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AN EXTENSION OF WEBER'S EQUATION

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Dedicated to Professor Tosihusa Kimura on his 60th birthday.

ABSTRACT. We consider the global behavior of the subdominant slotions of the differential equation $y''-(x^{2m}+ax^{m-1})y=0$, by utilizing the properties of hypergeomitric difference equations. When $m=1$, this is the Weber's equation. We derive a new extension of it and investigate zero-free domains, the distribution of zeros and other properties.

1. Introduction.

The linear differential equation under consideration is

$$\frac{d^2y}{dx^2}-(x^{2m}+ax^{m-1})y=0, \tag{1.1}$$

where x is a complex variable, a is a complex parameter and m is a positive integer. The point of infinity is an irregular singular point of rank $m+1$ and the origin is a transition point of order $2m$. Equation (1.1) is important because it is a simple example of a second order ordinary linear differential equation with such a transition point. For $m=1$, equation (1.1) is exactly the Weber's equation. Y. Sibuya [6] applied the Weber function to a differential equation with a transition point of order 2. Using the Whittaker's parabolic cylinder function, Y. Sibuya [5] investigated the properties of the subdominant solution of the Weber's equation.

For $a=0$, the equation (1.1) is the Airy equation which has a long history of investigations. C.A.Swanson and V.B.Headley [7] defined the Airy functions of the first and second kind in terms of the modified Bessel function of the first kind and investigated continuation formula, zero-free domains, the distribution of zeros and other properties. M.Kohno [4] defined the Airy function of the first kind as a paticular entire solution of linear differential equations which is principally recessive on the positive real axis arg $z=0$.

In this paper we shall define the Weber function as an entire solution of (1.1) which is subdominant on the positive real axis and investigate the global behavior of the Weber function, using hypergeometric difference equations. (cf.[2]) In section 5 and 6, by means of the asymptotic behavior of the Weber function and an application of the principal of the argument, we show that zeros of it are located in a small sector including the Stokes line. We know the zero-free domains, by using Lommel's method.

2. Subdominant solutions.

P. F. Hsieh and Y.Sibuya [3] constructed the unique solution $y(x, a)$ of the equation (1.1) such that

(i) $y(x, a)$ is an entire function of (x, a);

(ii) $y(x, a)$ and $y'(x, a)$ admit, respectively, the asymptotic representations

(2.1) $$y\ (x,\ a)=x^{\frac{1}{2}a-\frac{1}{2}m}\ \exp\ [-\frac{1}{m+1}x^{m+1}]\cdot\ [1+O(x^{-\frac{1}{2}})],$$

(2.2) $$y'\ (x,\ a)=x^{\frac{1}{2}a-\frac{1}{2}m}\ \exp\ [-\frac{1}{m+1}x^{m+1}]\cdot\ [-1+O(x^{-\frac{1}{2}})],$$

uniformly on each compact set in the a-space, as x tends to infinity in any closed subsector of the open sector $|\arg x|<\frac{3\pi}{2m+2}$. The solution $y(x,\ a)$ tends to zero as x tends to infinity in the sector $|\arg x|<\frac{\pi}{2m+2}$. Therefore, the solution $y(x,\ a)$ is called a subdominant solution in this sector and is uniquely determined.

If we put

(2. 3) $$y_k(x,\ a)=y\ (\omega^{-k}x,\ (-1)^k a),$$

where

(2. 4) $$\omega=\exp\ [\frac{1}{m+1}\pi i],$$

then, for each integer k, $y_k(x,\ a)$ is also a solution of the defferential equation (1.1). The solution $y_k(x,\ a)$ and its derivative $y'_k(x,\ a)$ admit, respectively, the asymptotic representations

(2. 5) $$y_k(x,\ a)=\omega^{-k[-\frac{1}{2}(-1)^k a-\frac{1}{2}m]}\ x^{-\frac{1}{2}(-1)^k a-\frac{1}{2}m}\ .$$
$$\cdot\ \exp[(-1)^{k+1}\frac{1}{m+1}\ x^{m+1}\ ]\cdot\ [1+O(x^{-\frac{1}{2}})],$$

(2. 6) $$y'_k(x,\ a)=\omega^{-k[-\frac{1}{2}(-1)^k a-\frac{1}{2}m]}\ x^{-\frac{1}{2}(-1)^k a+\frac{1}{2}m}\ .$$
$$\cdot\ \exp[(-1)^{k+1}\frac{1}{m+1}\ x^{m+1}]\cdot[(-1)^{k+1}+O(x^{-\frac{1}{2}})]$$

uniformly on each compact set in a-space, as x tends to infinity in any closed subsector of the open sector:

(2. 7) $$S_k:\ |\arg\ x-\frac{k}{m+1}\pi\ |<\frac{3\pi}{2m+2}.$$

The solution $y_k(x,\ a)$ is a subdominant solution in the sector

(2. 8) $$\tilde{S}_k:\ |\arg\ x-\frac{k}{m+1}\pi\ |<\frac{\pi}{2m+2}.$$

Note that

(2. 9) $$y_k(x,\ a)=y_h(x,\ a)\quad if\ k\equiv h\pmod{2m+2}.$$

We define $y(x,\ a)$ as an extended Weber function and in the following section we investigate the global behavior of the extended Weber function.

3. Difference equations. We shall consider the differential equation (1.1). The solution $y(x, a)$ is subdominant on the positive real axis, so its Mellin transformation

(3. 1) $$F(s)=\int_0^\infty y(x, a)\cdot x^{s-1}dx$$

exists as a holomorphic function in the right half-plane Re (s)>0. Moreover, it is a solution of a difference equation

(3. 2) $$s\ (s+1)\ F(s)-F(s+2m+2)-a\cdot F(s+m+1)=0.$$

Let us put

(3. 3) $$s=(m+1)t\ ,$$

and

(3. 4) $$F(s)=F((m+1)t)=\Gamma(t)W(t).$$

Then, the difference equation (3. 2) becomes

(3. 5) $$(t+1)W(t+2)+a\cdot W(t+1)-[(m+1)^2t+(m+1)]W(t)=0.$$

This equation is called the hypergeometric difference equation. (cf. P. M. Batchelder [1])

We define two constants ρ_1, ρ_2 by the roots of the characteristic equation of the difference equation (3. 5)

$$\rho^2-(m+1)^2=0.$$

That is

(3. 6) $$\rho_1=m+1,\quad \rho_2=-(m+1).$$

Furthermore, we define three constants by the equation

$$\beta_1+\beta_2+\beta_3+2=1,$$
$$\beta_1\rho_2+\beta_2\rho_1=-a,$$
$$\rho_1\rho_2\beta_3=-(m+1).$$

That is

(3. 7) $$\beta_1=\frac{a-m-2}{2(m+1)},\quad \beta_2=-\frac{a+m+2}{2(m+1)},\quad \beta_3=\frac{1}{m+1}.$$

Set

(3. 8) $$\tau=t+\beta_3$$

and

(3. 9) $$H(\tau)=W(\tau-\beta_3).$$

Then, $H(\tau)$ satisfies the difference equation

(3.10) $$(\tau+\beta_1+\beta_2+2)\ H(\tau+2)$$
$$-[(\rho_1+\rho_2)(\tau+\beta_3+1)+\beta_1\rho_2+\beta_2\rho_1]H(\tau+1)+\rho_1\rho_2\tau H(\tau)=0.$$

We need the following theorem. (P.M. Batchelder[1])

THEOREM 3.1. *There exist two solutions $h_1(\tau)$ and $h_2(\tau)$ analytic throughout the finite part of the plane except for pole, and such that, for $j=1,2$,*

$$h_j(\tau)\simeq\rho_j^\tau\cdot\tau^{-\beta_j-1}\{b+\frac{b'_j}{\tau}+\frac{b''_j}{\tau^2}+\cdots\cdots\}\quad ,$$

where b_j's are constants. These solutions called the principal solutions. Furthermore,

$h_j(\tau)$ *may be written in the form*

$$h_1(\tau)=(-\rho_1)^{\beta_1}(\rho_1-\rho_2)^{\beta_2}(\rho_1)^{\tau}\frac{\Gamma(\tau)\Gamma(\beta_1+1)}{\Gamma(\tau+\beta_1+1)} \cdot F\left(\beta_1+1,-\beta_2,\tau+b_1+1,\frac{1}{1-[\rho_2/\rho_1]}\right) \tag{3.11}$$

and

$$h_2(\tau)=(-\rho_2)^{\beta_1}(\rho_2-\rho_1)^{\beta_1}(\rho_2)^{\tau}\frac{\Gamma(\tau)\Gamma(\beta_2+1)}{\Gamma(\tau+\beta_2+1)} \cdot F\left(\beta_2+1,-\beta_1,\tau+\beta_2+1,\frac{[\rho_2/\rho_1]}{[\rho_2/\rho_1]-1}\right) \tag{3.12}$$

where $F(\cdot,\cdot,\cdot,\cdot)$ is the hypergeometric series

$$F(\alpha,\beta,\gamma,z)=1+\frac{\alpha\cdot\beta}{1\cdot\gamma}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}z^2+\cdots. \tag{3.13}$$

Using this theorem and (3. 3), (3. 4), (3. 5) and (3. 8), we obtain the general solution of the difference equation (3. 2) as follows;

$$\begin{aligned} F(s) &= \rho(s)\Gamma(t)h_1(t+\beta_3)+\rho'(s)\Gamma(t)h_2(t+\beta_3) \\ &= \rho(s)\,(-1)^{\beta_1}\,2^{\beta_2}\,(m+1)^{\beta_1+\beta_2+\beta_3}\,(m+1)^t \\ &\quad\cdot\frac{\Gamma(t)\Gamma(t+\beta_3)\Gamma(\beta_1+1)}{\Gamma(t+\beta_3+\beta_1+1)}\cdot F\left(\beta_1+1,-\beta_2,t+\beta_3+\beta_1+1,\frac{1}{2}\right) \\ &\quad+\rho'(s)\,(-1)^{\beta_1+\beta_2}2^{\beta_1}\,(m+1)^{\beta_1+\beta_2+\beta_3}\,[-(m+1)]^t \\ &\quad\cdot\frac{\Gamma(t)\Gamma(t+\beta_3)\Gamma(\beta_2+1)}{\Gamma(t+\beta_3+\beta_2+1)}\cdot F\left(\beta_2+1,-\beta_1,t+\beta_3+\beta_2+1,\frac{1}{2}\right), \end{aligned} \tag{3.14}$$

where $\rho(s)$ and $\rho'(s)$ are arbitrary periodic functions of period $m+1$. In order to seek a explicit solution of the difference equation (3. 2), we have to determine $\rho(s)$ and $\rho'(s)$. To do this, we use the asympotic representation (2. 1) of a subdominant solution $y(x, a)$. In order to know the asymptotic condition of the Mellin transformation $F(s)$ of $y(x, a)$, we need the following lemma. (See Wyrwich [8])

LEMMA 3.2. *Let* $G(t)$ *be summable in the sence of Lebesgue on each compact subset of the interval* $(0,\infty)$ *and satisfies the two conditions:*

(i) $G(t)=O(t^{-c})$ $t\to 0$, $c\in\mathbf{R}$;

(ii) $G(t)\simeq\exp[-\alpha t^{\beta}]t^{-\gamma}$, $t\to\infty$, $\alpha,\beta>0$, $\gamma\in\mathbf{C}$.

Then the Mellin transformation $g(s)$ *of* $G(t)$ *exists in the half plane* $\mathrm{Re}(s)>0$ *and satisfies*

$$g(s)\simeq\frac{1}{\beta}\alpha^{(\gamma-s)/\beta}\Gamma\left(\frac{s-\gamma}{\beta}\right),$$

as $s\to\infty$ in any half strip

$$\mathrm{Re}(s)>0,\ |\mathrm{Im}(s)|<d.$$

Now, we can easily obtain from (2. 1) and lemma 3.2 that

$$(3.16)\qquad F(s) \simeq (m+1)^{-1-\frac{a+m}{2m+2}} (m+1)^{\frac{s}{m+1}} \Gamma\left(\frac{1}{m+1}\left(s-\frac{1}{2}a-\frac{1}{2}m\right)\right).$$

Therefore, it follows from (3.14) and (3.15) that

$$\rho(s)\ (-1)^{\beta_1} 2^{\beta_1} (m+1)^{\frac{a+m}{2m+2}}\ \Gamma(\beta_1+1)\cdot \frac{\Gamma(t)\Gamma(t+\beta_3)}{\Gamma(t+\beta_3+\beta_1+1)\Gamma(t-\frac{a+m}{2m+2})}$$

$$(3.16)\qquad +\rho'(s)\ (-1)^{\beta_1+\beta_2}\ 2^{\beta_1}\ (m+1)^{\frac{a+m}{2m+2}}\ \Gamma(\beta_1+1)\cdot(-1)$$

$$\cdot\frac{\Gamma(t)\Gamma(t+\beta_3)}{\Gamma(t+\beta_3+\beta_2+1)\Gamma(t-\frac{a+m}{2m+2})} \simeq 1.$$

Since

$$\frac{\Gamma(t)\Gamma(t+\beta_3)}{\Gamma(t+\beta_3+\beta_1+1)\Gamma(t-\frac{a+m}{2m+2})} = 1+O(t^{-1})$$

and

$$\frac{\Gamma(t)\Gamma(t+\beta_3)}{\Gamma(t+\beta_3+\beta_2+1)\Gamma(t-\frac{a+m}{2m+2})} = t^{\frac{a}{m+1}}\ [1+O(t^{-1})].$$

we can get

$$(3.17)\qquad \rho(s) \simeq (-1)^{\frac{m+2-a}{2m+2}}\ 2^{\frac{m+2+a}{2m+2}}\ (m+1)^{-\frac{m+a}{2m+2}}\ \Gamma\left(\frac{2+m}{2m+2}\right)^{-1}$$

and

$$(3.18)\qquad \rho'(s) \simeq 0\ .$$

Here we used (3. 7) and the asympotic property of $\Gamma(z)$:

$$(3.19)\qquad \frac{\Gamma(z+c)}{\Gamma(z+d)} = z^{c-d}[1+O(z^{-1})].$$

Since $\rho(s)$ and $\rho'(s)$ were supposed to be periodic, we even have equalities in (3.17) and (3.18), and finally obtain from (3.14) that

$$(3.20)\quad F(s) = (m+1)^{-1-\frac{a+m}{2m+2}}$$

$$\cdot\ (m+1)^{t}\cdot\ \frac{\Gamma(t)\Gamma(t+\beta_3)}{\Gamma(t+\beta_3+\beta_1+1)}\cdot\ F(\beta_1+1, -\beta_2, t+\beta_3+\beta_1+1, \frac{1}{2})$$

$$= (m+1)^{-1-\frac{a+m}{2m+2}}\cdot\ (m+1)^{\frac{s}{m+1}}$$

$$\cdot\frac{\Gamma(\frac{s}{m+1})\Gamma(\frac{s+1}{m+1})}{\Gamma(\frac{2s+2+m+a}{2m+2})}\cdot F\left(\frac{m+a}{2m+2}, \frac{m+2+a}{2m+2}, \frac{2s+2+m+a}{2m+2}, \frac{1}{2}\right).$$

The subdominat solution $y(x, a)$ of the differential equation (1. 1) is holomorphic at $x=0$. Therefore, the residues of the transform $F(s)$ of $y(x, a)$ are given by

$$\text{Res }[F(s); s=-k] = \frac{1}{k!}\frac{d^k y\ (x,\ a)}{dx^k}\Big|_{x=0}. \tag{3.21}$$

Using the property of $\Gamma(z)$ and hypergeometric series:

$$\lim_{z\to -k}\ (z+k)\Gamma(z) = \frac{(-1)}{k!},$$

$$F(\alpha, \beta, \alpha;\ z) = F(\beta, \alpha, \alpha;\ z) = (1-z)^{-\beta},$$

we can get

$$\begin{aligned} y(0,\ a) &= \text{Res }[F(s); s=0] \\ &= (m+1)^{-\frac{1}{2}(m+1)(a+m)} \frac{\Gamma(\frac{1}{m+1})}{\Gamma(\frac{a+m+2}{2m+2})} \\ &\quad \cdot F(\frac{m+a}{2m+2}, \frac{a+m+2}{2m+2}, \frac{a+m+2}{2m+2}, \frac{1}{2}) \\ &= 2^{\frac{m+a}{2m+2}}(m+1)^{-\frac{m+a}{2m+2}} \frac{\Gamma(\frac{1}{m+1})}{\Gamma(\frac{a+m+2}{2m+2})} \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} y'(0,\ a) &= \text{Res }[F(s);\ s=-1] \\ &= (m+1)^{-1-\frac{1}{2}(m+1)\ (a+m)} \frac{\Gamma(\frac{-1}{m+1})}{\Gamma(\frac{m+a}{2m+2})} \\ &\quad \cdot F(\frac{a+m}{2m+2}, \frac{a+m+2}{2m+2}, \frac{a+m}{2m+2}, \frac{1}{2}) \\ &= 2^{\frac{a+m+2}{2m+2}}(m+1)^{-1-\frac{a+m}{2m+2}} \frac{\Gamma(\frac{-1}{m+1})}{\Gamma(\frac{a+m}{2m+2})}. \end{aligned} \tag{3.23}$$

Thus we solved the central connection problem for (1. 1).

4. Stokes phenomenon.

We shall now show linear dependence relations. Using (2. 5) and (2. 6), we can calculate the Wronskian

$$\text{Wron }[y_k(x,\ a),\ y_{k+1}(x,\ a); x] = 2\cdot\ \omega^{-(-1)^k\frac{1}{2}a+\frac{1}{2}m-k}.$$

From this, it follows that $y_k(x,\ a)$ and $y_{k+1}(x,\ a)$ $(k=0,1,\cdots,2m+1)$ make a fundamental set of solutions of the linear defferential equation (1. 1). Then we have

$$y(x,\ a) = c_k(a)y_k(x,\ a) + d_{k+1}(a)y_{k+1}(x,\ a), \tag{4. 1}$$

where

$$c_k(a) = (\frac{2}{m+1})^{\frac{(1-(-1)^k)a}{2m+2}}. \tag{4. 2}$$

$$\frac{\dfrac{\omega^{-k-1}}{\Gamma(\frac{m+(-1)^{k+1}a}{2m+2})\Gamma(\frac{m+a+2}{2m+2})}-\dfrac{1}{\Gamma(\frac{m+a}{2m+2})\Gamma(\frac{m+(-1)^{k+1}a+2}{2m+2})}}{\dfrac{\omega^{-k-1}}{\Gamma(\frac{m+(-1)^{k+1}a}{2m+2})\Gamma(\frac{m+(-1)^{k}a+2}{2m+2})}-\dfrac{\omega^{-k}}{\Gamma(\frac{m+(-1)^{k}a}{2m+2})\Gamma(\frac{m+(-1)^{k+1}a+2}{2m+2})}}$$

(4. 3) $$d_{k+1}(a)=\left(\frac{2}{m+1}\right)^{\frac{(1-(-1)^{k+1})a}{2m+2}}$$

$$\frac{\dfrac{\omega^{-k}}{\Gamma(\frac{m+(-1)^{k}a}{2m+2})\Gamma(\frac{m+a+2}{2m+2})}-\dfrac{1}{\Gamma(\frac{m+a}{2m+2})\Gamma(\frac{m+(-1)^{k}a+2}{2m+2})}}{\dfrac{\omega^{-k}}{\Gamma(\frac{m+(-1)^{k}a}{2m+2})\Gamma(\frac{m+(-1)^{k+1}a+2}{2m+2})}-\dfrac{\omega^{-k-1}}{\Gamma(\frac{m+(-1)^{k+1}a}{2m+2})\Gamma(\frac{m+(-1)^{k}a+2}{2m+2})}}$$

Here we used (3.22) and (3.23).

We shall now investigate the Stokes phenomenon of $y(x, a)$.

Note that

$$S_k=\tilde{S}_{k-1}\cup\tilde{S}_k\cup\tilde{S}_{k+1}\cup\{\arg x=\frac{k}{m+1}\pi-\frac{\pi}{2m+2}\}$$
$$\cup\{\arg x=\frac{k}{m+1}\pi-\frac{\pi}{2m+2}\}$$

and

$$S_k\cap S_{k+1}=S_k\cap\{\arg x=\frac{\pi}{2m+2}+\frac{k}{m+1}\pi\}\cap\tilde{S}_{k+1}\quad .$$

Using the fact that $y(x, a)$ is subdominat in the open sector $\tilde{S}_k$, it follows from (4. 1) that

(4. 4) $$y(x, a)\simeq\begin{cases}c_k(a)y_k(x, a) & \text{in } \tilde{S}_{k+1}\\ d_{k+1}(a)y_{k+1}(x, a) & \text{in } \tilde{S}_k\end{cases}$$
$(k\neq 0,-1)$

Therefore, the half-line

(4. 5) $$\theta_k=\arg x=\frac{\pi}{2m+2}+\frac{k}{m+1}\pi\qquad(k=1,2,\cdots,2m-1,2m)$$

are the actual Stokes lines of $y(x, a)$ in the whole comples plane $0\leq\arg x<2\pi$.

5. The distribution of zeros. In the following sections, we assume that the parameter a is a positive real number. From the global behavior derived in the preceeding section we can now investigate the distribution of zeros of $y(x, a)$ on the actual Stokes lines.

Let Q, P_1 and P_2 be points of intersection of the circle $|x|=\rho$ with one of the Stokes lines $\theta_k (k=1,2,\cdots,2m)$, the rays $\theta=\theta_k-\varepsilon$ and $\theta=\theta_k+\varepsilon$, ε being an arbitrarily small positive number, respectively. From (4. 1) and (4. 4), we then have for a sufficiently large positive number ρ

$$(5.\ 1)\quad \Delta_{P_1Q} \arg y(x, a) = \Big[\arg d_{k+1}(a) + \arg y_{k+1}(x, a)\Big]_{\theta=\theta_k-\varepsilon}^{\theta=\theta_k}$$

$$= \Big[\arg (\omega^{-k-1}x)^{-\frac{1}{2}(-1)^{k+1}a-\frac{1}{2}m} + \arg \{\exp [-\frac{1}{m+1}(\omega^{-k-1}x)^{m+1}]+o(1)\Big]_{\theta=\theta_k-\varepsilon}^{\theta=\theta_k}$$

$$=(-\frac{1}{2}(-1)^{k+1}a-\frac{1}{2}m)\,\varepsilon+\frac{\rho^{m+1}}{m+1}\{\sin((m+1)\theta_k-(k+1)\pi+\pi)$$
$$-\sin((m+1)(\theta_k-\varepsilon)-(k+1)\pi+\pi)\}+o(1)\ .$$

$$(5.\ 2)\quad \Delta_{P_2Q} \arg y(x, a)=(-\frac{1}{2}(-1)^{k}a-\frac{1}{2}m)(-\varepsilon)+\frac{\rho^{m+1}}{m+1}\{\sin((m+1)\theta_k-k\pi+\pi)$$
$$-\sin((m+1)(\theta_k+\varepsilon)-k\pi+\pi)\}+o(1)\ ,$$

$$(5.\ 3)\quad \Delta_{OP_1} \arg y(x, a)=\arg d_{k+1}(a)+(-\frac{1}{2}(-1)^{k+1}a-\frac{1}{2}m)\ (-\frac{k+1}{m+1}\pi+\theta_k-\varepsilon)$$
$$+\frac{\rho^{m+1}}{m+1}\cdot \sin((m+1)(\theta_k-\varepsilon)-(k+1)\pi+\pi)+o(1)-\arg y(x, a)\Big|_{x=0}$$

$$(5.\ 4)\quad \Delta_{OP_2} \arg y(x, a)=\arg c_k(a)+(-\frac{1}{2}(-1)^{k}a-\frac{1}{2}m)\ (-\frac{k}{m+1}\pi+\theta_k+\varepsilon)$$
$$+\frac{\rho^{m+1}}{m+1}\cdot \sin((m+1)(\theta_k+\varepsilon)-k\pi+\pi)+o(1)-\arg y(x, a)\Big|_{x=0}\ .$$

Hence we have for a sufficiently large ρ and a sufficiently small ε

$$(5.\ 5)\quad \Delta_{OP_1QP_2O} \arg y(x, a)$$
$$=\arg d_{k+1}(a)-\arg c_k(a)+(-\frac{1}{2}(-1)^{k+1}a-\frac{1}{2}m)\ (-\frac{k+1}{m+1}\pi+\theta_k)$$
$$-(-\frac{1}{2}(-1)^{k}a-\frac{1}{2}m)\ (-\frac{k}{m+1}\pi+\theta_k)+\frac{\rho^{m+1}}{m+1}\{\sin((m+1)\theta_k-k\pi)$$
$$-\sin((m+1)\theta_k-k\pi+\pi\}+o(1)$$

$$=\frac{2}{m+1}\rho^{m+1}+\frac{m\pi}{2m+2}+M(a)\ +o(1),$$

where

$$M(a)=\arg\ d_{k+1}(a)-\arg\ c_k(a)$$

$$=\arg\Big[\Gamma\Big(\frac{m+a}{2m+2}\Big)\Gamma\Big(\frac{m+(-1)^k a+2}{2m+2}\Big)\ \omega^{-k}-\Gamma\Big(\frac{m+(-1)^k a}{2m+2}\Big)\Gamma\Big(\frac{m+a+2}{2m+2}\Big)\ \Big]$$

$$-\arg\Big[\Gamma\Big(\frac{m+a}{2m+2}\Big)\Gamma\Big(\frac{m+(-1)^{k+1}a+2}{2m+2}\Big)\ \omega^{-k-1}-\Gamma\Big(\frac{m+(-1)^{k+1}a}{2m+2}\Big)\Gamma\Big(\frac{m+a+2}{2m+2}\Big)\ \Big]$$

and

$$M(0)=\pi+\frac{\pi}{2m+2.}$$

It follows from (5. 5) that putting

$$\rho=\Big[\frac{m+1}{2}\ \{2L\pi-\frac{m\pi}{2m+2}-M(a)\}\Big]^{\frac{1}{m+1}}, \tag{5. 6}$$

$$\Delta_{OP,QP,O}\arg\ y(x,\ a)\ =2L\pi+o(1). \tag{5. 7}$$

Consequently, we obtain the following result on the distribution of zeros of $y(x,\ a)$ on the Stokes lines.

THEOREM 5.1. *There exists an integer L_0 such that for any $L\geqq L_0$ exactly L zeros of $y(x,\ a)$ are located on each Stokes line* $\arg\ x=\dfrac{\pi}{2m+2}+\dfrac{k}{m+1}\pi$ $(k=1,2,\cdots,2m)$ *in the disk $|x|<\rho,\rho$ denoting the number (5. 6).*

6. Lommel's method. The well-known Lommel's method is very effective for the investigation of the location of zeros of $y(x,\ a)$ satisfying a second order linear differential equation of the form (1. 1). Suppose that $\alpha=re^{i\theta}$ is a non real zero of y or y'. The following identity can easily be obtained from the differential equations satisfied by $y(\alpha t)$ and $y(\beta t)$ and Green's symmetric identity:

$$(\alpha^{2m+2}-\beta^{2m+2})\int_c^1 t^{2m}\ y(\alpha t)y(\beta t)dt \tag{6. 1}$$

$$=(\alpha^{m+1}-\beta^{m+1})\{-\int_c^1 at^{m-1}\ y(\alpha t)y(\beta t)dt\}$$

$$+\big[\alpha y'(\alpha t)y(\beta t)-\beta y'(\beta t)\ y(\alpha t)\big]_c^1,$$

$$(\alpha^{2m+2}-\beta^{2m+2})\int_c^1\frac{d}{dt}\ y(\alpha t)\ \frac{d}{dt}y(\beta t)dt \tag{6. 2}$$

$$=(\alpha^{m+1}\beta^{2m+2}-\alpha^{2m+2}\beta^{m+1})\int_c^1 at^{m-1}\ y(\alpha t)y(\beta t)dt$$

$$+\left[\alpha^{2m+2}\beta y'(\beta t)y(\alpha t)-\alpha\beta^{2m+2}y'(\alpha t)y(\beta t)\right]_c^1,$$

where β is an arbitrary complex number and c is an arbitrary real number. Putting $\beta=\overline{\alpha}$ in these indentities (6. 1) and (6. 2), we can investigate zero-free domains of $y(x, a)$.

If $-\frac{\pi}{2m+2}<\theta<\frac{\pi}{2m+2}$, then, taking account of the fact that $y(x, a)$ is a subdominant solution and letting c tend to infinity in (6. 1), we have

$$2r^{m+1}\cos(m+1)\theta\cdot\int_1^\infty t^{2m}|y(\alpha t)|^2\,dt=-a\int_1^\infty t^{m-1}|y(\alpha t)|^2\,dt<0,$$

which is a contradiction. If x is positive real, $y''(x, a)$ and $y(x, a)$ have the same sign from (1. 1). Since $y(0, a)>0$, $\lim y(x, a)=0$ as $x\to\infty$, it follows that y and y' have no zeros. Thus there are no zeros of $y(x, a)$ in the sectorial domain: $-\frac{\pi}{2m+2}<\theta<\frac{\pi}{2m+2}$.

We now put $c=0$ and then obtain

$$\begin{aligned}(6.\ 3)\qquad & r^{2m+2}\sin(2m+2)\theta\int_0^1 t^{2m}|y(\alpha t)|^2\,dt\\ &=r^{m+1}\sin(m+1)\theta\{-\int_0^1 at^{m-1}|y(\alpha t)|^2\,dt\}\\ &+r\cdot\sin\theta\cdot y(0,a)y'(0,a),\end{aligned}$$

$$\begin{aligned}(6.\ 4)\qquad & \sin(2m+2)\theta\int_0^1|y'(\alpha t)|^2\,dt\\ &=r^{m+1}\sin(m+1)\theta\{-\int_0^1 at^{m-1}|y(\alpha t)|^2\,dt\}\\ &+r\cdot\sin(2m+1)\theta\cdot y(0,a)y'(0,a).\end{aligned}$$

Taking account of $y(0,a)y'(0,a)<0$, we can see that sectorial domains where one of the relations (6. 3) and (6. 4) does not hold are zero-free domains of $y(x, a)$. For instance, it is easily seen that the sectorial domains

$$(6.\ 5)\qquad \{\theta\mid\frac{\sin(m+1)\theta}{\sin(2m+2)\theta}>0,\quad\frac{\sin\theta}{\sin(2m+2)\theta}>0\}$$

and

$$(6.\ 6)\qquad \{\theta\mid\frac{\sin(m+1)\theta}{\sin(2m+2)\theta}>0,\quad\frac{\sin(2m+1)\theta}{\sin(2m+2)\theta}>0\}$$

are zero-free domains.

Next we shall investigate the location of zero on a ray: $\arg x=\phi$.
Let us put

$$(6.\ 7)\qquad x=\rho e^{i\phi}$$

where ϕ is constant. Then we have

$$e^{2i\phi}\int_0^\rho (x^{2m}+ax^{m-1})\,|\,y\,|^2\,d\rho=\left[\,y\frac{dy}{d\rho}\right]_0^\rho-\int_0^\rho\left|\frac{dy}{d\rho}\right|^2 d\rho. \tag{6. 8}$$

Putting $y(\rho e^{i\phi}, a)\cdot\dfrac{dy(\rho e^{i\phi},a)}{d\rho}=0$, we have from imaginary parts of (6. 8) that

$$\int_0^\rho \{\rho^{2m}\sin 2(m+1)\phi+a\rho^{m-1}\sin(m+1)\phi\}\,|\,y\,|^2\,d\rho=0. \tag{6. 9}$$

Therefore, if

$$\sin(m+1)\phi\cdot\{2\rho^{m+1}\cos(m+1)\phi+a\} \tag{6.10}$$

does not change sign sign on the ray $\arg x=\phi$, the product, $y\cdot\dfrac{dy}{dx}$ has no zero on the ray. Furthermore, if (6.10) does not change sign for $\rho_1<\rho<\rho_2$, there is at most one zero on this ray segment. As an example, we illustrate these results for the case in which $m=2$. See fig. 1.

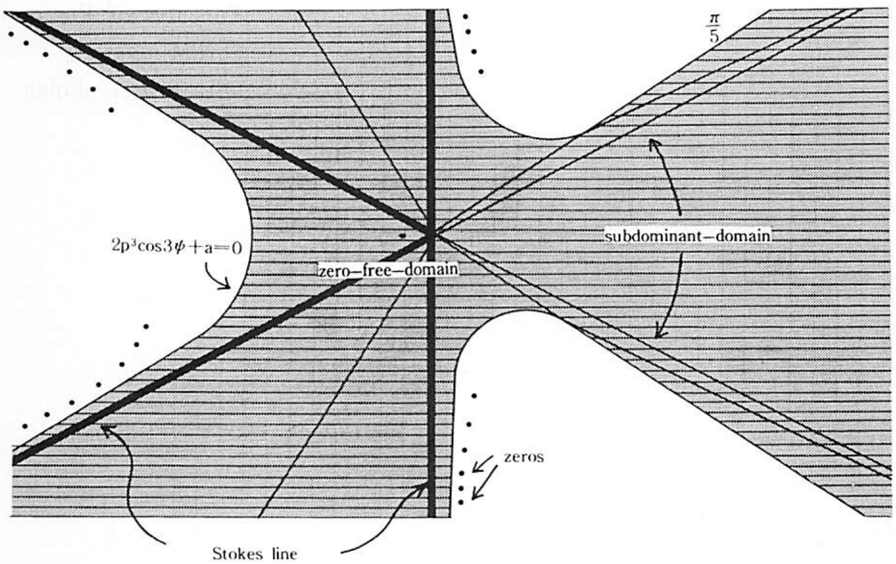

fig. 1

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