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A Note on Weakly Exchange and Weakly Clean Rings

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Abstract. We show that if R is a weakly exchange ring, then all idempotents of R can be lifted modulo the ideal $J(R)+2R$. Thus, if R is weakly clean, then $R/(J(R)+2R)$ is also weakly clean and idempotents lift modulo $J(R)+2R$. We also prove that if R is a ring of characteristic 2^n for some $n \in \mathbb{N}$, then R is weakly exchange (respectively, weakly clean) if, and only if, it is exchange (respectively, clean). This continues and extends own results from J. Math. Univ. Tokushima (2014) and J. Indones. Math. Soc. (2015).

1 Introduction and Backgrounds

Throughout this paper, let all rings be *associative*, not necessarily *commutative*, containing *identity* element. Our notations and terminology are classical and follow essentially those from [8]. For instance, for a ring R, the symbol $C(R)$ stands for the center of R, $U(R)$ stands for the unit group of R, and $J(R)$ stands for the Jacobson radical of R. It is well known that

$$J(R) = \{r \in R \ : \ 1 - trs \in U(R), \ \forall t, s \in R\}$$

and so, resultantly, $1 + J(R) \subseteq U(R)$.

The following concept has a key role in the contemporary ring theory.

• ([9]) A ring is called *clean* if, for every $r \in R$, there are a unit u and an idempotent e such that $r = u + e$.

Moreover, in [1] was introduced the important notion of a *weakly clean* ring in a commutative context. We, however, will state it in general.

• ([5], [6]) A ring R is called *weakly clean* if, for every $r \in R$, there are a unit u and an idempotent e such that either $r = u + e$ or $r = u - e$.

On the other hand, the next notion as stated in [9] also plays a key role in ring theory.

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• *A* ring R is said to be *exchange* if, for each $a \in R$, there is an idempotent $e \in aR$ such that $1-e \in (1-a)R$.

Generalizing the last, in [10] was defined the following:

• A ring R is said to be *weakly exchange* if, for each $a \in R$, there is an idempotent $e \in aR$ such that either $1-e \in (1-a)R$ or $1-e \in (1+a)R$.

Notice that clean rings are both weakly clean and exchange (see [9, Proposition 1.8 (1)]), while exchange rings are obviously weakly exchange. Both reversions are, however, false. In fact, as remarked in [1] and [4], there exists a weakly clean ring R in which 2 inverts that is not clean. Indeed, this is the ring $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$, where, for any prime p, $\mathbb{Z}_{(p)} = \{\frac{r}{s} \in \mathbb{Q} \ : \ p \nmid s\}$. This construction also manifestly illustrates that a commutative weakly clean ring is even not necessarily exchange. Likewise, weakly clean rings are weakly exchange (see, for example, [4]). On the other hand, the example in [2], using the ring of all integers $\mathbb{Z}$, is a clean ring whose center is exactly $\mathbb{Z}$ which is neither weakly clean nor weakly exchange. As for the classical clean, respectively exchange, rings we refer to Proposition 2.5 from [2] and Example 2.7 of [7], respectively.

Recall that idempotents of a ring R *lift modulo* the ideal I of R if whenever $x \in R$ with $x^2 - x \in I$ there exists an idempotent e of R such that $e - x \in I$. In that case it is said that e *lifts* x.

A brief history of the principal known results in this subject is as follows: In [5] (see [6] too) we obtained that if $R/J(R)$ is weakly exchange, respectively weakly clean, and idempotents lift modulo $J(R)$, then R is weakly exchange, respectively weakly clean. The converse holds if 2 is an element from $J(R)$. In particular, if $2 \in J(R)$, then the notions of being weakly exchange and exchange as well as weakly clean and clean, respectively, do coincide.

Thereby, it seems quite usual to ask whether we could weak the conditions on the element 2. Thus, the goal of the present paper is to establish two things: Firstly, that if R is weakly exchange, then all its idempotents lift modulo the ideal $J(R)+2R$; and secondly, that if $2^n \in J(R)$ for some arbitrary natural n, then any weakly exchange (resp., weakly clean) ring is exchange (resp., clean).

The article is organized as follows: In the first section, i.e. here, we provide the reader with some exposition of the already known results. In the next second section, we proceed by stating and proving our basic results. And finally, we finish our work with a few challenging problems of some interest and importance.

2 The Results

In [9] was proved that if R is an exchange ring, then idempotents lift modulo $J(R)$. However, for weakly exchange rings the following analogue holds.

Lemma 2.1 *If R is a weakly exchange ring, then all its idempotents are lifted modulo $J(R)+2R$.*

Proof. Given $x \in R$ such that $x^2 - x \in I = J(R) + 2R$. Exploiting Lemma 2.1 (b) in [5], we deduce that there is an idempotent e of R such that either $e - x \in (x^2 - x)R$ or $e + x \in (x^2 + x)R$. Furthermore, in the first situation, it follows at once that $e - x \in I$. As for the second one, one sees that $e - x = (e + x) - 2x \in (x^2 + x)R - 2x = (x^2 - x + 2x)R - 2x \subseteq (x^2 - x)R + 2xR - 2x \subseteq I$, as required. ∎

According to Lemma 2.1 it follows that if R is weakly clean, then $R/(J(R)+2R)$ is also weakly clean (and so it is clean being of characteristic 2) and idempotents lift modulo $J(R) + 2R$. The converse implication seems to be extremely difficult even in the case when 2 is non-invertible (compare with Problem 2 below). In fact, as emphasized above, there exists a weakly clean ring R with $2 \in U(R)$ which need not be clean.

Moreover, in regard to the preceding statement, we will show that the stated above "lifting modulo" property can be formulated in an equivalent way using the sign "+", but however this will not help us in proving that "weakly exchange" eventually implies "lifting modulo the Jacobson radical".

Proposition 2.2 *The following two conditions are tantamount for any ring R and its ideal I:*

(1) Whenever $x \in R$ with $x^2 - x \in I$ there is an idempotent e such that $e - x \in I$;

(2) Whenever $x \in R$ with $x^2 + x \in I$ there is an idempotent e such that $e + x \in I$.

Proof. (1) $\Rightarrow$ (2). Given $x^2 + x \in I$, we write $x^2 + x = (-x)^2 - (-x)$, whence there is an idempotent e such that $e - (-x) = e + x \in I$, as required.

(2) $\Rightarrow$ (1). Given $x^2 - x \in I$, we write $x^2 - x = (-x)^2 + (-x)$, whence there is an idempotent e such that $e + (-x) = e - x \in I$, as required. ∎

The following theorem strengthens the corresponding results in [5] and [6], respectively.

Theorem 2.3 *Suppose that R is a ring such that $2^n \in J(R)$ for some $n \geq 1$. Then the next two equivalencies hold:*

(i) R is weakly exchange if, and only if, R is exchange.

(ii) R is weakly clean, if and only if, R is clean.

Proof. As mentioned above, what suffices to demonstrate is that 2 belongs to $J(R)$, and hence we will use Propositions 2.5 and 2.6 from [5] to conclude the desired two points. And so, first, we shall assume that $\mathrm{char}(R) = 2^n$. In order to show that 2 lies in $J(R)$, it is necessary to check that, for any two elements t, s of R, the difference $1 - t2s = 1 - 2ts = 1 - 2z \in U(R)$ with $z = ts \in R$. To this aim, a straightforward computation shows that $(1 - 2z)^{2^n} = 1 - 2^{2^n}.z^{2^n}$ because the integer ${}_{2^n}C_i \cdot 2^i = 2^n \cdot \frac{2^i}{i} \cdot ({}_{2^n-1}C_{i-1})$ is divisible by 2^n for all $i \geq 1$. Since $2^{2^n} = 2^n y = 0$ for some $y \in R$, whence $(1 - 2z)^{2^n} = 1$, it readily follows that $1 - 2z$ inverts in R, as needed. As to treat the general case, R being either weakly exchange or weakly clean with $2^n \in J(R)$ implies with the aid of [5] that $R/J(R)$

is such a ring of characteristic 2^n. But by what we have established so far, 2 must lie in $J(R)$, as asserted. ■

Remark 1. It is worthwhile to noticing the interesting and surprising fact which was used in the proof of the above theorem that if $n \geq 1$ is an arbitrary integer, then 2^n lies in $J(R)$ exactly when 2 lies in $J(R)$. This can be established also directly by looking at $R/J(R)$ and especially that this quotient does not contain any non-zero central nilpotent elements. Actually, the essence of this fact is that 2 belongs to $C(R)$. This can be strengthened in the following way: Letting $c \in C(R)$, then $c^n \in J(R)$ for some $n \geq 2$ precisely when $c \in J(R)$. Indeed, for any $x \in R$, one has that

$$(1 - c^{n-1}x)(1 + c^{n-1}x) = 1 - c^n(c^{n-2}x^2) \in 1 + J(R) \subseteq U(R).$$

This implies that $1 - c^{n-1}x \in U(R)$, because $1 - c^{n-1}x$ and $1 + c^{n-1}x$ commute, and hence $c^{n-1} \in J(R)$. Continuing this procedure, we conclude that $c \in J(R)$, as promised.

Recollect that a ring R is said to be 2-*good* in the sense that each its element is the sum of two units. Such an element is also called 2-*good.* If an element in a ring R is a sum of three units, it is called 3-*good.* A ring R is said to be 3-*good* if every its element is 3-*good.*

A well-known result due to Camillo-Yu from [3] states that if R is a clean ring with $2 \in U(R)$, then R is a 2-good ring. The following claim somewhat settles the question of when a weakly clean ring is 2-good.

Proposition 2.4 *If R is a weakly clean ring such that 2 and 3 are both invertible in R, then R is 2-good.*

Proof. Let $x \in R$. Thus $\frac{x+1}{2} = u + e$ or $\frac{x+1}{2} = u - e$ for some unit u and an idempotent $e \in R$. If $\frac{x+1}{2} = u + e$, then $x = 2u + (2e - 1)$ where $(2e - 1)^2 = 1$. If $\frac{x+1}{2} = u - e$, then $x = 2u - (1 + 2e)$ where $(1 + 2e)(1 - \frac{2}{3}e) = 1$, that is, $1 + 2e$ is a unit. Since $2u \in U(R)$, in both cases x is a 2-good element, as required. ■

Remark 2. The above proof actually confirms once again the aforementioned result due to Camillo-Yu.

As an immediate consequence, we yield:

Corollary 2.5 *If R is a weakly clean ring for which 6 is invertible, then R is 2-good.*

Proof. It follows directly that $6 \in U(R)$ if and only if both $2 \in U(R)$ and $3 \in U(R)$, as needed. ■

If we drop the limitation on 3 to invert in R, the best we can offer at this stage is the following statement.

Proposition 2.6 *Let R be a weakly clean ring with 2 invertible in R. Then R is 3-good.*

Proof. Letting $x \in R$, we have two possibilities: $\frac{x+1}{2} = u + e$ or $\frac{x+1}{2} = u - e$ for some unit u and an idempotent e in R. If the first case occurs, then one writes $x = 2u + (2e-1) = u + u + (2e-1)$ and so as above x is 2-good and also 3-good. If now the second case occurs, then $x = 2u + (-2) + (1-2e)$ and thus x is obviously 3-good. Finally, in both cases, x is a 3-good element, as required. ■

3 Open Problems

We end the study with the next four questions of some interest and importance.

The definition of exchange rings turns out to be left-right symmetric (see, e.g., [9]). So, the next question seems to be adequate.

Problem 1. Does it follow that the definition of weakly exchange rings is also left-right symmetric?

Problem 2. Assume $2 \notin U(R)$. Is it true that R is weakly exchange (resp., weakly clean) if, and only if, $R/(J(R) + 2R)$ is weakly exchange (resp., weakly clean) and all idempotents lift modulo $J(R) + 2R$?

If $2 \in U(R)$, then the ideal $2R$ contains the invertible element 2 and thus $1 \in 2R$ which forces at once that $R = 2R$ and thereby there are no conditions to use.

Problem 3. Can the condition $3 \in U(R)$ be removed from the text of Proposition 2.4 and R to remain 2-good? In other words, is the ring R from Proposition 2.6 a 2-good ring?

Problem 4. Are Propositions 2.4 and 2.6 valid replacing "the invertibility of 2" by the weaker condition "1 is the sum of two units"?

Actually, the restriction $2 \in U(R)$ easily implies that $1 \in U(R) + U(R)$, while $3 \in U(R)$ will only imply that $2 \in U(R) + U(R)$ or $1 \in U(R) + U(R) + U(R)$.

Correction. In [5, Problem 6] the phrase "$R/M \cong B$, where B is Boolean" should be stated more precise as "$R/M \cong \mathbb{Z}_2$".

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