Kumamoto J. Math.
Vol.**30** (2017), 7–14

Symmetric differential polynomials

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(Received December 16, 2015)
(Accepted May 24, 2016)

Abstract. We show that any symmetric differential polynomial can be written as a ratio of differential polynomials in the elementary symmetric polynomials. An application to the twisted cohomology theory is presented.

1 Main theorem

It is well known that any symmetric polynomial in indeterminates $x_1, x_2, \ldots, x_n$ can be written as a polynomial in the elementary symmetric polynomials in the indeterminates. In this paper, we shall show a similar assertion for symmetric differential polynomials.

We fix a positive integer n greater than 1. Let (K, ∂) be an ordinary differential field of characteristic 0, and $x_1, x_2, \ldots, x_n$ be differential indeterminates. For $1 \leq j \leq n$, we denote by s_j the elementary symmetric polynomial in $x_1, x_2, \ldots, x_n$ of degree j:

$$s_j = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} x_{i_1} x_{i_2} \cdots x_{i_j}.$$

We regard s_j as a polynomial in the differential indeterminates. The symmetric group S_n acts on the set $K\{x_1, x_2, \ldots, x_n\}$ of differential polynomials by

$$\begin{aligned} &f^\sigma(x_1, \ldots, x_n, \partial x_1, \ldots, \partial x_n, \ldots, \partial^k x_1, \ldots, \partial^k x_n, \ldots) \\ &= f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \partial x_{\sigma(1)}, \ldots, \partial x_{\sigma(n)}, \ldots, \partial^k x_{\sigma(1)}, \ldots, \partial^k x_{\sigma(n)}, \ldots) \end{aligned}$$

for $f \in K\{x_1, \ldots, x_n\}$ and $\sigma \in S_n$. A differential polynomial is called symmetric if it is invariant under the action of S_n. We denote the set of symmetric differential polynomials in $K\{x_1, \ldots, x_n\}$ by $K\{x_1, \ldots, x_n\}^{S_n}$.

Our main theorem is the following.

Theorem 1. *We have*

$$K\{x_1, \ldots, x_n\}^{S_n} \subset K\langle s_1, \ldots, s_n \rangle,$$

Mathematical Subject Classification (2010): Primary 05E05; Secondary 13N99
Key words: symmetric polynomials, differential algebra, Picard-Fuchs equation
*Supported by the JSPS grant-in-aid for scientific research B, No.21340038 and the JSPS grant-in-aid for scientific research S, No. 24224001

where $K\langle s_1,\dots,s_n\rangle$ *denotes the quotient field of* $K\{s_1,\dots,s_n\}$.

Proof. We prove the assertion by induction on the number of differential indeterminates. We assume that the assertion holds when the number of differential indeterminates is $n-1$. We set

$$X=\{x_1,x_2,\dots,x_n\},$$

and

$$X_i=X\setminus\{x_i\}$$

for $1\le i\le n$. We denote by $s_{n,j}$ the elementary symmetric polynomial of degree j in n indeterminates in X, and by $s^{(i)}_{n-1,j}$ the elementary symmetric polynomial of degree j in $n-1$ indeterminates in X_i. We set

$$s_{n,0}=s^{(i)}_{n-1,0}=1.$$

Take any $i\in\{1,2,\dots,n\}$. Since the subgroup $S_{n-1}\subset S_n$ acts on $K\{X_i\}$, we have the inclusion

$$K\{X\}^{S_n}\subset K\{X_i\}^{S_{n-1}}\{x_i\}.$$

Then by the assumption, we get

$$K\{X\}^{S_n}\subset K\{X_i\}^{S_{n-1}}\{x_i\}\subset K\langle s^{(i)}_{n-1,1},\dots,s^{(i)}_{n-1,n-1}\rangle\{x_i\}.$$

From the identities

$$s_{n,p}=x_is^{(i)}_{n-1,p-1}+s^{(i)}_{n-1,p}\quad(1\le p\le n-1),$$

we obtain

$$s^{(i)}_{n-1,p}=\sum_{q=0}^{p}(-x_i)^q s_{n,p-q} \tag{1.1}$$

for $1\le p\le n-1$. Hence we have the inclusion

$$K\{X\}^{S_n}\subset K\langle s_{n,1},\dots,s_{n,n-1}\rangle\{x_i\}. \tag{1.2}$$

By using (1.1), we get

$$\begin{aligned}
s_{n,n}&=x_is^{(i)}_{n-1,n-1}\\
&=x_i\sum_{q=0}^{n-1}(-x_i)^q s_{n,n-1-q}\\
&=-\sum_{q=0}^{n-1}(-x_i)^{q+1}s_{n,n-1-q}.
\end{aligned}$$

Applying ∂^l for $l=1,2,\dots$, one gets

$$\partial^l s_{n,n}=\partial^l x_i\sum_{q=0}^{n-1}(q+1)(-x_i)^q s_{n,n-1-q}+R_l,$$

where

$$R_l \in K\{s_{n,1}, \ldots, s_{n,n-1}\}[x_i, \partial x_i, \ldots, \partial^{l-1} x_i].$$

Then we have

$$\partial^l x_i = \frac{\partial^l s_{n,n} - R_l}{\sum_{q=0}^{n-1} (q+1)(-x_i)^q s_{n,n-1-q}}. \tag{1.3}$$

Take any $f \in K\{X\}^{S_n}$. We regard f as an element in $K\langle s_{n,1}, \ldots, s_{n,n-1}\rangle\{x_i\}$ via the inclusion (1.2). Let k be the maximal integer such that $\partial f/\partial(\partial^k x_i) \neq 0$. We replace $\partial^k x_i$ in f by the right hand side of (1.3) for $l = k$, so that we have

$$f \in K\langle s_{n,1}, \ldots, s_{n,n}\rangle(x_i, \partial x_i, \ldots, \partial^{k-1} x_i).$$

In a similar way, by applying (1.3) repeatedly, we come to the expression

$$f = \frac{u(x_i)}{v(x_i)}$$

with

$$u(T), v(T) \in K\langle s_{n,1}, \ldots, s_{n,n}\rangle[T].$$

Since f is symmetric, it can be expressed as

$$f = \frac{1}{n} \sum_{i=1}^{n} \frac{u(x_i)}{v(x_i)}.$$

If we reduce the right hand side into the form U/V, we see

$$U, V \in K\{s_{n,1}, \ldots, s_{n,n}\}[X]^{S_n}.$$

Evidently we have $K\{s_{n,1}, \ldots, s_{n,n}\}[X]^{S_n} = K\{s_{n,1}, \ldots, s_{n,n}\}$, which completes the proof. □

Note that the above proof gives an effective algorithm to express a symmetric differential polynomial as a differential polynomial in $s_1, s_2, \ldots, s_n$. For example, by applying the process in the proof, we get

$$x_1' x_2' = \frac{s_1 s_1' s_2' - {s_1'}^2 s_2 - {s_2'}^2}{{s_1}^2 - 4 s_2},$$

where we denote $\partial(a) = a'$.

Corollary 2. *We have*

$$K\langle x_1, x_2, \ldots, x_n\rangle^{S_n} = K\langle s_1, s_2, \ldots, s_n\rangle.$$

Proof. Clearly we have $K\langle x_1, x_2, \ldots, x_n\rangle^{S_n} \supset K\langle s_1, s_2, \ldots, s_n\rangle$.

Take any $f = u/v \in K\langle x_1, x_2, \ldots, x_n\rangle^{S_n}$ with $u, v \in K\{x_1, \ldots, x_n\}$. If $u \in K\{x_1, \ldots, x_n\}^{S_n}$, we have $v \in K\{x_1, \ldots, x_n\}^{S_n}$, which shows $f \in K\langle s_1, s_2, \ldots, s_n\rangle$ by Theorem 1. If $u \notin K\{x_1, \ldots, x_n\}^{S_n}$, we rewrite f as

$$f = \frac{u \prod_{\sigma \in S_n \setminus \{1\}} u^\sigma}{v \prod_{\sigma \in S_n \setminus \{1\}} u^\sigma}.$$

Then the numerator of the right hand side is in $K\{x_1, \ldots, x_n\}^{S_n}$, and hence it is reduced to the first case. □

Historical Remark. After obtaining our results, we noticed some results concerning a similar problem from another viewpoint. Consider a linear ordinary differential equation

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n(t)x = 0, \tag{1.4}$$

where $a_1(t), \ldots, a_n(t)$ are elements of a differential field K. Let $(x_1, x_2, \ldots, x_n)$ be a fundamental system of solutions of (1.4). We define the action of $\mathrm{GL}(n, C_K)$ by

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_n)g, \quad g \in \mathrm{GL}(n, C_K),$$

where C_K is the field of constants of K. Appell [1] showed that, if a polynomial in $x_1, \ldots, x_n$ and their derivatives is invariant under the action of $\mathrm{GL}(n, C_K)$ up to scalar multiplication, the polynomial is a polynomial in $a_1(t), \ldots, a_n(t)$, their derivatives and the inverse of the Wronskian $W(a_1, \ldots, a_n)(t)$. The same assertion was shown in purely algebraic way by Kung and Rota [4], who noted that the coefficients $a_1(t), \ldots, a_n(t)$ can be written as ratios of determinants of matrices with entries in $x_1, \ldots, x_n$ and their derivatives. They call the determinants the generalized Wronskians. Then it may be natural to thinks of Schur functions which are defined by using determinants and appear in the invariant theory. In this direction, Kung [3] defined differential Schur functions, and showed several fundamental properties. In particular, differential Schur functions are invariants of $\mathrm{GL}(n, C_K)$ action. Thus, these results are concerned with the invariant theory for the action of the general linear group, while our results are concerned with the invariant theory for the action of the symmetric group.

Since the symmetric group S_n can be regarded as a subgroup of $\mathrm{GL}(n, C_K)$, differential Schur functions are symmetric in our sense. Then, thanks to Corollary 2, we see that differential Schur functions can be written as ratios of differential polynomials of elementary symmetric functions.

2 Application

Our result can be applied to explicit calculations of twisted de Rham cohomologies. As an example, we derive the Picard-Fuchs equations satisfied by periods of elliptic curves. We consider an elliptic curve

$$y^2 = x^3 - s_1 x^2 + s_2 x - s_3, \tag{2.1}$$

where s_1, s_2, s_3 are functions in some variable t. The periods of (2.1) are given by the integral

$$u(t) = \int_\Delta \frac{dx}{\sqrt{x^3 - s_1x^2 + s_2x - s_3}}, \tag{2.2}$$

where Δ is a path which connects two of the zeros of the right hand side of (2.1).

Theorem 3. *Let (2.1) be an elliptic curve with holomorphic functions s_1, s_2, s_3 in t as coefficients. Then the Picard-Fuchs equation satisfied by the periods $u(t)$ in (2.2) is given by*

$$\frac{d^2u}{dt^2} + p(t)\frac{du}{dt} + q(t)u = 0,$$

where

$$\begin{aligned}
p(t) = &-\frac{2({s_2}^2 - 3s_1s_3)s_1'' - (s_1s_2 - 9s_3)s_2'' + 2({s_1}^2 - 3s_2)s_3''}{(2({s_2}^2 - 3s_1s_3)s_1' - (s_1s_2 - 9s_3)s_2' + 2({s_1}^2 - 3s_2)s_3')} \\
&- \Big(2(2s_1{s_2}^4 - 15{s_1}^2{s_2}^2s_3 + 6{s_2}^3s_3 + 24{s_1}^3{s_3}^2 - 81{s_3}^3)(s_1')^2 \\
&- ({s_1}^3{s_2}^2 - 8s_1{s_2}^3 + 4{s_1}^4s_3 - 18{s_1}^2s_2s_3 + 108{s_2}^2s_3 - 135s_1{s_3}^2)(s_2')^2 \\
&- 4({s_1}^2 - 3s_2)(2{s_1}^3 - 9s_1s_2 + 27s_3)(s_3')^2 \\
&- ({s_1}^2{s_2}^3 + 12{s_2}^4 - 12{s_1}^3s_2s_3 - 54s_1{s_2}^2s_3 + 216{s_1}^2{s_3}^2 - 243s_2{s_3}^2)s_1's_2' \\
&- 2({s_1}^3{s_2}^2 - 8s_1{s_2}^3 + 4{s_1}^4s_3 - 18{s_1}^2s_2s_3 + 108{s_2}^2s_3 - 135s_1{s_3}^2)s_1's_3' \\
&+ (8{s_1}^4s_2 - 57{s_1}^2{s_2}^2 + 84{s_2}^3 + 12{s_1}^3s_3 + 54s_1s_2s_3 - 405{s_3}^2)s_2's_3'\Big) \\
&/\Big(({s_1}^2{s_2}^2 - 4{s_2}^3 - 4{s_1}^3s_3 + 18s_1s_2s_3 - 27{s_3}^2) \\
&\times (2({s_2}^2 - 3s_1s_3)s_1' - (s_1s_2 - 9s_3)s_2' + 2({s_1}^2 - 3s_2)s_3')\Big),
\end{aligned}$$

$$\begin{aligned}
q(t) = &-\frac{(s_2s_2' - 2s_1s_3')s_1'' - (s_2s_1' - 3s_3')s_2'' + (2s_1s_1' - 3s_2')s_3''}{2(2({s_2}^2 - 3s_1s_3)s_1' - (s_1s_2 - 9s_3)s_2' + 2({s_1}^2 - 3s_2)s_3')} \\
&+ \Big(-2({s_2}^4 - 9s_1{s_2}^2s_3 + 12{s_1}^2{s_3}^2 + 18s_2{s_3}^2)(s_1')^3 \\
&- 3(s_1{s_2}^2 + 4{s_1}^2s_3 - 21s_2s_3)(s_2')^3 + 30({s_1}^2 - 3s_2)(s_3')^2 \\
&- (s_1s_2 - 9s_3)({s_2}^2 + 12s_1s_3)(s_1')^2s_2' \\
&+ 2(5{s_1}^2{s_2}^2 - 7{s_2}^3 - 4{s_1}^3s_3 - 18s_1s_2s_3 + 54{s_3}^2)(s_1')^2s_3' \\
&+ 2({s_1}^2{s_2}^2 + {s_2}^3 + 4{s_1}^3s_3 - 18s_1s_2s_3 - 54{s_3}^2)(s_2')^2s_1' \\
&+ 6(4{s_1}^2s_2 - 9{s_2}^2 - 9s_1s_3)(s_2')^2s_3' \\
&+ 2(8{s_1}^4 - 36{s_1}^2s_2 + 45{s_2}^2 - 27s_1s_3)(s_3')^2s_1' \\
&- 3(8{s_1}^3 - 21s_1s_2 - 27s_3)(s_3')^2s_2 \\
&- 2(8{s_1}^3s_2 - 15s_1{s_2}^2 - 36{s_1}^2s_3 + 27s_2s_3)s_1's_2's_3'\Big) \\
&/\Big(4({s_1}^2{s_2}^2 - 4{s_2}^3 - 4{s_1}^3s_3 + 18s_1s_2s_3 - 27{s_3}^2) \\
&\times (2({s_2}^2 - 3s_1s_3)s_1' - (s_1s_2 - 9s_3)s_2' + 2({s_1}^2 - 3s_2)s_3')\Big).
\end{aligned}$$

Proof. Let $f(x)$ denote the right hand side of (2.1). We factorize $f(x)$ as

$$f(x) = (x - e_1)(x - e_2)(x - e_3),$$

and then have

$$\begin{aligned} e_1 + e_2 + e_3 &= s_1, \\ e_1e_2 + e_2e_3 + e_3e_1 &= s_2, \\ e_1e_2e_3 &= s_3. \end{aligned}$$

We regard e_1, e_2, e_3 as elements in a differential field extension of $(\mathbb{Q}(s_1, s_2, s_3), d/dt)$. Let D be a divisor defined by $f(x)$. We set

$$\begin{aligned} U(x) &= (x - e_1)^{\frac{1}{2}}(x - e_2)^{-\frac{1}{2}}(x - e_3)^{-\frac{1}{2}}, \\ \omega &= \frac{dU}{U}, \\ \nabla_\omega &= d + \omega \wedge . \end{aligned}$$

We can take

$$\varphi_1 = \frac{dx}{x - e_1}, \ \varphi_2 = \frac{dx}{x - e_2}$$

as a basis of the twisted cohomology group $H^1(\Omega^\bullet(*D), \nabla_\omega)$. By using the basis, we define

$$u_i(t) = \int_\Delta U\varphi_i \quad (i = 1, 2),$$

so that we have $u(t) = u_1(t)$.

In the following we sometimes use $'$ for d/dt. We have

$$\begin{aligned} \frac{du_1}{dt} &= \int_\Delta U \left(\frac{1}{2}\frac{e_1'}{(x - e_1)^2} + \frac{1}{2}\frac{e_2'}{(x - e_2)(x - e_1)} + \frac{1}{2}\frac{e_3'}{(x - e_3)(x - e_1)} \right) dx, \\ \frac{du_2}{dt} &= \int_\Delta U \left(-\frac{1}{2}\frac{e_1'}{(x - e_1)(x - e_2)} + \frac{3}{2}\frac{e_2'}{(x - e_2)^2} + \frac{1}{2}\frac{e_3'}{(x - e_3)(x - e_2)} \right) dx. \end{aligned}$$

Then, by expressing the twisted 1-forms in the above integrand as linear combinations of the basis φ_1, φ_2, we obtain the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{2.3}$$

where

$$\begin{aligned} a &= -\frac{1}{2}\frac{e_2' - e_1'}{e_2 - e_1}, & b &= \frac{1}{2}\left(\frac{e_2' - e_1'}{e_2 - e_1} - \frac{e_3' - e_1'}{e_3 - e_1} \right), \\ c &= \frac{1}{2}\left(-\frac{e_1' - e_2'}{e_1 - e_2} + \frac{e_3' - e_2'}{e_3 - e_2} \right), & d &= \frac{1}{2}\frac{e_1' - e_2'}{e_1 - e_2} - \frac{e_3' - e_2'}{e_3 - e_2}. \end{aligned}$$

The Picard-Fuchs equation for the integral (2.2) is now derived from the system (2.3) as a single differential equation satisfied by u_1. Thus we obtain

$$\frac{d^2u}{dt^2} + p\frac{du}{dt} + qu = 0, \tag{2.4}$$

where

$$p = -\left(a + d + \frac{b'}{b}\right),\ q = -a' + \frac{ab'}{b} + ad - bc.$$

Since $u(t) = u_1(t)$ is symmetric in e_1, e_2, e_3, we know a priori that p, q are ratios of symmetric differential polynomials in e_1, e_2, e_2. In fact, we see that p and q are rational functions in $e_1, e_2, e_3, e_1', e_2', e_3', e_1'', e_2'', e_3''$ which are invariant under the action of the symmetric group S_3.

By using the result in the previous section, we have

$$\begin{aligned} e_1' &= \frac{{e_1}^2 s_1' - e_1 s_2' + s_3'}{3{e_1}^2 - 2e_1 s_1 + s_2}, \\ e_2' &= \frac{{e_2}^2 s_1' - e_2 s_2' + s_3'}{3{e_2}^2 - 2e_2 s_1 + s_2}, \\ e_3' &= \frac{{e_3}^2 s_1' - e_3 s_2' + s_3'}{3{e_3}^2 - 2e_3 s_1 + s_2}, \end{aligned}$$

and

$$\begin{aligned} e_1'' &= \frac{{e_1}^2 s_1'' - e_1 s_2'' + s_3'' + 4e_1 e_1' s_1' - 2e_1' s_2' + 2(e_1')^2 s_1 - 6e_1(e_1')^2}{3{e_1}^2 - 2e_1 s_1 + s_2}, \\ e_2'' &= \frac{{e_2}^2 s_1'' - e_2 s_2'' + s_3'' + 4e_2 e_2' s_1' - 2e_2' s_2' + 2(e_2')^2 s_1 - 6e_2(e_2')^2}{3{e_2}^2 - 2e_2 s_1 + s_2}, \\ e_3'' &= \frac{{e_3}^2 s_1'' - e_3 s_2'' + s_3'' + 4e_3 e_3' s_1' - 2e_3' s_2' + 2(e_3')^2 s_1 - 6e_3(e_3')^2}{3{e_3}^2 - 2e_3 s_1 + s_2}. \end{aligned}$$

Put these expressions into p and q. Then we have rational functions symmetric in e_1, e_2, e_3 with coefficients in $\mathbb{Q}\{s_1, s_2, s_3\}$. Then they can be written as elements in $\mathbb{Q}\langle s_1, s_2, s_3\rangle$. The explicit forms of p, q in the theorem can be easily obtained from the above process by the help of Symmetric Reduction Algorithm of any symbolic computing systems. $\square$

Theorem 3 gives one way to get the Picard-Fuchs equation, where we need only a formal computation. There is no need to study the behavior of the integral (2.2). We note that one factor in the denominator of p and q is just the discriminant of $f(x)$. Namely we have

$${s_1}^2{s_2}^2 - 4{s_2}^3 - 4{s_1}^3 s_3 + 18 s_1 s_2 s_3 - 27{s_3}^2 = (e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2.$$

The other factor comes from b, whose zeros may give apparent singular points for the Picard-Fuchs equation.

Beukers [2] showed the irrationality of $\zeta(2)$ by using the integral (2.2) with

$$s_1 = -\frac{t^2 + 6t + 1}{4},\ s_2 = \frac{t(t+1)}{2},\ s_3 = -\frac{t^2}{4} \tag{2.5}$$

(without using the expression $\zeta(2) = \pi^2/6$, of course). In this case, the Picard-Fuchs equation for the integral has one accessory parameter. Beukers determined the special value of the accessory parameter by calculating a Taylor expansion of the integral. We can apply Theorem 3 directly to this case. Namely we put (2.5) into $p(t)$ and $q(t)$ in Theorem 3, and get the Picard-Fuchs equation

$$t(t^2 + 11t - 1)\frac{d^2u}{dt^2} + (3t^2 + 22t - 1)\frac{du}{dt} + (t + 3)u = 0.$$

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