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THE DETERMINATION OF A GENERAL SOLUTION FOR GAUSS DIFFERENTIAL EQUATION REPRESENTED BY RATIONAL FUNCTIONS

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1. Gauss differential equation is given by

$$x(1-x)\frac{d^2y}{dx^2}+(\gamma-(\alpha+\beta+1)x)\frac{dy}{dx}-\alpha\beta y=0, \tag{G}$$

where x is a complex independent variable; α, β and γ are complex parameters. In 1872, H. A. Schwarz determined, by means of the Kummer's solutions, in his famous paper (Über diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Funktion ihres vierten Elementes darstellt, Jour. für die reine und angew. Math.,75 (1872), pp.292-335), the form of a general solution of equation (G) generated by rational functions. However, the proof does not seem to be easy to understand. So, the author would like to reproduce his result in a readable way.

2. As the well known twenty four representations of particular solutions of equation (G) due to E. E. Kummer suggest, we see that a rational function solution of equation (G) is written as

$$x^a(1-x)^c g(x),$$

where $a\in Z$, $c\in Z$ and $g(x)$ is a polynomial in x. Here, Z denotes the set of integers. The exponent a is one of the characteristic roots of equation (G) at a regular singular point $x=0$ and the exponent c is one of those at a regular singular point $x=1$. Hence, we have

$$a=0 \text{ or } 1-\gamma, \qquad c=0 \text{ or } \gamma-\alpha-\beta.$$

The characteristic roots at a regular singular point $x=\infty$ are α and β.

Let n be the degree of the polynomial $g(x)$. Then, we must have either

$$\alpha=-a-c-n \quad \text{or} \quad \beta=-a-c-n.$$

Since equation (G) is symmetric with respect to the parameters α and β, we can assume without loss of generality that

$$\alpha=-a-c-n.$$

The possible combinations of the parameters a, c, α, β, γ are, as was done by Schwarz, given in the table below.

	a	c	α	β	γ
Case 1	0	0	$-n$	β	γ
Case 2	0	$\gamma-\alpha-\beta$	$-n-c$	$n+\gamma$	γ
Case 3	$1-\gamma$	0	$-n-a$	β	$1-a$
Case 4	$1-\gamma$	$\gamma-\alpha-\beta$	$-n-a-c$	$n+1$	$1-a$

In Case 2, the relation $\beta = n+\gamma$ is obtained as follows. Since $c = \gamma - \alpha - \beta$ and $\alpha = -n - c$, we have

$$\beta = \gamma - \alpha - c = \gamma - (-n - c) - c = n+\gamma.$$

In Case 4, the relation $\beta = n+1$ is derived as follows. Since $c = \gamma - \alpha - \beta$, $a = 1-\gamma$ and $\alpha = -n-a-c$, we have

$$\beta = \gamma - \alpha - c = (1-a) - (-n-a-c) - c = n+1.$$

3. The hypergeometric function $F(\alpha, \beta, \gamma; x)$ is defined for $x \in C \setminus \{0, 1\}$, where C denotes the set of complex numbers. For $|x|<1$, this function is defined as a holomorphic solution of equation (G) at $x=0$ and is represented by the uniformly convergent power series in x:

$$\text{(F)} \quad F(\alpha, \beta, \gamma; x) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\cdot \beta(\beta+1)\cdots(\beta+n-1)}{1\cdot 2\cdot \cdots \cdot n\cdot \gamma(\gamma+1)\cdots(\gamma+n-1)} x^n$$

provided that every coefficient takes on a finite value. If we substitute for y in equation (G) the power series in x:

$$\text{(3. 1)} \qquad y = 1 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots,$$

we find the recursive formula for the coefficients a_k's

$$\text{(3. 2)} \qquad (k+1)(k+\gamma) a_{k+1} = (k+\alpha)(k+\beta) a_k, \quad (a_0 = 1).$$

In this way, we have the power series (F) in generic.

We investigate a degenerated case such that the F is reduced to a polynomial in x.

(i) $\gamma \notin Z \setminus N$. *Here, N is the set of natural numbers. The power series* (F) *is always well defined. Moreover, if and only if at least one of the α and β is equal to a nonpositive integer, the power series* (F) *is reduced to a polynomial. When $\alpha = -n \in Z \setminus N$ and $\beta \notin Z \setminus N$ (or $\beta = -n' \in Z \setminus N$ and $\alpha \notin Z \setminus N$), the degree of this polynomial is equal to n (or n'). When $\alpha = -n \in Z \setminus N$ and $\beta = -n' \in Z \setminus N$, it is given by* $\min\{n, n'\}$.

This proposition will be obvious from recursive formula (3. 2).

(ii) *Assume that $\gamma = -n'' \in Z \setminus N$. If and only if either α and β belongs to a set of nonpositive integers $\{0, -1, \cdots, -n''\}$, the power series* (F) *has the meaning.*

(ii−1) *If $\alpha = -n$ $(0 \leqq n \leqq n'')$ and $\beta \notin Z \setminus N$, the power series* (F) *is reduced to a polynomial in x of degree n.*

In fact, recursive formula (3. 2) is given by

$$\text{(3. 3)} \qquad (k+1)(k-n'')a_{k+1} = (k-n)(k+\beta)a_k.$$

Hence, when $n<n''$, the a_{n+1} must be zero. In the case of $n=n''$, since the a_{n+1} can take on an arbitrary value, we put $a_{n+1} = 0$. Thus, for $n \leqq n''$, we have a polynomial solution of degree n, which is expressed by $F(-n, \beta, -n''; x)$.

(ii−2) *If $\beta = -n'$ $(0 \leqq n' \leqq n'')$ and $\alpha \notin Z \setminus N$, the power series* (F) *becomes a polynomial in x of degree n'.*

This proposition can be verified in quite a similar way.

(ii−3) *If $\alpha = -n$, $\beta = -n'$ and* $\min\{n, n'\} \leqq n''$, *the power series* (F) *is reduced to a polynomial in x of degree* $\min\{n, n'\}$.

Indeed, recursive formula (3. 2) is written as

(3. 4) $$(k+1)(k-n'')a_{k+1} = (k-n)(k-n')a_k, \ (a_0=1).$$

When $\min\{n, n'\} \leqq n''-1$, as can be easily verified from (3. 4), we have $a_k=0$ for $k=\min\{n, n'\}+1$. This proves our proposition. When $\min\{n, n'\}=n''$, let $n''=n' \leqq n$. Then, the a_k's satisfy

(3. 4-bis) $$(k+1)(k-n')a_{k+1} = (k-n)(k-n')a_k, \ (a_0=1).$$

Therefore, the values of a_k $(1 \leqq k \leqq n')$ are uniquely determined. But, the $a_{n'+1}$ can take on an arbitrary value. And, the values of a_k for $n'+2 \leqq k \leqq n+1$ are uniquely determined except for the multiplicative constant and, in particular, we have $a_{n+1}=0$. Thus, according to Kummer, this polynomial solution can be written as

$$F(-n, -n', -n'; x) + a_{n'+1}x^{n'+1}F(-n+n'+1, 1, 2+n'; x).$$

The degree of the polynomial $F(-n, -n', -n'; x)$ is equal to $n'(=\min\{n, n'\})$.

4. In 1832, E. E. Kummer discovered twenty four representations of particular solutions for Gauss differential equation (G). We denote them by $K_{0j}(x)$ $(j=1, 2,\cdots,8)$, $K_{1j}(x)$ $(j=1, 2,\cdots,8)$ and $K_{\infty j}(x)$ $(j=1, 2, \cdots,8)$. The first four solutions are

$$K_{01}(x) = F(\alpha,\beta,\gamma; x), \ K_{02}(x) = x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; x),$$
$$K_{03}(x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma; x),$$
$$K_{04}(x) = x^{1-\gamma} (1-x)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, 2-\gamma; x),$$

where the F's are represented by uniformly convergent power series in x for $|x|<1$, if we substitute the corresponding parameters for (α,β,γ) in power series (F). These solutions can be easily obtained as follows. The $K_{01}(x)$ is a holomorphic solution at $x=0$. If we apply the transformation $y=x^{1-\gamma}z$ for equation (G), then the z satisfies Gauss differential equation with the parameters $(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma)$ instead of (α, β,γ) and we have the solution $K_{02}(x)$ by means of its solution holomorphic at $x=0$. The substitution from y to z by $y=(1-x)^{\gamma-\alpha-\beta}z$ yields Gauss differential equation with the parameters $(\gamma-\alpha, \gamma-\beta, \gamma)$ and we have the solution $K_{03}(x)$ holomorphic at $x=0$. Finally, if we apply the transformation from y to z by $y=x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}z$, the z satisfies Gauss differential equation with the parameters $(1-\alpha, 1-\beta, 2-\gamma)$ which yields the solution $K_{04}(x)$.

It is known that the substitution from x to ξ which transforms the set of singularities $\{0, 1, \infty\}$ into itself makes a group, named the group of anharmonic ratio. Corresponding to each substitution of the independent variable, a suitable substitution of the dependent variable transforms Gauss differential equation into itself with a suitable choice of the parameters. The substitutions, the parameters and the Kummer's solutions are listed in the table below.

Substitutions	Parameters	Kummer's solutions
$\xi = x,\ y=z$	(α,β,γ)	$K_{01},\ K_{02},\ K_{03},\ K_{04}$
$\xi = 1-x,\ y=z$	$(\alpha,\beta,\ \alpha+\beta-\gamma+1)$	$K_{11},\ K_{12},\ K_{13},\ K_{14}$
$\xi = \frac{1}{x},\ y=x^{-\alpha}z$	$(\alpha,\alpha-\gamma+1,\ \alpha-\beta+1)$	$K_{\infty 1},\ K_{\infty 2},\ K_{\infty 3},\ K_{\infty 4}$
$\xi = \frac{1}{1-x},\ y=(1-x)^{-\alpha}z$	$(\alpha,\gamma-\beta,\ \alpha-\beta+1)$	$K_{\infty 5},\ K_{\infty 6},\ K_{\infty 7},\ K_{\infty 8}$
$\xi = \frac{x-1}{x},\ y=x^{-\alpha}z$	$(\alpha,\alpha-\gamma+1,\ \alpha+\beta-\gamma+1)$	$K_{15},\ K_{16},\ K_{17},\ K_{18}$
$\xi = \frac{x}{x-1},\ y=(1-x)^{-\alpha}z$	$(\alpha,\gamma-\beta,\ \gamma)$	$K_{05},\ K_{06},\ K_{07},\ K_{08}$

Here, the K_{ab}'s are given by the expressions

$$K_{11}(x) = F(\alpha,\beta,\alpha+\beta-\gamma+1;\ 1-x),$$
$$K_{12}(x) = x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1;\ 1-x),$$
$$K_{13}(x) = (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta,\ \gamma-\alpha-\beta+1;\ 1-x),$$
$$K_{14}(x) = x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\alpha,\ 1-\beta,\gamma-\alpha-\beta+1;\ 1-x),$$

where the power series F's are uniformly convergent for $|x-1|<1$;

$$K_{\infty 1}(x) = x^{-\alpha}F(\alpha,\ \alpha-\gamma+1,\ \alpha-\beta+1;\ \frac{1}{x}),$$
$$K_{\infty 2}(x) = x^{-\beta}F(\beta,\ \beta-\gamma+1,\ \beta-\alpha+1;\ \frac{1}{x}),$$
$$K_{\infty 3}(x) = x^{-\alpha}\Big(1-\frac{1}{x}\Big)^{\gamma-\alpha-\beta}F(-\beta+1,\ \gamma-\beta,\ \alpha-\beta+1;\ \frac{1}{x}),$$
$$K_{\infty 4}(x) = x^{-\beta}\Big(1-\frac{1}{x}\Big)^{\gamma-\alpha-\beta}F(-\alpha+1,\ \gamma-\alpha,\ -\alpha+\beta+1;\ \frac{1}{x}),$$

where the power series F's are uniformly convergent for $|x|>1$;

$$K_{\infty 5}(x) = (1-x)^{-\alpha}F(\alpha,\ \gamma-\beta,\ \alpha-\beta+1;\frac{1}{1-x}),$$
$$K_{\infty 6}(x) = (1-x)^{-\beta}F(\beta,\ \gamma-\alpha,\ \beta-\alpha+1;\frac{1}{1-x}),$$
$$K_{\infty 7}(x) = \Big(\frac{x}{x-1}\Big)^{\gamma-1}\Big(\frac{1}{1-x}\Big)^{\beta}F(\beta-\gamma+1,\ -\alpha+1,\ \beta-\alpha+1;\frac{1}{1-x}),$$
$$K_{\infty 8}(x) = \Big(\frac{x}{x-1}\Big)^{\gamma-1}\Big(\frac{1}{1-x}\Big)^{\alpha}F(\alpha-\gamma+1,\ -\beta+1,\ \alpha-\beta+1;\frac{1}{1-x}),$$

where the power series F's are uniformly convergent for $|x-1|>1$;

$$K_{15}(x) = x^{-\alpha}F(\alpha,\ \alpha-\gamma+1,\ \alpha+\beta-\gamma+1;\frac{x-1}{x}),$$

$$K_{16}(x) = x^{-\beta}F(\beta,\ \beta-\gamma+1,\ \beta+\alpha-\gamma+1;\ \frac{x-1}{x}),$$

$$K_{17}(x) = \left(\frac{x-1}{x}\right)^{\gamma-\alpha-\beta} x^{-\beta}F(-\alpha+1,\ \gamma-\alpha,\ \gamma-\alpha-\beta+1; \frac{x-1}{x}),$$

$$K_{18}(x) = \left(\frac{x-1}{x}\right)^{\gamma-\alpha-\beta} x^{-\alpha}F(-\beta+1,\ \gamma-\beta,\ \gamma-\beta-\alpha+1; \frac{x-1}{x}),$$

where the power series F's are uniformly convergent for $\left|\frac{x-1}{x}\right| < 1$ or Re $x > \frac{1}{2}$;

$$K_{05}(x) = (1-x)^{-\alpha}F(\alpha,\ \gamma-\beta,\ \gamma; \frac{x}{x-1}),$$

$$K_{06}(x) = (1-x)^{-\beta}F(\beta,\ \gamma-\alpha,\ \gamma; \frac{x}{x-1}),$$

$$K_{07}(x) = \left(\frac{x}{x-1}\right)^{1-\gamma} (1-x)^{-\alpha}F(\alpha-\gamma+1,\ -\beta+1,\ -\gamma+2; \frac{x}{x-1}),$$

$$K_{08}(x) = \left(\frac{x}{x-1}\right)^{1-\gamma} (1-x)^{-\beta}F(\beta-\gamma+1,\ -\alpha+1,\ -\gamma+2; \frac{x}{x-1}),$$

where the power series F's are uniformly convergent for $\left|\frac{x}{x-1}\right| < 1$ or Re $x < \frac{1}{2}$.

For the symbol $K_{ab}(x)$, the first letter a ($=0$ or 1 or ∞) in the subindices indicates a singularity such that the corresponding power series $F(\alpha',\ \beta',\ \gamma';\ \xi)$ can be represented by a uniformly convergent power series in $\xi = \xi(x)$ for the values of x near the singularity "a". These twenty four expressions can be easily obtained if we use the transformation formulas for Riemann's P-functions.

By an easy consideration, we see that these twenty four expressions are separated into six groups with identical relations:

$$K_{01}(x) = K_{03}(x) = K_{05}(x) = K_{06}(x);$$

$$K_{02}(x) = K_{04}(x) = e^{\pi(1-\gamma)i}K_{07}(x) = e^{\pi(1-\gamma)i}K_{08}(x),\ (i = \sqrt{-1}),$$

with $\arg(1-x) = 0$, $\arg(x-1) = \pi$ for $0 < x < \frac{1}{2}$;

$$K_{11}(x) = K_{12}(x) = K_{15}(x) = K_{16}(x);$$

$$K_{13}(x) = K_{14}(x) = e^{-\pi(\gamma-\alpha-\beta)i}K_{17}(x) = e^{-\pi(\gamma-\alpha-\beta)i}K_{18}(x)$$

with $\arg(1-x) = 0$, $\arg(x-1) = \pi$ for $\frac{1}{2} < x < 1$;

$$K_{\infty 1}(x) = K_{\infty 3}(x) = e^{\pi\alpha i}K_{\infty 5}(x) = e^{\pi\alpha i}K_{\infty 8}(x)$$

with $\arg(x-1) = 0$, $\arg(1-x) = \pi$ for $1 < x < \infty$;

$$K_{\infty 2}(x) = K_{\infty 4}(x) = e^{\pi\beta i}K_{\infty 6}(x) = e^{\pi\beta i}K_{\infty 7}(x)$$

with $\arg(x-1) = 0$, $\arg(1-x) = \pi$ for $1 < x < \infty$.

5. Case 1. In generic, the functions

$K_{01}(x) = F(\alpha,\beta,\gamma;\ x)$, $K_{02}(x) = x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma;\ x)$

represent linearly independent solutions around the singular point $x = 0$. Observe that $\alpha=-n$.

If $\gamma \neq 0, -1, \cdots, -n+1$, the function $K_{01}(x) = F(-n, \beta, \gamma; x)$ is a polynomial in x with degree $\leqq n$. If there is another rational function solution, it must be the $K_{02}(x)$. Since the function F is single valued around $x=0$, the quantity $1-\gamma$ must be an integer.

Let $1-\gamma = n' \in Z$. Since $\alpha=-n$ and $\gamma=1-n'$, we have

$$(5.1)\quad \begin{cases} K_{01}(x) = F(-n, \beta, 1-n'; x), \\ K_{02}(x) = x^{n'}F(-n+n', \beta+n', n'+1; x). \end{cases}$$

(i) Assume that $1 \leqq n' \leqq n$. Then the F appearing in the $K_{02}(x)$ is well defined independently of the value of β. The $K_{02}(x)$ is a polynomial in x. But, the F appearing in the $K_{01}(x)$ is well defined if and only if $\beta=-n'' \in Z \setminus N$ and $0 \leqq n'' \leqq n'-1$. In this case, the $K_{01}(x)$ is also a polynomial in x of degree n''. Then, the degree of the polynomial $K_{02}(x)$ is equal to n. Thus, we have two polynomial solutions $Y_1(x)$ and $Y_2(x)$, linearly independent. We name this type of solutions Type I :

$$(5.2)\quad (\text{Type I})\quad \begin{cases} Y_1(x) = F(-n, -n'', 1-n'; x), \\ Y_2(x) = x^{n'}F(-n+n', -n''+n', n'+1; x). \end{cases}$$

Here, n, n' and n'' are nonnegative integers, and $1 \leqq n' \leqq n$, $0 \leqq n'' < n'$. The degree of the polynomial $Y_1(x)$ is equal to n'' and that of the polynomial $Y_2(x)$ is equal to n.

(ii−1) Assume that $n'=0$. Since we have $K_{01}(x) = K_{02}(x)$, a pair of linearly independent solutions is a linear form of $\log x$ with coefficients holomorphic at $x=0$. So, this case must be omitted.

(ii−2) Assume that $n' < 0$. If we write as $-n'$ instead of n', we have

$K_{01}(x) = F(-n, \beta, n'+1; x)$, $K_{02}(x) = x^{-n'}F(-n-n', \beta-n', 1-n'; x)$.

Observe that $n' \in N$, $1-n' \in Z \setminus N$. The F appearing in the $K_{01}(x)$ is well defined independently of the value of β. But, the F in the $K_{02}(x)$ has the meaning if and only if $\beta=n'' \in N$ and $1 \leqq n'' \leqq n'$. Then, this power series F becomes a polynomial in x of degree $n'-n''$. And, the $K_{01}(x)$ is a polynomial in x of degree n. Thus, we have linearly independent solutions which are represented by rational functions.

We put

$$n+n' = n_1,\ n' = n_1',\ n'-\beta(=-n'-n'') = n_1''.$$

Clearly, $0 \leqq n_1'' \leqq n_1'-1$. Thus, we have linearly independent solutions $Y_1(x)$ and $Y_2(x)$, which are rational functions, of the form

$$(5.3)\quad \begin{cases} Y_1(x) = F(-n_1+n_1', n_1'-n_1'', n_1'+1; x), \\ Y_2(x) = x^{-n_1'}F(-n_1, -n_1'', 1-n_1'; x). \end{cases}$$

This case is considered as solutions of Type III, which will appear in the section 7. The polynomial $Y_1(x)$ is of degree n $(=n_1-n_1')$. But, the degree of the polynomial appearing in the $Y_2(x)$ is equal to $n'-n''(=n_1'')$ and it may exceed n.

(ii−3) Assume that $n'=n+m$ and $m \in N$. We have

$K_{01}(x) = F(-n, \beta, -n-m+1; x)$,

$K_{02}(x) = x^{n+m}F(m,\ \beta+n+m,\ n+m+1;\ x)$.

The F appearing in the $K_{01}(x)$ is a polynomial in x independently of the value of β. The F in the $K_{02}(x)$ is reduced to a polynomial if and only if $\beta+n+m=-n''\in Z\setminus N$.

Observe that

$K_{01}(x) = F(\beta,-n,-n-m+1;\ x) = F(-(n+m)-n'',-n,-(n+m)+1;\ x)$,

$K_{02}(x) = x^{n+m}F(\beta+n+m,\ m,\ n+m+1;\ x) = x^{n+m}F(-n'',\ m,\ (n+m)+1;\ x)$,

because of the symmetricity property $F(\alpha,\beta,\gamma;\ x) = F(\beta,\alpha,\gamma;\ x)$. We put

$n+m+n''=n_1,\ n+m=n_1',\ n=n_1''$,

so that

$n''=n_1-n_1',\ m=-n+n_1'=-n_1''+n_1'$.

Thus we have two linearly independent solutions, which are expressed by rational functions,

$$(5.4)\qquad \begin{cases} Y_1(x) = F(-n_1,\ -n_1'',\ 1-n_1';\ x), \\ Y_2(x) = x^{n_1'}F(-n_1+n_1',\ -n_1''+n_1',\ n_1'+1;\ x). \end{cases}$$

This case is regarded as solutions of Type I. The polynomial $Y_1(x)$ admits n as its degree. But, the polynomial $Y_2(x)$ is of degree $(n+m)+n''$ and this quantity exceeds n.

For Type I, we have

$$\alpha=-n,\ \beta=-n'',\ \gamma=1-n'.$$

The characteristic exponent $(\lambda,\ \mu,\ \nu)$ is defined by the triple

$$(5.5)\qquad (\lambda,\ \mu,\ \nu) = (1-\gamma,\ \gamma-\alpha-\beta,\ \alpha-\beta),$$

where every entry is a difference of the two characteristic roots at the corresponding singularity. Hence,

$$(5.6)\qquad (\lambda,\ \mu,\ \nu) = (n',\ 1-n'+n+n'',\ n''-n).$$

Therefore, the four quantities Q_j $(j=1,\ 2,\ 3,\ 4)$ defined by

$$(5.7)\qquad \begin{cases} Q_1=\lambda+\mu+\nu\ \ (=1+2n''),\ Q_2=-\lambda+\mu+\nu\ \ (=1-2n'+2n''), \\ Q_3=\lambda-\mu+\nu\ \ (=-1+2n'-2n),\ Q_4=\lambda+\mu-\nu\ \ (=1+2n) \end{cases}$$

are all odd integers.

6. Case 2. The solution has the form

$$x^a(1-x)^c g(x) = (1-x)^c g(x).$$

In generic, the two solutions given by

$K_{03}(x) = (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta,\ \gamma;\ x)$,

$K_{04}(x) = x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\alpha,\ 1-\beta,\ 2-\gamma;\ x)$

are linearly independent. Since, as is shown in the table,

$$\alpha=-n-c,\qquad \beta=n+\gamma,$$

the two solutions become

$$(6.1)\qquad \begin{cases} K_{03}(x) = (1-x)^c F(n+c+\gamma,-n,\gamma;\ x), \\ K_{04}(x) = x^{1-\gamma}(1-x)^c F(n+c+1,\ 1-n-\gamma,\ 2-\gamma;\ x). \end{cases}$$

The function $(1-x)^c$ and the hypergeometric function F are single valued around $x=0$. So, the quantity $1-\gamma$ must be an integer, because we look for rational function solutions.

Let $1-\gamma = n'\in Z$. Then, we have

$$(6.\ 1\text{-bis})\quad \begin{cases} K_{03}(x) = (1-x)^c F(n+1+c-n', -n, 1-n'; x), \\ K_{04}(x) = x^{n'}(1-x)^c F(n+1+c, n'-n, n'+1; x). \end{cases}$$

(i) Assume that $1 \leqq n' \leqq n$. The F appearing in the $K_{03}(x)$ becomes a polynomial in x if and only if

$$n+1+c-n' \in Z \setminus N \text{ and } n+1+c-n' = 0, -1, \cdots, -(n'-1)$$

or, what it is the same thing,

$$n+c=n'' \in \{N, 0\} \text{ and } 0 \leqq n'' < n'.$$

Thus, we have linearly independent solutions $Y_1(x)$ and $Y_2(x)$ which are rational functions. We call this type of solutions Type II:

$$(6.\ 2)\ (\text{Type II})\quad \begin{cases} Y_1(x) = (1-x)^{n''-n} F(n''+1-n', -n, 1-n'; x), \\ Y_2(x) = (1-x)^{n''-n} x^{n'} F(n''+1, n'-n, n'+1; x). \end{cases}$$

Here, n, n' and n'' are nonnegative integers and $1 \leqq n' \leqq n$, $0 \leqq n'' < n'$. Obviously, the degree of the polynomial in x appearing in the $Y_1(x)$ is equal to $n'-n''-1$. The $Y_2(x)$ involves a polynomial in x of degree n.

(ii−1) Assume that $n'=0$. Then, a pair of linearly independent solutions is a linear form of $\log x$ with coefficients holomorphic at $x=0$. Hence, this case must be excluded.

(ii−2) Assume that $n'=n+m$ and $m \in N$. The two solutions $K_{03}(x)$ and $K_{04}(x)$ are written in the form

$$K_{03}(x) = (1-x)^c F(c+1-m, -n, 1-n-m; x) = (1-x)^c F(-n, c+1-m, 1-n-m; x),$$
$$K_{04}(x) = x^{n+m}(1-x)^c F(n+c+1, m, 1+n+m; x) = x^{n+m}(1-x)^c F(m, n+c+1, 1+n+m; x).$$

The F appearing in the $K_{04}(x)$ is reduced to a polynomial if and only if

$$n+c+1 = -n'' \in Z \setminus N.$$

Put

$$n+n''+m = n_1,\quad n+m = n_1',\quad m-1 = n_1''.$$

An easy consideration implies that

$$c = -n-1-n'' = -(n+n''+m) + (m-1) = n_1''-n_1,$$
$$-n = m-n_1' = (m-1)+1-n_1' = n_1''+1-n_1',$$
$$c+1-m = -n-n''-m = -n_1,$$
$$n+c+1 = -n'' = n+m-n_1 = n_1'-n_1.$$

Thus, we have a pair of linearly independent solutions

$$(6.\ 3)\quad \begin{cases} Y_1(x) = (1-x)^{n_1''-n_1} F(n_1''+1-n_1', -n_1, n_1'+1; x), \\ Y_2(x) = (1-x)^{n_1''-n_1} x^{n_1'} F(n_1''+1, n_1'-n_1, n_1'+1; x). \end{cases}$$

This case is considered as solutions of Type II. The degree of the polynomial appearing in the $Y_1(x)$ is equal exactly to n $(=n_1'-n_1''-1)$. But, that in the $Y_2(x)$ is equal to $n_1 (=n_1'+(n_1-n_1'))$ and exceeds n.

(ii−3) Assume that $n'<0$. Changing the sign of the number n', we have the solutions, owing to (6. 1−bis),

$$K_{03}(x) = (1-x)^c F(n+1+c+n', -n, n'+1; x),$$
$$K_{04}(x) = x^{-n'}(1-x)^c F(n+1+c, -n-n', 1-n'; x).$$

Here, $n' \in N$.

The power series F appearing in the $K_{04}(x)$ is well defined if and only if

$$n+1+c=-n''\in Z\setminus N \quad \text{and} \quad 0\leqq n''\leqq n'-1.$$

We put

$$n+n'=n_1,\ n'=n_1',\ n''=n_1''.$$

Then we have

$$c=-n-1-n''=(-n_1+n')-n''-1=-n_1+n_1'-n_1''-1.$$

Thus we have linearly independent solutions, which are represented by rational functions, of the form

$$(6.\ 4)\quad \begin{aligned} Y_1(x) &= (1-x)^{-n_1+n_1'-n_1''-1}\ F(n_1'-n_1'',-n_1+n_1',\ n_1'+1;\ x),\\ Y_2(x) &= x^{-n_1'}(1-x)^{-n_1+n_1'-n_1''-1}\ F(-n_1'',-n_1,\ 1-n_1';\ x), \end{aligned}$$

with $n_1,\ n_1',\ n_1''\in\{N,\ 0\}$. This type of solutions is considered as Type IV, which will appear in the section 8. The degree of the polynomial appearing in the $Y_1(x)$ is exactly equal to n $(=n_1-n_1')$. And, the $Y_2(x)$ involves a polynomial in x of degree $n_1''(=n'')$.

For Type II, we have

$$(6.\ 5)\qquad \alpha=-n'',\ \beta=n+1-n',\ \gamma=1-n'.$$

Hence, the characteristic exponent is given by

$$(6.\ 6)\quad (\lambda,\mu,\nu)=(1-\gamma,\gamma-\alpha-\beta,\alpha-\beta)=(n',-n+n'',-n+n'-n''-1).$$

So, the four quantities $Q_j(j=1,\ 2,\ 3,\ 4)$ defined by (5. 7) become

$$(6.\ 7)\qquad \begin{cases} Q_1=-2n+2n'-1,\ Q_2=-2n-1,\\ Q_3=\ 2n'-2n''-1,\ Q_4=2n''+1 \end{cases}$$

and they are all odd integers.

7. Case 3. The solutions under consideration have the form

$$x^a(1-x)^c g(x)=x^a g(x).$$

In generic, the two solutions due to Kummer

$$K_{01}(x)=F(\alpha,\beta,\gamma;\ x),\ K_{02}(x)=x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma;\ x)$$

are linearly independent. As was shown in the table, we have

$$\alpha=-n-a,\ \gamma=1-a.$$

Henca, $\alpha-\gamma+1=-n$. Thus, the solutions are written as

$$K_{01}(x)=F(-n-a,\ \beta,\ 1-a;\ x),\ K_{02}(x)=x^aF(-n,\ \beta+a,\ 1+a;\ x).$$

Observe that the F is a single valued function around $x=0$. Hence, the quantity a must be an integer. Let $a=-n'\in Z$. Then, we have

$$(7.\ 1)\quad \begin{cases} K_{01}(x)=F(-n+n',\ \beta,\ 1+n';\ x),\\ K_{02}(x)=x^{-n'}F(-n,\ \beta-n',\ 1-n';\ x). \end{cases}$$

(i) Assume that $1\leqq n'\leqq n$. The function F appearing in the $K_{02}(x)$ is reduced to a polynomial, if and only if

$$\beta-n'=-n''\in Z\setminus N \quad \text{and} \quad 0\leqq n''\leqq n'-1.$$

Notice that $\beta=n'-n''\in N$. The $K_{01}(x)$ is a polynomial in x of degree $n-n'$. Thus we have two linearly independent solutions $Y_1(x)$ and $Y_2(x)$, which are expressed by rational functions. We name this type of solutions Type III:

$$(7.2)\quad (\text{Type III})\quad \begin{cases} Y_1(x) = F(-n+n',\ n'-n'',\ n'+1;\ x), \\ Y_2(x) = x^{-n'}\ F(-n, -n'', 1-n';\ x). \end{cases}$$

Here, n, n' and n'' are nonnegative integers, and $1\leqq n'\leqq n$, $0\leqq n''\leqq n'-1$. The F appearing in the $Y_1(x)$ is a polynomial in x of degree $n-n'$. The degree of the polynomial appearing in the $Y_2(x)$ is equal to n'', since $\min\{n, n''\} = n''\leqq n'-1$.

(ii−1) Assume that $n'=0$. Since we have $K_{01}(x)=K_{02}(x)$, a pair of linearly independent solutions is necessarily a linear form of $\log x$. So, this case must be excluded.

(ii−2) Assume that $n'<0$. If we write $-n'$ instead of n', the solutions (7. 1) are written as

$$K_{01}(x) = F(-n-n',\ \beta,\ 1-n';\ x),$$
$$K_{02}(x) = x^{n'}\ F(-n,\ \beta+n',\ n'+1;\ x),$$

where $n'\in N$. The power series F appearing in the $K_{01}(x)$ is well defined if and only if

$$\beta=-n''\in Z\setminus N \text{ and } 0\leqq n''\leqq n'-1.$$

Then, the F becomes a polynomial in x of degree n''. Obviously, the F appearing in the $K_{02}(x)$ is a polynomial in x of degree n.

Now we put

$$n+n'=n_1,\ n'=n_1',\ n''=n_1''.$$

Then, we have linearly independent solutions of the form

$$(7.3)\quad \begin{cases} Y_1(x) = F(-n_1,\ -n_1'',\ 1-n_1';\ x), \\ Y_2(x) = x^{n_1'}F(-n_1+n_1',\ n_1'-n_1'',\ n_1'+1;\ x). \end{cases}$$

This pair of solutions is regarded as solutions of Type I. The degree of the polynomial $Y_2(x)$ is equal to $n_1(=n+n')$ and exceeds n, but that of the F appearing in it is exactly equal to n.

(ii−3) Assume that $n'>n$. The solutions are rewritten as

$$K_{01}(x) = F(-n+n',\ \beta,\ n'+1;\ x) = F(\beta,\ -n+n',\ n'+1;\ x),$$
$$K_{02}(x) = x^{-n'}\ F(-n,\ \beta-n',\ 1-n';\ x) = x^{-n'}\ F(\beta-n',\ -n,\ 1-n';\ x).$$

Note that $n'-1\geqq n$. Hence, the F appearing in the $K_{02}(x)$ is a polynomial in x independently of the value of β. The F in the $K_{01}(x)$ is reduced to a polynomial in x if and only if

$$\beta=-n''\in Z\setminus N.$$

The degree of the $K_{01}(x)$ is equal to n''. That of the polynomial appearing in the $K_{02}(x)$ is equal to n, since $\min\{n'-\beta,\ n\}=n\leqq n'-1$.

We set

$$n'+n''=n_1,\ n'=n_1',\ n=n_1''.$$

An easy consideration yields the relations

$$\beta=-n''=-n_1+n'=-n_1+n_1'$$
$$-n+n'=n_1'-n_1'',$$
$$\beta-n'=-n''-n'=-n_1.$$

Thus we have a pair of linearly independent solutions of the form

$$(7.4)\quad \begin{cases} Y_1(x) = F(-n_1+n_1',\ n_1'-n_1'',\ n_1'+1;\ x), \\ Y_2(x) = x^{-n_1'}F(-n_1, -n_1'',\ 1-n_1';\ x). \end{cases}$$

This pair of solutions can be regarded as solutions of Type III. The degree of the polynomial

$Y_1(x)$ is equal to n'' $(=n_1-n_1')$. If $n''>n$, the degree of this polynomial does exceed n. The degree of the polynomial appearing in the $Y_2(x)$ is exactly equal to n, because of the inequalities $\min\{n_1, n_1''\}=n\leqq n'-1=n_1'-1$.

For Type III, we have

(7. 5) $$\alpha=-n+n',\ \beta=n'-n''>0,\ \gamma=1+n'.$$

Hence, the characteristic exponent (λ,μ,ν) is given by

(7. 6) $$(\lambda,\mu,\nu)=(1-\gamma,\gamma-\alpha-\beta,\alpha-\beta)=(-n',\ n-n'+n''+1,\ -n+n'').$$

The four quantities Q_j given by (5. 7) are calculated as

(7. 7) $$\begin{cases} Q_1=-2n'+2n''+1, & Q_2=2n''+1,\\ Q_3=-2n-1, & Q_4=2n-2n'+1\end{cases}$$

and they are all odd integers.

8. Case 4. The solutions have the form $x^a(1-x)^c g(x)$. In generic, the solutions due to Kummer

$$K_{03}(x)=(1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\ \gamma-\beta,\ \gamma;\ x),$$
$$K_{04}(x)=x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta}F(1-\alpha,\ 1-\beta,\ 2-\gamma;\ x)$$

are linearly independent. As is seen from the table, the parameters α,β,γ are related with the relations

$$\alpha=-n-a-c,\ \beta=n+1,\quad \gamma=1-a,\ \gamma-\alpha-\beta=c.$$

Hence, these solutions are written as

$$K_{03}(x)=(1-x)^cF(n+c+1,\ -n-a,\ 1-a;\ x),$$
$$K_{04}(x)=x^a(1-x)^cF(n+a+c+1,\ -n,\ 1+a;\ x).$$

Since the functions $(1-x)^c$ and F are both single valued around $x=0$, the quantity a must be an integer. Let $a=-n'\in Z$. Then, the solutions can be given by

(8. 1) $$\begin{cases} K_{03}(x)=(1-x)^cF(n+c+1,\ -n+n',\ 1+n';\ x),\\ K_{04}(x)=(1-x)^cx^{-n'}F(n-n'+c+1,\ -n,\ 1-n';\ x).\end{cases}$$

(i) Assume that $1\leqq n'\leqq n$. The power series F appearing in the $K_{01}(x)$ has the meaning if and only if

$$n-n'+c+1=-n''\in Z\setminus N \text{ and } 0\leqq n''\leqq n'-1.$$

Hence, the quantity c is also an integer and

$$c=-n+n'-n''-1\in Z.$$

Thus we have two linearly independent solutions $Y_1(x)$ and $Y_2(x)$, which are expressed by rational functions. We call this type of solutions Type IV:

(8. 2) (Type IV) $$\begin{cases} Y_1(x)=(1-x)^{-n+n'-n''-1}F(n'-n'',\ -n+n',\ n'+1;\ x),\\ Y_2(x)=x^{-n'}(1-x)^{-n+n'-n''-1}F(-n'',\ -n,\ 1-n';\ x).\end{cases}$$

Here, n, n' and n'' are nonnegative integers, and $1\leqq n'\leqq n$, $0\leqq n''\leqq n'-1$. The degree of the polynomial appearing in the $Y_1(x)$ is equal to $n-n'$ and that in the $Y_2(x)$ is equal to n'' since $\min\{n'', n\}=n''\leqq n'-1$.

(ii−1) Assume that $n'=0$. The both solutions coincide. Hence, a pair of linearly independent solutions is necessarily a linear form of $\log x$. This case must be omitted.

(ii−2) Assume that $n'<0$. By changing the sign of the number n', we have the solutions, by virtue of (8. 1),

$$K_{03}(x) = (1-x)^c F(n+c+1,\ -n-n',\ 1-n';\ x),$$
$$K_{04}(x) = (1-x)^c x^{n'} F(n+n'+c+1,\ -n,\ 1+n';\ x).$$

with $n' \in N$. The power series F appearing in the $K_{03}(x)$ is well defined, if and only if

$$n+c+1=-n'' \in Z \setminus N \quad \text{and} \quad 0 \leqq n'' \leqq n'-1.$$

Since then

$$c=-n-n''-1 \in Z \setminus N,$$

the solutions are written as

$$K_{03}(x) = (1-x)^{-n-n''-1} F(-n'',\ -n-n',\ 1-n';\ x),$$
$$K_{04}(x) = (1-x)^{-n-n''-1} x^{n'} F(n'-n'',\ -n,\ n'+1;\ x).$$

Put

$$n+n'=n_1 \in N,\ n'=n_1' \in N,\ n'-1-n''=n_1'' \in \{N,\ 0\}.$$

An easy consideration implies that

$$n+n''+1=(n_1-n')+(n'-n_1'')=n_1-n_1'',$$
$$-n''=-n_1'+n_1''+1,$$
$$n'-n''=n_1''+1,$$
$$-n=-n_1+n'=-n_1+n_1'.$$

Thus, the above solutions can be written in the form

$$(8.\ 3) \quad \begin{cases} Y_1(x) = (1-x)^{n_1'-n_1} F(-n_1'+n_1''+1,\ -n_1,\ 1-n_1';\ x), \\ Y_2(x) = (1-x)^{n_1'-n_1} x^{n_1'} F(n_1''+1,\ -n_1+n_1',\ n_1'+1;\ x), \end{cases}$$

which can be regarded as solutions of Type II. The degree of the polynomial appearing in the $Y_1(x)$ is equal to the number n'', because of $n_1'-n_1''-1=n'-(n'-1-n'')-1=n''$. The degree of the polynomial in the $Y_2(x)$ is equal to $n_1(=n_1'+(n_1-n_1'))$ and exceeds n, but that of the F is exactly equal to n.

(ii−3) Assume that $n'>n$. Note that the solutions (8. 1) can be written as

$$K_{03}(x) = (1-x)^c F(n+c+1,\ -n+n',\ n'+1;\ x)$$
$$= (1-x)^c F(-n+n',\ n+c+1,\ n'+1;\ x)$$
$$K_{04}(x) = x^{-n'} (1-x)^c F(n-n'+c+1,\ -n,\ 1-n';\ x)$$
$$= x^{-n'} (1-x)^c F(-n,\ n-n'+c+1,\ 1-n';\ x).$$

Since $-n+n' \in N$, the F appearing in the $K_{03}(x)$ becomes a polynomial in x if and only if

$$n+c+1=-n'' \in Z \setminus N \quad \text{or} \quad c=-n-n''-1.$$

Note that $n-n'+c+1=-n'-n''$. If we put

$$n'+n''=n_1,\ n'=n_1',\ n=n_1'',$$

it follows by an elementary calculation that

$$c=-n-n''-1=-n-(n'+n'')+n'-1=-n_1''-n_1+n_1'-1$$
$$=-n_1+n_1'-n_1''-1,$$
$$n+c+1=-n''=-(n'+n'')+n'=-n_1+n_1',$$
$$n-n'+c+1=-n_1.$$

Thus, we have a pair of solutions of the form

$$(8.4)\quad \begin{cases} Y_1(x) = (1-x)^{-n_1+n_1'-n_1''-1} F(n_1'-n_1'',\ -n_1+n_1',\ n_1'+1;\ x), \\ Y_2(x) = x^{-n_1'}(1-x)^{-n_1+n_1'-n_1''-1} F(-n_1'',\ -n_1,\ 1-n_1';\ x). \end{cases}$$

Since the power exponent of $(1-x)$ is a negative integer, this pair of solutions is included in Type IV. The degree of the polynomial appearing in the $Y_1(x)$ is equal to $n''(=n_1-n_1')$. That in the $Y_2(x)$ is equal to $n_1''(=n)$, because we have $\min\{n_1'', n_1\}=n\leqq n'-1=n_1'-1$.

For Type IV, as was shown, we have

$$\begin{aligned} &\gamma-\alpha-\beta = c = -n+n'-n''-1, \\ &\gamma-\alpha = n'-n'', \\ &\gamma-\beta = -n+n', \\ &1-\gamma = -n'. \end{aligned}$$

Hence, the characteristic exponent given by (5. 5) is

$$(8.5)\quad (\lambda, \mu, \nu) = (-n',\ -n+n'-n''-1,\ -n+n'')$$

and the four quantities Q_j defined by (5. 7) become

$$(8.6)\quad \begin{cases} Q_1=-2n-1, & Q_2=-2n+2n'-1 \\ Q_2=-2n'+2n''+1, & Q_4=-2n''-1, \end{cases}$$

which are all odd integers.

9. Conclusion. Summarizing the above discussions, we have the following.

Theorem. *When Gauss differential equation admits a pair of linearly independent solutions $Y_1(x)$ and $Y_2(x)$, which are represented by rational functions, the solutions must be one of the four types:*

Type I: $Y_1(x)$ and $Y_2(x)$ are both polynomials in x;

Type II: $Y_1(x)$ and $Y_2(x)$ are both polynomials in x divided by a monomial in $1-x$ with the same form;

Type III: $Y_1(x)$ is a polynomial in x, while $Y_2(x)$ is a polynomial in x divided by a monomial in $1-x$;

Type IV: $Y_1(x)$ is a polynomial in x divided by a monomial in $1-x$, and $Y_2(x)$ is a polynomial in x divided by the product of a monomial in x and a monomial in $1-x$ with the same form as in $Y_1(x)$.

In any type, the four quantities Q_j $(j=1, 2, 3, 4)$ defined by (5. 7) are all odd integers.

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