUU-  %%

 FOR J:=N-1 STEP -1 UNTIL 2 DO
 BEGIN
  FOR K:=1:J-1 DO
   BEGIN
```

```
  FOR R:=0:Q-1 DO
   BEGIN
    G:=EE(J,J-K-1,1,S)-EE(J+1,J-K,1,S)
        -(FOR L:0:K-1 SUM BEO(J,J-L,J-K,1));
    BB(J,J-K,R):=SUB(S=0,DF(G,S,R));
    CLEAR G;
   END;

 FOR L:=1:Q-1 DO
   BEGIN
     FOR M:=1:J-K-1 DO
      BEGIN
       KSI(J,J-K-1,L,1):=V;
       H:=FOR I:=1:K-1 SUM
             (BEI(J,J-I,J-K-M,M)
                +BEO(J,J-I,J-K-M,M+1))
            +BEO(J,J,J-K-M,M+1)
            +EE(J+1,J-K-M,M+1,S);
       KSI(J,J-K-M,L,M+1)
         :=(P(J)-K*Q-L-(J-K-M-1))
              *KSI(J,J-K-M,L,M)
            BB(J,J-K,L)*KSI(J-K,JJ-K-M,0,M)
            +(FOR I:=1:L SUM
                 (KSI(J,J-1,L,1)+KSI(J+1,J,L,1))
                    *KSI(J,J-K-M,L-I,M))
            +(FOR I:=1:L SUM
                 KSI(J-K,J-K-M,I,M)*BB(J,J-K,L-I))
            SUB(S=0,DF(H,S,L))*
                (FOR I:=1:L PRODUCT I);
       CLEAR H;
       KSI(J,J-K-1,L,1)
          :=-SUB(V=0,KSI(J,0,L,J-K))
              /(DF(KSI(J,0,L,J-K),V));
      END;
     END;
    END;
   END;
END;
```

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References

[1] M.Kohno: A Simple Reduction of Single Linear Differential Equations to Birkoff and Schlesinger's Cannonical System, Kumamoto J.Math., 2(1989),9−27

[2] M.Kohno: A two point connection problem for general linear ordinary differential equations, Hiroshima Math.J., 4(1974), 293−338

[3] M.Kohno: A two point connection problem, Hiroshima Math.J., 4(1979),61−135

[4] M.Kohno: A multi-point connection problem, Lecture note in Math. **1015** Springer-Verlag, 1983, 136−171

[5] M.Kohno & T.Suzuki: Reduction of single Fuchsian differential equations to hypergeometric system, Kumamoto Journal of Science (Mathematics)**17**, No 2(1987),26−74

[6] H. L. Turrittin: Solvable related equation pertaining to turnig point problems, Asymptotic solutions of differential equations and their applications, by C.H.Wilcox, John Wiley,1964,27−52

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