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Point-arrangements in the real projective spaces and the Fibonacci polynomials

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Abstract. We find a relation between the Fibonacci polynomials and arrangements of $n+3$ points in the real projective n-space admitting an action of the cyclic group of order $n+3$. We also describe explicitly the rational curve of degree n passing through these $n+3$ points, and determine the permutation of the $n+3$ points induced by this curve.

Introduction

Arrangements of $n+2$ points in general position in the real projective n-space $\mathbb{P}^n = \mathbb{P}^n(\mathbb{R})$ are unique up to projective transformations. Those of $m := n+3$ points are projectively not unique, but they are combinatorially unique. We are interested in arrangements of m points which admit an action of the cyclic group of order m.

Let $p_1, \ldots, p_{n+2}$ be $n+2$ points in $\mathbb{P}^n$ in general position. We add another point p_m, and require that the m points $p_1, \ldots, p_{n+2}, p_m$ admit a projective transformation σ inducing the cyclic permutation:

$$\sigma : \; p_1 \to p_2 \to \cdots \to p_{n+2} \to p_m \to p_1.$$

There always exist such p_m and σ, and in fact there are several choices in general. Our first theorem (Theorem 1 in §2) asserts that such choices exactly correspond to the roots of the *Fibonacci polynomial* $F_n(t)$ of degree $[n/2]+1$. And moreover, the resulting m points $p_1, \ldots, p_{n+2}, p_m$ are in general position if and only if the corresponding root is "primitive", i.e., a root of the *core Fibonacci polynomial* $f_n(t)$, which is an irreducible factor of $F_n(t)$ of degree $\varphi(m)/2$. Here, $\varphi(m)$ denotes Euler's function counting the number of positive integers less than m and co-prime to m.

On the other hand, for m points in $\mathbb{P}^n$ in general position, there is a unique rational curve C of degree n passing through these points. When we view the curve

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C as an image of $\mathbb{P}^1(\mathbb{R})$, the natural order in $\mathbb{R}$ determines a cyclic permutation of these points. For the points corresponding to a root of the core Fibonacci polynomial as above, we can explicitly compute this permutation (Corollary 2 to Theorem 2 in §3). More precisely, let $-|1+\zeta|^{-2}$ be a root of $f_n(t)$, where ζ is a *primitive* m-th root of unity (see §1 for the description of roots of $f_n(t)$), and p_m the m-th point associated to this root by Theorem 1. For each j $(1 \le j \le m)$, denote by q_j the point in $\mathbb{P}^1$ such that $C(q_j) = p_j$. Without loss of generality, we may assume $q_1 = \infty$, $q_2 = 0$, $q_1 = 1$. Then we show in Theorem 2 that, there exits a linear fractional transformation R from $\mathbb{P}^1(\mathbb{R})$ to the unit circle in the complex plane, preserving the natural orientation of $\mathbb{P}^1(\mathbb{R})$ and the unit circle (counter clock-wise), such that

$$R(q_1) = \zeta^{-1},\ R(q_2) = 1,\ R(q_3) = \zeta,\ R(q_4) = \zeta^2,\ \ldots,\ R(q_m) = \zeta^{m-2}.$$

Since ζ is a primitive m-th root of unity, the m points

$$1,\ \zeta,\ \zeta^2,\ \ldots,\ \zeta^{m-2},\ \zeta^{m-1} = \zeta^{-1}$$

form vertices of a regular m-gon on the unit circle. From this, if we write $\zeta = \zeta_m^i$ with $\zeta_m = e^{2\pi\sqrt{-1}/m}$ and $(i,m) = 1$, we see that the cyclic permutation determined by the curve C is the 'i-skip mod m', i.e., the permutation of $\{1, 2, \ldots, m\}$ given by $\{\overline{0\cdot i}+1, \overline{1\cdot i}+1, \ldots, \overline{(m-1)\cdot i}+1\}$, where $\bar{l}$ denotes the residue of l mod m such that $0 \le \bar{l} \le m-1$.

After introducing the necessary properties of Fibonacci polynomials in §1, we state and prove Theorem 1 in §2 and Theorem 2 in §3. In the final section §4, we discuss fixed points of the transformation σ.

1 Fibonacci polynomials

In this section, we summarize properties of the polynomials $F_k(t)$ and $f_k(t)$ that we need in this paper.

Definition 1. The Fibonacci polynomials $F_k(t)$ are defined as

$$F_{-2} = F_{-1} = 1, \quad F_k = F_{k-1} + tF_{k-2}, \quad k = 0, 1, 2, \ldots$$

The degree of F_k is $[k/2] + 1$.

Remark 1. In the literature (e.g., [Ko]), the Fibonacci polynomial $\widetilde{F}_k(t)$ is defined by $\widetilde{F}_0 = 0,\ \widetilde{F}_1 = 1,\ \widetilde{F}_k = t\widetilde{F}_{k-1}(t) + \widetilde{F}_{k-2}(t)$ $(k \ge 2)$. The relation to our $F_k(t)$ is $F_k(t) = \sqrt{t}^{k+2}\widetilde{F}_{k+3}(1/\sqrt{t})$. From this, all properties described in the sequel should in principle follow from known properties of $\widetilde{F}_k(t)$. We nevertheless supply proofs for the convenience of the reader. (In [AJ] the Fibonacci polynomial $F_k(s,t)$ is defined by $F_0 = 1,\ F_1 = s,\ F_k = sF_{k-1} - tF_{k-2}$ $(k \ge 2)$.)

For notational simplicity, put $G_k = F_{k-3}$ $(k \ge 1)$. Of course the G_k's satisfy the same recursion with $G_1 = G_2 = 1$.

Proposition 1. *$G_k(t)$ is a polynomial of degree $[(k-1)/2]$ and is explicitly given as*

$$G_k(t) = \sum_{i=0}^{[(k-1)/2]} \binom{k-1-i}{i} t^i, \quad k \geq 1.$$

Also, $G_k(t)$ admits the following expression:

$$G_k(t) = \frac{\alpha^k - \beta^k}{\sqrt{1+4t}}, \tag{1}$$

where

$$\alpha = \frac{1+\sqrt{1+4t}}{2}, \quad \beta = \frac{1-\sqrt{1+4t}}{2}.$$

Proof. The first formula is easily proved by induction. The second can be shown either by the generating function $\sum_{k=0}^{\infty} G_{k+1}(t)X^k = 1/(1-X-tX^2) = 1/(1-\alpha X)(1-\beta X)$ or by checking the right-hand side satisfies the same recurrence relation as $G_k(t)$.

We introduce a new polynomial (a priori, a rational function) $g_k(t)$. The core Fibonacci polynomial $f_k(t)$ is defined as $f_k(t) = g_{k+3}(t)$.

Definition 2. Put

$$g_k(t) = \prod_{d|k} G_d(t)^{\mu(k/d)}, \quad k \geq 1,$$

where d runs over all positive divisors of k, and μ is the Möbius function[1]. Note that $g_1 = g_2 = 1$.

Proposition 2. *1) For $k \geq 3$, $g_k(t)$ is a polynomial of degree $\varphi(k)/2$, and is irreducible over $\mathbb{Q}$.*

2) The irreducible decomposition of $G_k(t)$ over $\mathbb{Q}$ is given by

$$G_k(t) = \prod_{2<d|k} g_d(t).$$

In terms of $F_k(t)$ and $f_k(t)$, this can be written as

$$F_k(t) = \prod_{2<d|k+3} f_{d-3}(t).$$

3) The $g_k(t)$ is expressed as

$$g_k(t) = \beta^{\varphi(k)} \Phi_k(\alpha/\beta),$$

where

$$\Phi_k(t) = \prod_{d|k} (t^d - 1)^{\mu(k/d)}$$

is the k-th cyclotomic polynomial.

[1] $\mu(n) = 0$ if n has a square factor and $\mu(n) = (-1)^\nu$ if n is a product of ν distinct primes. $\mu(1) = 1$.

Proof. By (1), we have

$$\begin{aligned}
g_k &= \prod_{d|k} G_d(t)^{\mu(k/d)} = \prod_{d|k} \left\{ \frac{\alpha^d - \beta^d}{\sqrt{1+4t}} \right\}^{\mu(k/d)} \\
&= \left(\frac{1}{\sqrt{1+4t}} \right)^{\sum_{d|k} \mu(k/d)} \prod_{d|k} \left(\alpha^d - \beta^d \right)^{\mu(k/d)} \\
&= \prod\nolimits_{d|k} \left(\alpha^d - \beta^d \right)^{\mu(k/d)} = \beta^{\sum_{d|k} d\mu(k/d)} \prod\nolimits_{d|k} \left\{ \left(\frac{\alpha}{\beta} \right)^d - 1 \right\}^{\mu(k/d)} \\
&= \beta^{\varphi(k)} \Phi_k(\alpha/\beta).
\end{aligned}$$

Here, we have used the well-known identities $\sum_{d|k} \mu(k/d) = 0$ and $\sum_{d|k} d\mu(k/d) = \varphi(k)$. This proves 3). Since the cyclotomic polynomial Φ_k is of degree $\varphi(k)$, $g_k(t)$ is a polynomial in α and β of total degree $\varphi(k)$, which is symmetric in α and β because of the expression $g_k = \prod_{d|k} \left(\alpha^d - \beta^d \right)^{\mu(k/d)}$ as above and $(-1)^{\sum_{d|k} \mu(k/d)} = 1$. Therefore, $g_k(t)$ is a polynomial in t, of degree at most $\varphi(k)/2$ because $\alpha + \beta = 1$ and $\alpha\beta = -t$. The formula in 2) follows from the definition of $g_k(t)$ and the Möbius inversion formula. To prove the irreducibility of $g_k(t)$ and find the exact degree, we look at the roots of $g_k(t)$. By the formula in 3), we have

$$g_k(t) = \beta^{\varphi(k)} \prod_{\zeta:\ \text{primitive } k\text{-th root of unity}} (\alpha/\beta - \zeta).$$

Because $G_k(0) = 1$ for all k, we have $g_k(0) = 1$ and so β cannot be zero ($\beta = 0 \Leftrightarrow t = 0$). Hence,

$$\begin{aligned}
g_k(t) = 0 \quad &\Leftrightarrow \frac{1+\sqrt{1+4t}}{1-\sqrt{1+4t}} = \zeta : \quad \text{primitive } k\text{-th root of unity} \\
&\Leftrightarrow t = \frac{1}{4} \left\{ \left(\frac{1-\zeta}{1+\zeta} \right)^2 - 1 \right\} = -\frac{1}{\zeta + \zeta^{-1} + 2} = -\frac{1}{|1+\zeta|^2}.
\end{aligned}$$

Assume $k \geq 3$, and write $\zeta = e^{2\pi l \sqrt{-1}/k}$ with an integer l, so that $\zeta + \zeta^{-1} = 2\cos(2l\pi/k)$. Since ζ and ζ^{-1} give the same root, and ζ is primitive, we see that exactly $\varphi(k)/2$ values

$$t = -\frac{1}{2\cos\frac{2l\pi}{k} + 2} = -\frac{1}{4\cos^2\frac{l\pi}{k}}, \quad (l,k) = 1, \quad 1 \leq l \leq \left[\frac{k-1}{2} \right]$$

give distinct roots of $g_k(t)$. Hence $g_k(t)$ is of degree $\varphi(k)/2$ (remember we have shown the degree is at most $\varphi(k)/2$), and has distinct roots. Since its splitting field is $\mathbb{Q}(\cos(2\pi/k)) = \mathbb{Q}(\zeta + \zeta^{-1})$, which is the maximal real subfield of degree $\varphi(k)/2$ of the cyclotomic field $\mathbb{Q}(\zeta)$, we conclude that the polynomial g_k is irreducible over $\mathbb{Q}$.

Corollary 1. *The roots of $g_k(t)$ are given by*

$$-\frac{1}{|1+\zeta_k^i|^2} = -\frac{1}{4\cos^2\frac{i\pi}{k}}, \quad (i,k) = 1, \quad 1 \leq i \leq \left[\frac{k-1}{2} \right].$$

Here, $\zeta_k = e^{2\pi\sqrt{-1}/k}$. *In particular, all roots are negative real numbers, and if* $k \neq k'$, *roots of* $g_k(t)$ *and* $g_{k'}(t)$ *never coincide.*

The roots of $G_k(t)$ *are given by*

$$-\frac{1}{|1+\zeta_k^i|^2} = -\frac{1}{4\cos^2\frac{i\pi}{k}}, \quad 1 \leq i \leq \left[\frac{k-1}{2}\right].$$

Examples: Factorizations of the first several Fibonacci polynomials are as follows:

$$\begin{array}{llll}
F_0 = f_0, & F_1 = f_1, & F_2 = f_2, & F_3 = f_0 f_3, \\
F_4 = f_4, & F_5 = f_1 f_5, & F_6 = f_0 f_6, & F_7 = f_2 f_7, \\
F_8 = f_8, & F_9 = f_0 f_1 f_3 f_9, & F_{10} = f_{10}, & F_{11} = f_4 f_{11}, \\
F_{12} = f_0 f_2 f_{12}, & F_{13} = f_1 f_5 f_{13}, & F_{14} = f_{14}, & F_{15} = f_0 f_3 f_6 f_{15}, \\
F_{16} = f_{16}, & F_{17} = f_1 f_2 f_7 f_{17}, & F_{18} = f_0 f_4 f_{18}, & F_{19} = f_8 f_{19}, \\
F_{20} = f_{20}, & F_{21} = f_0 f_1 f_3 f_5 f_9 f_{21}, & F_{22} = f_2 f_{22}, & F_{23} = f_{10} f_{23}, \ldots,
\end{array}$$

whereas the 'core Fibonacci polynomials' are given by

$$\begin{array}{ll}
f_0 = t+1, & f_1 = 2t+1, \\
f_2 = t^2+3t+1, & f_3 = 3t+1, \\
f_4 = t^3+6t^2+5t+1, & f_5 = 2t^2+4t+1, \\
f_6 = t^3+9t^2+6t+1, & f_7 = 5t^2+5t+1, \\
f_8 = t^5+15t^4+35t^3+28t^2+9t+1, & f_9 = t^2+4t+1, \\
f_{10} = t^6+21t^5+70t^4+84t^3+45t^2+11t+1, & f_{11} = 7t^3+14t^2+7t+1, \ldots
\end{array}$$

When $n = 18$, we have $3, 7 \mid 21 = 18+3$, and the twelve numbers $1, 2, \ldots, 19, 20$ are coprime to 21. So

$$F_{18} = f_0 f_4 \cdot f_{18}, \quad \deg\, f_{18} = 12/2 = 6.$$

When $n = 21$, we have $2, 3, 4, 6, 8, 12 \mid 24 = 21+3$, and the eight numbers $1, 5, \ldots, 19, 23$ are coprime to 24. So

$$F_{21} = f_0 f_1 f_3 f_5 f_9 \cdot f_{21}, \quad \deg\, f_{21} = 8/2 = 4.$$

Finally, we give the following lemma which will be used in the proof of Theorem 1.

Lemma 1. *For* $-1 \leq i < j$, *we have*

$$F_i F_{j-1} - F_j F_{i-1} = (-1)^i t^{i+2} F_{j-i-3}.$$

Proof. We proceed by induction on i. For $i = -1$, the identity becomes the recursion of F_j. Suppose the identity is true up to i (and for all j). Then, by the recursion and the induction hypothesis, we have

$$\begin{aligned}
F_{i+1}F_{j-1} - F_j F_i &= (F_i + tF_{i-1})F_{j-1} - (F_{j-1} + tF_{j-2})F_i \\
&= -t(F_i F_{j-2} - F_{j-1}F_{i-1}) = (-1)^{i+1} t^{i+3} F_{j-i-4}.
\end{aligned}$$

2 $n+3$ points in $\mathbb{P}^n$ admitting a cyclic group action

For $n+2$ points $p_1, \dots, p_{n+2}$ in the real projective n-space in general position (no $n+1$ points are collinear), we would like to add another point p_m, and require that the m points admit a projective transformation σ inducing the cyclic action:

$$\sigma : \ p_1 \to p_2 \to \cdots \to p_{n+2} \to p_m \to p_1. \tag{2}$$

Without loss of generality, we put $n+3$ points in the projective n-space $\mathbb{P}^n$ coordinatized by $x_1 : \cdots : x_{n+1}$ as:

$$\begin{array}{rccccc}
 & x_1 : & x_2 & : \cdots : & x_n : & x_{n+1} \\
 & & & & & \\
p_1 = & 1 : & 1 & : \cdots : & 1 : & 1, \\
p_2 = & 1 : & 0 & : \cdots : & 0 : & 0, \\
p_3 = & 0 : & 1 & : \cdots : & 0 : & 0, \\
\vdots & & & & & \\
p_{n+1} = & 0 : & 0 & : \cdots : & 1 : & 0, \\
p_{n+2} = & 0 : & 0 & : \cdots : & 0 : & 1, \\
p_m = & \xi_1 : & \xi_2 & : \cdots : & \xi_n : & \xi_{n+1}.
\end{array}$$

In the following, we sometimes use the abbreviation $x_1 x_2 \cdots x_{n+1}$ for a point $[x_1 : x_2 : \cdots : x_{n+1}]$ in $\mathbb{P}^n$.

Theorem 1. *There exists a one-to-one correspondence between the m-th points p_m admitting the projective transformation σ as above and the roots of $F_n(t)$. Moreover, under this correspondence, the m points $\{p_1, \dots, p_{n+2}, p_m\}$ in $\mathbb{P}^n$ are in general position if and only if the associated root is a root of $f_n(t)$.*

Proof.

Since the inverse of σ induces the move $0\cdots 01 \to 0\cdots 010 \to \cdots \to 10\cdots 0 \to 1\cdots 1$, we have

$$\sigma^{-1} : \ x_1' = x_1 + b_1 x_2, \ \dots, \ x_n' = x_1 + b_n x_{n+1}, \quad x_{n+1}' = x_1, \tag{3}$$

for some non-zero b_j's. Because the last coordinate of the image of $1\cdots 1$ is 1, we may and shall assume $\xi_{n+1} = 1$. Then from the move $1\cdots 1 \to \xi_1 \cdots \xi_n 1$, we have

$$\xi_j = 1 + b_j, \quad (1 \le j \le n), \tag{4}$$

and from the move $\xi_1 \cdots \xi_n 1 \to 0 \cdots 01$, we get a system of equations in b_j:

$$\begin{aligned}
1 + b_1 + b_1(1 + b_2) &= 0, \\
1 + b_1 + b_2(1 + b_3) &= 0, \\
&\ \,\vdots \\
1 + b_1 + b_{n-1}(1 + b_n) &= 0, \\
1 + b_1 + b_n &= 0.
\end{aligned}$$

Set $b := b_n$. Then by the last equation we have

$$1 + b_1 = -b \tag{5}$$

and by solving the other equations we obtain

$$b_1 = \frac{b}{1+b_2},\ b_2 = \frac{b}{1+b_3},\ \ldots,\ b_{n-1} = \frac{b}{1+b_n} = \frac{b}{1+b}. \tag{6}$$

In particular, every b_j is written in terms of b (as a rational function) and so is ξ_j $(1 \le j \le n)$ by (4). Equations (5) and (6) in terms of ξ_j's can be written as

$$\xi_1 = -b \quad \text{and} \quad \xi_j = \frac{\xi_1}{1-\xi_{j-1}} \quad (2 \le j \le n). \tag{7}$$

Now define rational functions $h_k(t)$ in t recursively by

$$h_0 = t, \quad h_k = \frac{t}{1+h_{k-1}} \quad (k = 1, 2, \ldots).$$

We easily see from (6) that $b_j = h_{n-j}(b)$ $(1 \le j \le n)$. The h_k's and Fibonacci polynomials are related as

Lemma 2.

$$1 + h_k = \frac{F_k}{F_{k-1}}, \quad h_k = t\frac{F_{k-2}}{F_{k-1}}, \quad k = 0, 1, \ldots.$$

Proof. The first identity is easily proved by induction on k. When $k = 0$, the both sides are equal to $1 + t$. Assuming the validity for k, we have

$$1 + h_{k+1} = 1 + \frac{t}{1+h_k} = \frac{F_k/F_{k-1} + t}{F_k/F_{k-1}} = \frac{F_k + tF_{k-1}}{F_k} = \frac{F_{k+1}}{F_k}.$$

The second follows from the first by the recurrence for F_k.

By the relations $1 + b_1 + b = 0$, $b_1 = h_{n-1}(b)$ and by Lemma 2, we obtain

$$0 = 1 + h_{n-1}(b) + b = \frac{F_{n-1}(b)}{F_{n-2}(b)} + b = \frac{F_n(b)}{F_{n-2}(b)}.$$

Therefore $b = -\xi_1$ is a root of the Fibonacci polynomial $F_n(t)$.

Conversely, let b be any root of $F_n(t)$ and b_j $(1 \le j \le n)$ be determined by $b_n = b$ and (6). Then the point $p_m = \xi_1 \cdots \xi_n 1$ and the projective transformation σ determined by (4) and (3) satisfy the desired condition. That the different b's give different p_m's is clear. We note that the σ is uniquely determined by the p_m. This concludes the proof of the first half of the theorem.

For the second half, suppose first a root b of $F_n(t)$ is not a root of $f_n(t)$. Then by 2) of Proposition 2, b must be a root of some $f_{d-3}(t)$ with $d < m$. This means that b is a root of some $F_j(t)$ with $j < n$. By the identity

$$\xi_{n-j} = 1 + b_{n-j} = 1 + h_j(b) = \frac{F_j(b)}{F_{j-1}(b)}, \tag{8}$$

we conclude that $\xi_{n-j} = 0$, and so the points $p_m = \xi_1 \cdots \xi_n 1$ and $p_1, \dots, p_{n+2}$ are not in general position (points other than p_1 and p_{n-j+1} are on the hyperplane $x_{n-j} = 0$). Next suppose b is a root of $f_n(t)$. Then b is never a root of any $F_j(t)$ with $j < n$ by 2) of Proposition 2, and so by (8), no ξ_j $(1 \le j \le n)$ can be zero. Also, by the same identity (8), if $\xi_{n-i} = \xi_{n-j}$ for some $i < j$, we have $F_i(b)F_{j-1}(b) - F_j(b)F_{i-1}(b) = 0$ and hence by Lemma 1 $F_{j-i-3}(b) = 0$ (note that b is never zero). This contradicts to the fact that b is a root of $f_n(t)$. Therefore we have $\xi_i \ne \xi_j$ whenever $i \ne j$ and hence we conclude $\{p_1, \dots, p_{n+1}, p_m\}$ is in general position. This completes the proof of Theorem 1.

3 The rational curve of degree n passing through $n+3$ points

Let $p_1, \dots, p_{n+2}, p_m$ be $m = n+3$ points in general position admitting a projective cyclic permutation. Without loss of generality, we assume $n+2$ points $p_1, \dots, p_{n+2}$ are as in §2, the m-th point p_m has coordinates $\xi_1 : \cdots : \xi_n : 1$ with $\xi_i \ne 0$ and $\xi_i \ne \xi_j$ $(i \ne j)$, and the cyclic permutation is as (2).

It is known that there exits a unique rational curve C of degree n passing through m points in $\mathbb{P}^n$ in general position (see for example [CYY]). Thus let

$$C : t \longmapsto x_1(t) : \cdots : x_{n+1}(t) \in \mathbb{P}^n$$

be the curve such that each $x_j(t)$ is a polynomial in t of degree n, and

$$C(q_1) = p_1, \quad C(q_2) = p_2, \ \dots, \ C(q_{n+2}) = p_{n+2}, \quad C(q_m) = p_m \tag{9}$$

for some $q_j \in \mathbb{P}^1$. We may normalize $\{q_j\}$ so that

$$q_1 = \infty, \quad q_2 = 0, \quad q_3 = 1.$$

Our second theorem describes q_j explicitly in terms of the root of $f_n(t)$ $(= g_m(t))$.

Theorem 2. *Let $-|1+\zeta|^{-2}$ be the root of $f_n(t)$ corresponding to the m-th point p_m as in Theorem 1, where ζ is a primitive m-th root of unity. Then, q_j is given by*

$$q_j = (1+\zeta) \cdot \frac{1-\zeta^{j-2}}{1-\zeta^{j-1}} \quad (1 \le j \le m).$$

The linear fractional transformation

$$z = \frac{x-(1+\zeta)}{\zeta x-(1+\zeta)} \tag{10}$$

from the real x-line to the complex z-plane sends q_j to ζ^{j-2}, hence $q_1, q_2, \dots, q_m$ are inverse images of $\zeta^{-1}, 1, \zeta, \dots, \zeta^{m-2}$, vertices of a regular m-gon on the unit circle.

Corollary 2. *Take $\zeta = \zeta_m^l$, $(l,m)=1$ in the theorem ($\zeta_m = e^{2\pi\sqrt{-1}/m}$), then q_j can also be written as*

$$q_j = 1 + \frac{\sin\left(\frac{(j-3)l}{m}\pi\right)}{\sin\left(\frac{(j-1)l}{m}\pi\right)}.$$

If we arrange $q_1, q_2, \ldots, q_m$ according to magnitude as

$$q_1 = r_1 = -\infty < r_2 < r_3 < \cdots < r_m,$$

then the permutation of indices is given by

$$q_j = r_{(j-1)l+1} \quad (1 \le j \le m),$$

where the index of r should be taken modulo m with value in the interval $[1,m]$. In particular, if $\zeta = \zeta_m$ $(l=1)$, then $q_j = r_j$.

Proof. With our normalization, the condition (9) is equivalent to the system of equations

$$\begin{array}{lll}
(\, x_1(r) =) & c(r-q_3)(r-q_4)(r-q_5)\cdots(r-q_{n+2}) & = \xi_1, \\
(\, x_2(r) =) & c(r-q_2)(r-q_4)(r-q_5)\cdots(r-q_{n+2}) & = \xi_2, \\
 & \vdots & \\
(\, x_{j-1}(r) =) & c(r-q_2)\cdots(r-q_{j-1})(r-q_{j+1})\cdots(r-q_{n+2}) & = \xi_{j-1}, \\
 & \vdots & \\
(\, x_{n+1}(r) =) & c(r-q_2)(r-q_3)(r-q_4)\cdots(r-q_{n+1}) & = \xi_{n+1} = 1,
\end{array}$$

with $n+1$ unknowns $q_4, \ldots, q_{n+2}, r = q_m$ and c. The value of c is determined by the rest from the last equation. From the first and the $(j-1)$-st equations, by taking the ratio, we have

$$\frac{r-q_j}{r} = \frac{\xi_1}{\xi_{j-1}} \quad (3 \le j \le n+2)$$

and thus

$$q_j = r\,\frac{\xi_{j-1}-\xi_1}{\xi_{j-1}} = r\xi_{j-2} \quad (3 \le j \le n+2).$$

For the last equality, we have used (7) (with $j \to j-1$) in the previous section. Since $q_3 = 1$, the case $j=3$ gives $r\,(=q_m) = 1/\xi_1$ and so we obtain

$$q_j = \frac{\xi_{j-2}}{\xi_1} \quad (3 \le j \le m), \tag{11}$$

where the case $j=m$ is included because $\xi_{m-2} = \xi_{n+1} = 1$. From this, by writing the relation $\xi_{j-1} = \xi_1/(1-\xi_{j-2})$ in (7) (j being replaced by $j-1$) as

$$\frac{\xi_{j-1}}{\xi_1} = \frac{1}{1-\xi_{j-2}} = \frac{\frac{1}{\xi_1}}{\frac{1}{\xi_1} - \frac{\xi_{j-2}}{\xi_1}},$$

we obtain the relation

$$q_{j+1} = \frac{|1+\zeta|^2}{|1+\zeta|^2 - q_j} \quad (1 \le j \le n+2). \tag{12}$$

Here, we have used $\xi_1 = |1+\zeta|^{-2}$ from (7) (note b is the root of $f_n(t)$), and note that (12) is valid also for $j = 1, 2$ because of our normalization.

Now, let $R = R(x)$ be the map given by (10):

$$z = R(x) = \frac{x - (1+\zeta)}{\zeta x - (1+\zeta)}.$$

This gives an orientation preserving homeomorphism from $\mathbb{P}^1(\mathbb{R})$ to the unit circle (counter clock-wise) in the complex z-plane, its inverse being given by

$$x = R^{-1}(z) = (1+\zeta) \cdot \frac{1-z}{1-\zeta z}.$$

Straightforward computation shows that the rotation $z \mapsto \zeta z$ in the z-plane corresponds under R the map

$$x \mapsto \frac{|1+\zeta|^2}{|1+\zeta|^2 - x} \quad \left(= R^{-1}\big(\zeta\, R(x)\big)\right). \tag{13}$$

By our normalization, $R(q_1) = R(\infty) = \zeta^{-1}$. We therefore conclude that, by (12) and (13), the point q_j is the image of R^{-1} of ζ^{j-2}, which is the image of ζ^{-1} under the $(j-1)$-st iteration of the rotation $z \to \zeta z$, and so

$$q_j = R^{-1}(\zeta^{j-2}) = (1+\zeta) \cdot \frac{1-\zeta^{j-2}}{1-\zeta^{j-1}}.$$

This concludes the proof of Theorem 2.

When $\zeta = \zeta_m^l$, we compute

$$\begin{aligned} q_j &= \frac{1+\zeta-\zeta^{j-2}-\zeta^{j-1}}{1-\zeta^{j-1}} = 1 + \frac{\zeta - \zeta^{j-2}}{1-\zeta^{j-1}} \\ &= 1 + \frac{\zeta^{(j-3)/2} - \zeta^{-(j-3)/2}}{\zeta^{(j-1)/2} - \zeta^{-(j-1)/2}} = 1 + \frac{\sin\left(\frac{(j-3)l}{m}\pi\right)}{\sin\left(\frac{(j-1)l}{m}\pi\right)}. \end{aligned}$$

The other assertion in the corollary is clear from the description above.
Examples: When $m = 5$ and 6, the vertices of the regular m-gon in $\mathbb{P}^1(\mathbb{R})$ are

$$\begin{array}{llllllll} m=5: & x = & 0, & 1, & \frac{1+\sqrt{5}}{2}, & \frac{3+\sqrt{5}}{2}, & \infty, & \\ m=5: & x = & \frac{1-\sqrt{5}}{2}, & 0, & \frac{3-\sqrt{5}}{2}, & 1, & \infty, & \\ m=6: & x = & 0, & 1, & 3/2, & 2, & 3, & \infty, \end{array}$$

where the last example plays an important role in [AMY].

Remark 2. We see that all q_j's are in $\mathbb{Q}(\zeta+\zeta^{-1})$, and so are the ξ_j's. This is a geometric explanation of the fact that this field is the splitting field of the core Fibonacci polynomial f_n.

4 Fixed points

Let τ be a root of f_n. We find fixed points of σ (notation as in §1):

$$\lambda x_1 = x_1 + b_1 x_2,\ \lambda x_2 = x_1 + b_2 x_3,\ \ldots,\ \lambda x_n = x_1 + b_n x_{n+1},\ \lambda x_{n+1} = x_1.$$

If we put $x_{n+1} = 1$, then λ must satisfy

$$\widetilde{H}_n(\lambda, \tau) := -\lambda^{n+1} + \lambda^n + b_1(\lambda^{n-1} + b_2(\cdots(\lambda^2 + b_{n-1}(\lambda + b_n))\cdots)) = 0.$$

If there is a real λ solving this equation, then the coordinates $\lambda_1 : \cdots : \lambda_n : 1$ of the fixed point are

$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda\frac{\lambda_1 - 1}{b_1}, \quad \lambda_3 = \lambda\frac{\lambda_2 - 1}{b_2}, \ldots,$$

or equivalently, $\lambda_n = 1 + b_n\lambda_{n+1}/\lambda, \quad \lambda_{n-1} = 1 + b_{n-1}\lambda_n/\lambda, \ldots.$

We can express the polynomial $\widetilde{H}_n$ in terms of the Fibonacci polynomials. Since $b_n = h_0(\tau) = \tau$ and

$$b_1 = h_{n-1}(\tau) = \tau\frac{F_{n-3}(\tau)}{F_{n-2}(\tau)}, \quad b_1 b_2 = h_{n-1}(\tau)h_{n-2}(\tau) = \tau^2\frac{F_{n-4}(\tau)}{F_{n-2}(\tau)}, \ldots,$$
$$b_1\cdots b_n = h_{n-1}\cdots h_0 = \tau^n\frac{F_{-2}(\tau)}{F_{n-2}(\tau)},$$

and $0 = F_n(\tau) = F_{n-1}(\tau) + \tau F_{n-2}(\tau)$, we see that, if we put $x = \lambda/\tau$, $\widetilde{H}_n(\lambda, \tau)$ is a constant multiple of $H_n(x, \tau)$, where

$$H_n(x, t) := F_{n-1}(t)x^{n+1} + F_{n-2}(t)x^n + \cdots + F_{-1}x + F_{-2}, \quad F_{-1} = F_{-2} = 1.$$

Theorem 3. *Let τ be a root of f_n. When n is odd, $H_n(x, \tau)$ has no real root. When $n = 2k$ is even, $H_n(x, \tau)$ has a unique real root*

$$-\frac{F_{k-2}(\tau)}{F_{k-1}(\tau)}.$$

Proof. Substituting

$$F_i = G_{i+3} = \frac{1}{\sqrt{1+4t}}(\alpha^{i+3} - \beta^{i+3})$$

into $H_n = \sum_{i=-2}^{n-1} F_i x^{i+2}$, we have

$$\begin{aligned} H_n &= \sum_{i=-2}^{n-1}\frac{1}{\sqrt{1+4t}}(\alpha^{i+3} - \beta^{i+3})x^{i+2} = \frac{1}{\sqrt{1+4t}}\sum_{i=0}^{n+1}(\alpha^{i+1} - \beta^{i+1})x^i \\ &= \frac{1}{\sqrt{1+4t}}\left\{\alpha\frac{\alpha^{n+2}x^{n+2} - 1}{\alpha x - 1} - \beta\frac{\beta^{n+2}x^{n+2} - 1}{\beta x - 1}\right\} \\ &= \frac{\alpha\beta(\alpha^{n+2} - \beta^{n+2})x^{n+3} - (\alpha^{n+3} - \beta^{n+3})x^{n+2} + \alpha - \beta}{\sqrt{1+4t}(\alpha x - 1)(\beta x - 1)}. \end{aligned}$$

Since $\alpha+\beta=1, \alpha\beta=-t$ and $\alpha-\beta=\sqrt{1+4t}$, we have

$$H_n = \frac{-tF_{n-1}(t)x^{n+3} - F_n(t)x^{n+2}+1}{1-x-tx^2}.$$

If τ is a root of $F_n(t)$ (which is always negative real), the equation $H_n=0$ in x is equivalent to

$$x^{n+3} = \frac{1}{\tau F_{n-1}(\tau)}.$$

If n is even, this has a unique real solution, and if n is odd, since $F_{n-1}(\tau)$ is positive (next Lemma), it has no real solution. The theorem follows from the two lemmas below.

Lemma 3. *If n is odd and if τ is a root of $f_n(t)$, then $F_{n-1}(\tau)$ is positive.*

Proof. Recall that $F_n = G_{n+3}$, and the roots of $G_{n+3}(t)$ are given as (Corollary 1)

$$\tau_i = -\frac{1}{4\cos^2\frac{i\pi}{n+3}}, \quad 1 \le i \le \left[\frac{n}{2}\right]+1,$$

and the roots of $G_{n+2}(t)$ are

$$t_j = -\frac{1}{4\cos^2\frac{j\pi}{n+2}}, \quad 1 \le j \le \left[\frac{n+1}{2}\right].$$

Since $\tau_i - t_j < 0$ if and only if

$$\frac{j}{n+2} < \frac{i}{n+3},$$

the number of roots t_j such that $\tau_i - t_j < 0$ (for fixed i) is $i-1$. So if i is odd, $F_{n-1}(\tau) > 0$. If n is odd and τ is a root of $f_n(t)$, we must have $(i, n+3) = 1$, which implies i is odd.

Lemma 4. *Let $n = 2k$ is even, and let τ be a root of $F_{2k}(t)$. Then we have*

$$\left(-\frac{F_{k-2}(\tau)}{F_{k-1}(\tau)}\right)^{n+3} = \frac{1}{\tau F_{n-1}(\tau)}.$$

Proof. Recall that, if we set $a = (1+\sqrt{1+4\tau})/2$ and $b = (1-\sqrt{1+4\tau})/2$,

$$F_j(\tau) = \frac{a^{j+3}-b^{j+3}}{\sqrt{1+4\tau}}.$$

By assumption, we have $a^{2k+3} = b^{2k+3}$. We first note

$$-\frac{F_{k-2}(\tau)}{F_{k-1}(\tau)} = -\frac{a^{k+1}-b^{k+1}}{a^{k+2}-b^{k+2}} = \frac{a^{k+1}}{b^{k+2}},$$

because

$$(a^{k+2}-b^{k+2})a^{k+1}+(a^{k+1}-b^{k+1})b^{k+2}=a^{2k+3}-b^{2k+3}=0.$$

Hence, we have

$$\left(-\frac{F_{k-2}(\tau)}{F_{k-1}(\tau)}\right)^{n+3}=\left(\frac{a^{k+1}}{b^{k+2}}\right)^{2k+3}=\left(\frac{a^{2k+3}}{b^{2k+3}}\right)^{k}\frac{a^{2k+3}}{b^{2(2k+3)}}=\frac{1}{b^{2k+3}}.$$

On the other hand, by using $\tau=-ab$, $\sqrt{1+4\tau}=a-b$, and $a^{2k+3}=b^{2k+3}$, we obtain

$$\frac{1}{\tau F_{n-1}(\tau)}=\frac{-(a-b)}{ab(a^{2k+2}-b^{2k+2})}=\frac{-(a-b)}{b(b^{2k+3}-ab^{2k+2})}=\frac{-(a-b)}{b^{2k+3}(b-a)}=\frac{1}{b^{2k+3}}.$$

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