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Doubly resolvent Feller property of generalized Feynman-Kac functionals

Mila KURNIAWATY

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Abstract. In this paper we will apply the stability of doubly Feller property of resolvent with multiplicative functionals in our previous paper [10] to generalized Feynman-Kac functionals.

1 Introduction

Let (E, d) be a locally compact separable metric space, $E_\partial := E \cup \{\partial\}$ its one point compactification, $\mathcal{B}(E)$ its Borel σ-field on E, and $\mathcal{B}(E_\partial)$ Borel σ-field on E_∂. It is well-known that $\mathcal{B}(E_\partial) = \mathcal{B}(E) \cup \{B \cup \{\partial\} \mid B \in \mathcal{B}(E)\}$. Any function f defined on E is extended to E_∂ by setting $f(\partial) = 0$. Denote by $\mathcal{B}_b(E)$ (resp. by $C_b(E)$, the family of bounded Borel functions on E (resp. the family of bounded continuous functions on E), and by $C_0(E)$ (resp. by $C_\infty(E)$), the family of continuous functions on E with compact support (resp. the family of continuous functions on E vanishing at infinity).

We consider a Hunt process $\mathbf{X} = (\Omega, \mathcal{F}_t, \mathcal{F}_\infty, X_t, \zeta, \mathbf{P}_x)_{x \in E_\partial}$ defined on E_∂ and denote by $(P_t)_{t \geq 0}$ (resp. $(R_\alpha)_{\alpha > 0}$) its transition semigroup (resp. its resolvent kernel), that is, $P_t f(x) = \mathbf{E}_x[f(X_t)] = \int_\Omega f(X_t(\omega))\mathbf{P}_x(\mathrm{d}\omega)$ (resp. $R_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x)\mathrm{d}t$) for $f \in \mathcal{B}_b(E_\partial)$. Here $\zeta := \inf\{t \geq 0 \mid X_t = \partial\}$ is the life time of $\mathbf{X}$ and ∂ is a cemetery point of $\mathbf{X}$, that is, $X_t = \partial$ for all $t \geq \zeta$ under $\mathbf{P}_x$ for

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$x \in E$. The transition semigroup $(P_t)_{t\geq 0}$ of $\mathbf{X}$ is said to have the *Feller property* if the following two conditions are satisfied:

(i) For each $t > 0$ and $f \in C_\infty(E)$, we have $P_t f \in C_\infty(E)$.

(ii) For each $f \in C_\infty(E)$ and $x \in E$, we have $\lim_{t\to 0} P_t f(x) = f(x)$.

The resolvent $(R_\alpha)_{\alpha>0}$ of $\mathbf{X}$ is said to have the *Feller property* if the following two conditions are satisfied:

(i)' For each $\alpha > 0$ and $f \in C_\infty(E)$, we have $R_\alpha f \in C_\infty(E)$.

(ii)' For each $f \in C_\infty(E)$ and $x \in E$, we have $\lim_{\alpha\to\infty} \alpha R_\alpha f(x) = f(x)$.

It is known that (i) and (ii) together imply

(iii) For each $f \in C_\infty(E)$, we have $\lim_{t\to 0} \|P_t f - f\|_\infty = 0$.

Since

$$R_\beta f = \int_0^\infty e^{-\beta t} P_t f \mathrm{d}t \quad \text{and} \quad P_t f = \lim_{\beta\to\infty} e^{-t\beta} \sum_{n=0}^\infty \frac{(t\beta)^n}{n!} (\beta R_\beta)^n f, \quad f \in C_\infty(E)$$

hold in $(C_\infty(E), \|\cdot\|_\infty)$, the condition (i) is equivalent to (i)'. It is easy to see that (ii) implies (ii)'. Conversely, the conditions (i)' and (ii)' imply (ii). Indeed, it is known that these conditions together imply

(iii)' For each $f \in C_\infty(E)$, we have $\lim_{\beta\to\infty} \|\beta R_\beta f - f\|_\infty = 0$.

From (iii)', we have

$$\begin{aligned}
&|P_t f(x) - f(x)| \\
&\leq |P_t f(x) - \beta R_\beta P_t f(x)| + |\beta R_\beta P_t f(x) - f(x)| \\
&\leq 2\|\beta R_\beta f - f\|_\infty + \beta \left| (e^{\beta t} - 1) \int_t^\infty e^{-\beta s} P_s f(x) \mathrm{d}s - \int_0^t e^{-\beta s} P_s f(x) \mathrm{d}s \right| .
\end{aligned}$$

Hence, $\limsup_{t\to 0} |P_t f(x) - f(x)| \leq 2\|\beta R_\beta f - f\|_\infty \to 0$ as $\beta \to \infty$. Consequently, the Feller property of $(P_t)_{t\geq 0}$ is equivalent to the Feller property of $(R_\alpha)_{\alpha>0}$. So we can say that $\mathbf{X}$ has the *Feller property* if (i) and (ii), or (i)' and (ii)' hold.

The semigroup $(P_t)_{t\geq 0}$ is said to have the *strong Feller property* if

(iv) For each $f \in \mathcal{B}_b(E)$ and $t > 0$, we have $P_t f \in C_b(E)$.

The resolvent $(R_\alpha)_{\alpha>0}$ is said to have the *strong Feller property* if

(iv)' For each $f \in \mathcal{B}_b(E)$ and $\alpha > 0$, we have $R_\alpha f \in C_b(E)$.

When E is compact, ∂ is an isolated point of E_∂, hence any function f on E_∂ with $f(\partial)=0$, which is continuous on E, belongs to $C_\infty(E)$. In this case, the strong Feller property of semigroup (resp. resolvent) implies (i) (resp. (i)').

Remark 1.1 *It is well-known that the strong Feller property of $(P_t)_{t\geq 0}$ implies the strong Feller property of $(R_\alpha)_{\alpha>0}$, but the converse assertion is not true. Indeed, the following semigroups are not strong Feller, but their resolvents enjoy strong Feller:*

(1) *The shift semigroup $(P_t)_{t\geq 0}$ on $\mathbb{R}$ defined by $P_tf(x):=f(x+t\ell)$, $x\in\mathbb{R},\ell\in\mathbb{R}\setminus\{0\}$ does not enjoy the strong Feller property, but the resolvent strong Feller property in view of $R_\alpha f(x)=\frac{e^{\frac{\alpha x}{\ell}}}{\ell}\int_x^\infty e^{-\frac{\alpha y}{\ell}}f(y)\mathrm{d}y$ (resp. $R_\alpha f(x)=\frac{e^{\frac{\alpha x}{\ell}}}{-\ell}\int_{-\infty}^x e^{-\frac{\alpha y}{\ell}}f(y)\mathrm{d}y$) for $f\in\mathcal{B}_b(\mathbb{R})$ and $\ell>0$ (resp. $\ell<0$).*

(2) *The semigroup $(P_t)_{t\geq 0}$ of space-time Brownian motion (B_t,t) under $\mathbf{P}_{(x,\tau)}:=\mathbf{P}_x^{(1)}\otimes\mathbf{P}_\tau^{(2)}$ for $(x,\tau)\in\mathbb{R}^2$ defined by*

$$P_tf(x,\tau):=\mathbf{E}_{(x,\tau)}[f(B_t,t)]=\mathbf{E}_x^{(1)}\otimes\mathbf{E}_\tau^{(2)}[f(B_t,t)],\quad f\in\mathcal{B}_b(\mathbb{R}^2)$$

does not enjoy the strong Feller property, but the resolvent strong Feller property. Here $\mathbf{P}_x^{(1)}$ is the law for 1-dimensional Brownian motion starting from x and $\mathbf{P}_\tau^{(2)}$ is the law for uniform motion to the right starting from τ with speed 1, i.e. $\mathbf{E}_\tau^{(2)}[g(t)]=g(\tau+t)$. More generally, the product semigroup of strong Feller semigroup and the semigroup of uniform motion to the right is not a strong Feller semigroup, but its resolvent enjoys the strong Feller property. Indeed, for $x_n\to x$ and $\tau_n\to\tau$ and $f\in\mathcal{B}_b(\mathbb{R}^2)$, we see

$$\begin{aligned}
&|R_\alpha f(x_n,\tau_n)-R_\alpha f(x,\tau)|\\
&\leq |e^{\alpha\tau_n}-e^{\alpha\tau}|\int_{\tau_n}^\infty e^{-\alpha s}\mathbf{E}_{(x_n,0)}[|f|(B_{s-\tau_n},s)]\mathrm{d}s\\
&\qquad+e^{\alpha\tau}\left|\int_{\tau_n}^\infty e^{-\alpha s}\mathbf{E}_{(x_n,0)}[f(B_{s-\tau_n},s)]\mathrm{d}s-\int_\tau^\infty e^{-\alpha s}\mathbf{E}_{(x,0)}[f(B_{s-\tau},s)]\mathrm{d}s\right|\\
&\leq |e^{\alpha\tau_n}-e^{\alpha\tau}|\|f\|_\infty/\alpha+e^{\alpha\tau}\int_0^\infty e^{-\alpha s}\Big|\mathbf{1}_{]\tau_n,\infty[}(s)P^{(1)}_{s-\tau_n}(f(\cdot,s))(x_n)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad-\mathbf{1}_{]\tau,\infty[}(s)P^{(1)}_{s-\tau}(f(\cdot,s))(x)\Big|\,\mathrm{d}s\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Here we use that the strong Feller property of the semigroup $(P_t^{(1)})_{t\geq 0}$ of 1-dimensional Brownian motion implies the continuity of $]0,\infty[\times\mathbb{R}\ni(t,x)\mapsto P_t^{(1)}g(x)$ for any $g\in\mathcal{B}_b(\mathbb{R})$. On the other hand, $P_tf(x,\tau)=P_t^{(1)}f_1(x)f_2(\tau+$

s) for $f = f_1 \otimes f_2$, $f_i \in \mathcal{B}_b(\mathbb{R})$ $(i = 1, 2)$ *does not enjoy* $P_t f \in C_b(\mathbb{R}^2)$ *for* $f_2 \notin C_b(\mathbb{R})$.

Under (iv)', we see that $R_1 1 \in C_\infty(E)$ implies (i)'. Moreover, under (iv), $R_1 1 \in C_\infty(E)$ implies (i) (see [1, Proposition 1]). The semigroup $(P_t)_{t\geq 0}$ or $\mathbf{X}$ is said to have the *doubly Feller property* if it enjoys the both of Feller property and strong Feller property. The Hunt process $\mathbf{X}$ is said to have the *doubly resolvent Feller property* if its resolvent enjoys the both of Feller property and strong Feller property. $\mathbf{X}$ is said to be a *Feller process* (resp. *strong Feller process*, *doubly Feller process*) if it enjoys the Feller property (resp. strong Feller property, doubly Feller property). $\mathbf{X}$ is said to be a *resolvent strong Feller process* (resp. *doubly resolvent Feller process*) if it enjoys the resolvent strong Feller property (resp. doubly resolvent Feller property).

In [4], the stability of the doubly Feller property of a semigroup (of a part process) with multiplicative functionals was presented, and its stability was discussed under Feynman-Kac and Girsanov transformations. In [10], the same conditions as in [4] also remain valid for the stability of the doubly Feller property of the resolvent of a part process with multiplicative functionals. In this paper, we apply the stability of the doubly Feller property of the resolvent to the generalized Feynman-Kac functionals.

Let μ_1 (resp. μ_2) be smooth measures in the strict sense corresponding to a positive continuous additive functional A^{μ_1} (resp. A^{μ_2}) in the strict sense with respect to $\mathbf{X}$ and let $\mu := \mu_1 - \mu_2$ be the signed smooth measure in the strict sense. We denote A by $A^\mu := A^{\mu_1} - A^{\mu_2}$ to emphasize the correspondence between μ and A. Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form of $\mathbf{X}$ on $L^2(E; \mathfrak{m})$. For a bounded finely continuous (nearly) Borel function u on E which is strictly $\mathcal{E}$-quasi continuous on E_∂ and locally in $\mathcal{F}$, let N^u be the continuous additive functional of zero quadratic variation appeared in the Fukushima decomposition $u(X_t) - u(X_0)$ (see (3.2)). Note that N^u is not necessarily of bounded variation in general. Let F_1, F_2 be non-negative bounded functions on $E \times E_\partial$ which are symmetric on $E \times E$. F_1 and F_2 are extended to $E_\partial \times E_\partial$ by setting $F_i(\partial, x) = F_i(x, \partial) = F_i(x, x) = 0$ for $x \in E_\partial$ for each $i = 1, 2$ (actually there is no need to define the value $F_i(\partial, x)$ for $x \in E$, $i = 1, 2$). We set $F := F_1 - F_2$. Then $A_t^F := \sum_{0<s\leq t} F(X_{s-}, X_s)$ (whenever it is summable) is an additive functional of $\mathbf{X}$. It is natural to consider the following generalized non-local Feynman-Kac transforms by the additive functionals $A := N^u + A^\mu + A^F$ of the form

$$e_A(t) := \exp(A_t), \quad t \geq 0, \tag{1.1}$$

because the process $\mathbf{X}$ admits many continuous additive functionals which do not have bounded variations, and many discontinuous additive functionals.

Let us briefly state the constitution of this paper. In Section 2, we give the stability of doubly Feller property of the resolvent with multiplicative functionals as [10]. In Section 3, we introduce classes of local potentials and properties of generalized Feynman-Kac and Girsanov transform. Section 4 includes the main result of this paper, we show the stability of doubly Feller property of generalized Feynman-Kac functionals.

2 Doubly Feller property of transformed resolvent

In this section, we summarize the content of [10, Section 6]. Let $(Z_t)_{t\geq 0}$ be a multiplicative functional associated with $\mathbf{X}$. Namely, for each $x \in E$, $\mathbf{P}_x$-a.s.: $Z_0 = 1$, $0 \leq Z_t < \infty$, $Z_t \in \mathcal{F}_t$ for $t \geq 0$; and

$$Z_{t+s} = Z_s \cdot (Z_t \circ \theta_s), \qquad \text{for all} \qquad t, s \geq 0. \tag{2.1}$$

Throughout this section, we fix a non-empty open set B. The following conditions are dependent of B:

(a)$_B$ For some $t > 0$, $a_t^B := \sup_{x\in B} \sup_{s\in[0,t]} \mathbf{E}_x[Z_s : s < \tau_B] < \infty$.

(a)$_B^*$ There exists $p > 1$ such that $a_t^B(p) := \sup\limits_{x\in B} \sup\limits_{s\in[0,t]} \mathbf{E}_x[Z_s^p : s < \tau_B] < \infty$ for some $t > 0$.

Note that (a)$_B^*$ implies (a)$_B$. As shown in the previous paper ([10]), we can deduce that $t \mapsto a_t^B$ is submultiplicative under (a)$_B$. This implies that $t \mapsto \log(a_t^B)^{1/t}$ is decreasing, $a_t^B \leq (a_{t_0}^B)^{t/t_0}$ for all $t \geq t_0$ with given $t_0 > 0$, and for any $\alpha > \alpha_0^B := \inf_{s\in]0,\infty[} \log(a_s^B)^{1/s} \geq 0$, $\int_0^\infty e^{-\alpha t} a_t^B \mathrm{d}t < \infty$. Under (a)$_B^*$, we have a similar statement including $\alpha > \alpha_0^B(p) := \inf_{s\in]0,\infty[} \log(a_s^B(p))^{1/s} \geq 0$, $\int_0^\infty e^{-\alpha t} a_t^B(p)\mathrm{d}t < \infty$.

(b)$_B^w$ For each $t > 0$ and any compact subset K of B, there exists a number $p = p(K,t) > 1$ such that $\sup\limits_{x\in K} \mathbf{E}_x\left[Z_t^p\right] < \infty$.

(b)$_B^s$ For each $t > 0$, there exists a number $p = p(t) > 1$ such that $\sup\limits_{x\in B} \mathbf{E}_x\left[Z_t^p\right] < \infty$.

$(\mathrm{c})^w_B$ For any relatively compact open subset D of B, we have

$$\lim_{t\to 0}\sup_{x\in D}\mathbf{E}_x[|Z_t-1|:t<\tau_D]=0.$$

$(\mathrm{c})^s_B$ $\lim_{t\to 0}\sup_{x\in B}\mathbf{E}_x[|Z_t-1|:t<\tau_B]=0.$

When $B=E$, we omit B like (a) for $(\mathrm{a})_E$.

Remark 2.1 (1) *The condition* $(\mathrm{b})^s_B$ *(resp.,* $(\mathrm{c})^s_B$*) is stronger than* $(\mathrm{b})^w_B$ *(resp.,* $(\mathrm{c})^w_B$*).*

(2) *The condition* $(\mathrm{c})^s_B$ *implies the condition* $(\mathrm{a})_B$*. Indeed, under* $(\mathrm{c})^s_B$*, for any given* $\varepsilon>0$*, there exists* $\delta>0$ *such that* $\sup_{t\in[0,\delta]}\sup_{x\in B}\mathbf{E}_x[|Z_t-1|:t<\tau_B]\le\varepsilon$*. Then we have* $\sup_{t\in[0,\delta]}\sup_{x\in B}\mathbf{E}_x[|Z_t|:t<\tau_B]\le\varepsilon+1<\infty$*, which shows* $(\mathrm{a})_B$*.*

(3) *In view of [4, Remark 1.6(ii)], the conditions* $(\mathrm{a})^*_B$ *and* $(\mathrm{c})^w_B$ *imply the condition* $(\mathrm{c})^s_B$

Define $(Q^B_t)_{t\ge 0}$ as follows: for $f\in\mathcal{B}_b(E)$,

$$Q^B_t f(x):=\mathbf{E}_x[t<\tau_B:Z_t f(X_t)].$$

By means of (2.1), we can verify that $(Q^B_t)_{t\ge 0}$ forms a semigroup, not necessarily sub-Markovian. But we have, for each $t>0$,

$$\|Q^B_t f\|_\infty\le\left(\sup_{x\in B}\mathbf{E}_x[Z_t:t<\tau_B]\right)\|f\|_\infty\le a^B_t\|f\|_\infty$$

by $(\mathrm{a})_B$, so that each Q^B_t maps $\mathcal{B}_b(E)$ into $\mathcal{B}_b(E)$. For $\alpha>\alpha_0:=\inf_{s\in]0,\infty[}\log a_s^{1/s}$ and $f\in\mathcal{B}_b(E)$, we set

$$S^B_\alpha f(x):=\mathbf{E}_x\left[\int_0^{\tau_B}e^{-\alpha t}Z_t f(X_t)\mathrm{d}t\right]$$

and call $(S^B_\alpha)_{\alpha>\alpha_0}$ the *resolvent of the transformed process* from $\mathbf{X}$ by the multiplicative functional $(Z_t)_{t\ge 0}$ for B open subset of E.

The following two theorems have been proved in [10].

Theorem 2.1 (cf. [10, Theorem 5.1]) *Suppose that* $\mathbf{X}$ *has the resolvent strong Feller property and condition* (a) *and* $(\mathrm{c})^w$ *hold. Then, for* $f\in\mathcal{B}_b(E)$ *and* $\alpha>\alpha_0$*,* $S_\alpha f\in C_b(E)$*.*

Theorem 2.2 (cf. [10, Theorem 6.2]) *Let $\mathbf{X}$ be a doubly resolvent Feller process, and let B be an open subset of E. Suppose that B is regular. Under* $(\mathrm{a})_B$, $(\mathrm{b})_B^s$, *and* $(\mathrm{c})_B^s$, *for any $\alpha > \alpha_0^B := \inf_{s\in]0,\infty[}\log(a_s^B)^{1/s} \geq 0$, we have $S_\alpha^B f \in C_\infty(B)$ and $\lim_{\alpha\to\infty}\alpha S_\alpha^B f(x) = f(x)$ for $f \in C_\infty(B)$ and $x \in B$. Suppose further that B is relatively compact, and assume* $(\mathrm{a})_B^*$ *or there exists an open set C with $\overline{B} \subset C$ such that* $(\mathrm{c})_C^s$ *holds. Then $S_\alpha^B f \in C_\infty(B)$ for $f \in \mathcal{B}_b(B)$.*

3 Feynman-Kac and Girsanov transform

In this section, we summarize the content of [7, Sections 3 and 4]. Let $\mathbf{X} = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, \theta_t, X_t, \mathbf{P}_x, x \in E_\partial)$ be an $\mathfrak{m}$-symmetric Hunt process on E and $\{\mathcal{F}_t\}_{t\geq 0}$ is the minimal (augmented) admissible filtration and $\theta_t, t \geq 0$ is the shift operator satisfying $X_s(\theta_t) = X_{s+t}$ identically for $s, t \geq 0$.

An $(\mathcal{F}_t)$-adapted process $\{A_t\}_{t\geq 0}$ with values in $[-\infty, \infty]$ is said to be an *additive functional* (AF in abbreviation) if the following conditions hold:

(i) $A_t(\cdot)$ if $\mathcal{F}_t$-measurable for all $t \geq 0$,

(ii) there exists a set $\Lambda \in \mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$ such that $\mathbf{P}_x(\Lambda) = 1$, for all $x \in X$, $\theta_t\Lambda \subset \Lambda$ for all $t > 0$, and for each $\omega \in \Lambda, A.(\omega)$ is a function satisfying $A_0 = 0, A_t(\omega) < \infty$ for $t < \zeta(\omega)$ for $t \geq 0$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s, t \geq 0$.

An AF A is said to be a *continuous additive functional* (CAF in abbreviation) if $t \mapsto A_t(\omega)$ is continuous on $[0, \infty[$ for each $\omega \in \Lambda$. A $[0, \infty[$-valued AF is called a *positive additive functional* (PAF in abbreviation). If an AF $\{A_t\}_{t\geq 0}$ is positive and continuous with respect to t for each $\omega \in \Lambda$, the AF is called a *positive continuous additive functional* (PCAF in abbreviation).

Let $S_1(\mathbf{X})$ be the family of positive smooth measures in the strict sense ([6]). We say that a PCAF in the strict sense A^ν of $\mathbf{X}$ and a positive measure $\nu \in S_1(\mathbf{X})$ are in the Revuz correspondence if they satisfy for any $t > 0$, $f \in \mathcal{B}_+(E)$,

$$\int_E f(x)\nu(\mathrm{d}x) =\uparrow \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_{\mathfrak{m}}\left[\int_0^t f(X_s)\mathrm{d}A_s^\nu\right].$$

It is known that the family of equivalence classes of the set of PCAFs in the strict sense and the family of positive measures belonging to $S_1(\mathbf{X})$ are in one to one correspondence under the Revuz correspondence ([6, Theorem 5.1.4]).

A PAF B_t in the classical sense is said to be in the *Kato class* of $\mathbf{X}$ if $\lim_{t\to 0}\sup_{x\in E}\mathbf{E}_x[B_t] = 0$. A PAF B_t in the classical sense is said to be in the *local Kato class* if, for any relatively compact open subset G of E, $(\mathbf{1}_G * B)_t := \int_0^t \mathbf{1}_G(X_s)\mathrm{d}B_s$ is of Kato class (see [4, Section 2] for the PAF of (local) Kato class).

Corollary 3.1 (cf. [10, Corollary 6.1]) *Let $\mathbf{X}$ be a doubly resolvent Feller process. Let B_t be a PAF of local Kato class. Then the subprocess killed by e^{-B_t} enjoys the doubly Feller property of the resolvent.*

For $\alpha > 0$, there exists an α-order resolvent kernel $R_\alpha(x,y)$ which is defined for all $x, y \in E$ (see Lemma 4.2.4 in [6]). Since $\alpha \mapsto R_\alpha(x,y)$ is decreasing for each $x, y \in E$, we can define the 0-order resolvent kernel $R(x,y) := R_0(x,y) := \lim_{\alpha\to 0} R_\alpha(x,y)$. $R(x,y)$ is called the Green function of $\mathbf{X}$. For a non-negative Borel measure ν, we write $R_\alpha\nu(x) := \int_E R_\alpha(x,y)\nu(\mathrm{d}y)$ and $R\nu(x) := R_0\nu(x)$. Note that $R_\alpha f(x) = R_\alpha(f\mathfrak{m})(x)$ for any $f \in \mathcal{B}_+(E)$ or $f \in \mathcal{B}_b(E)$. A measure $\nu \in S_1(\mathbf{X})$ is said to be in the *Dynkin class* (resp. *Green-bounded*) of $\mathbf{X}$ if $\sup_{x\in E} R_\alpha\nu(x) < \infty$ for some $\alpha > 0$ (resp. $\sup_{x\in E} R\nu(x) < \infty$). We also can define Kato class and local Kato class in the sense of measures as in AFs. A measure $\nu \in S_1(\mathbf{X})$ is said to be in the *Kato class* (resp. *extended Kato class*) with respect to $\mathbf{X}$ if $\lim_{\alpha\to\infty} \sup_{x\in E} R_\alpha\nu(x) = 0$ (resp. < 1). A measure $\nu \in S_1(\mathbf{X})$ is said to be in the *local Kato class* if for any compact subset K of E, $\mathbf{1}_K\nu$ is of Kato class. Denote by $S_D^1(\mathbf{X})$ (resp. $S_{D_0}^1(\mathbf{X})$) the family of measures of Dynkin class (resp. Green-bounded), and by $S_K^1(\mathbf{X})$ (resp. $S_{EK}^1(\mathbf{X})$, $S_{LK}^1(\mathbf{X})$) the family of measures of Kato class (resp. extended Kato class, local Kato class). Clearly, $S_K^1(\mathbf{X}) \subset S_{EK}^1(\mathbf{X}) \subset S_D^1(\mathbf{X})$, $S_K^1(\mathbf{X}) \subset S_{LK}^1(\mathbf{X})$ and $S_{D_0}^1(\mathbf{X}) \subset S_D^1(\mathbf{X})$.

Let $(N(x,\mathrm{d}y), H_t)$ be a Lévy system for $\mathbf{X}$, that is, $N(x,\mathrm{d}y)$ is a kernel on $(E_\partial, \mathcal{B}(E_\partial))$ and H_t is a PCAF with bounded 1-potential such that for any non-negative Borel function ϕ on $E_\partial \times E_\partial$ vanishing on the diagonal and any $x \in E_\partial$, non-negative Borel function g on $[0,\infty[$ and $(\mathcal{F}_t)$-stopping time T,

$$\mathbf{E}_x\left[\sum_{s\le T} g(s)\phi(X_{s-}, X_s)\right] = \mathbf{E}_x\left[\int_0^T \int_{E_\partial} g(s)\phi(X_s, y)N(X_s, \mathrm{d}y)\mathrm{d}H_s\right] \tag{3.1}$$

(see [3, A.3.33]). To simplify notation, we will write

$$N\phi(x) := \int_{E_\partial} \phi(x,y)N(x,\mathrm{d}y).$$

Let μ_H be the Revuz measure of the PCAF H. Then the jumping measure J and the killing measure κ of $\mathbf{X}$ are given by

$$J(\mathrm{d}x\mathrm{d}y) = \frac{1}{2}N(x,\mathrm{d}y)\mu_H(\mathrm{d}x) \quad \text{and} \quad \kappa(\mathrm{d}x) = N(x,\{\partial\})\mu_H(\mathrm{d}x).$$

These measures feature in the Beurling-Deny decomposition of $\mathcal{E}$:

$$\mathcal{E}(f,g)=\mathcal{E}^c(f,g)+\int_{(E\times E)\setminus\mathsf{diag}}(f(x)-f(y))(g(x)-g(y))J(\mathrm{d}x\mathrm{d}y)+\int_E f(x)g(x)\kappa(\mathrm{d}x)$$

for $f,g\in\mathcal{F}_e$. Here $\mathcal{E}^c$ is the strongly local part of $\mathcal{E}$ and diag be the diagonal set in $E\times E$ defined by $\mathsf{diag}:=\{(x,y)\in E\times E\mid x=y\}$.

A function f on E is said to be *locally in $\mathcal{F}$ in the broad sense* if there exists a nest $\{G_n\}$ of finely open (nearly) Borel sets and a sequence $\{f_n\}$ of elements in $\mathcal{F}$ such that $f=f_n$ $\mathfrak{m}$-a.e. on G_n. Let $\dot{\mathcal{F}}_{\mathrm{loc}}$ be the family of functions on E locally in $\mathcal{F}$ in the broad sense. It is known that any $f\in\dot{\mathcal{F}}_{\mathrm{loc}}$ admits $\mathcal{E}$-quasi continuous $\mathfrak{m}$-version.

An increasing sequence $\{F_k\}$ of closed sets is said to be a *strict $\mathcal{E}$-nest* if $\mathbf{P}_x(\lim_{k\to\infty}\sigma_{E\setminus F_k}=\infty)=1$ $\mathfrak{m}$-a.e. $x\in E$. A function f defined on E_∂ is said to be *strictly $\mathcal{E}$-quasi continuous* if there exists a strict $\mathcal{E}$-nest $\{F_k\}$ of closed sets such that $f|_{F_k\cup\{\partial\}}$ is continuous for each $k\in\mathbb{N}$. Denote by $QC(E_\partial)$ the totality of strictly $\mathcal{E}$-quasi continuous functions on E_∂. We consider a bounded finely continuous (nearly) Borel function $u\in\dot{\mathcal{F}}_{\mathrm{loc}}\cap QC(E_\partial)$ satisfying $\mu_{\langle u\rangle}\in S^1_D(\mathbf{X})$. In [9, Theorem 6.2(2)], we proved that the additive functional $u(X_t)-u(X_0)$ admits the following strict decomposition:

$$u(X_t)-u(X_0)=M^u_t+N^u_t\quad t\in[0,\infty[\quad \mathbf{P}_x\text{-a.s. for all } x\in E, \tag{3.2}$$

where M^u is a square integrable martingale additive functional in the strict sense, and N^u is a CAF in the strict sense which is locally of zero energy. M^u can be decomposed as

$$M^u_t=M^{u,c}_t+M^{u,j}_t+M^{u,\kappa}_t, \tag{3.3}$$

where $M^{u,j}_t$, $M^{u,\kappa}_t$ and $M^{u,c}_t$ are the jumping, killing and continuous part of M^u respectively. Those are defined $\mathbf{P}_x$-a.s. for all $x\in E$ by [9, Theorem 6.2(2)]. The strict decompositions (3.2) and (3.3) on $[0,\infty[$ guarantee the extension of the supermartingale multiplicative functional Y_t on $[\![0,\zeta[\![$ up to $[0,\infty[$ (see [8, Proposition 3.1]). Let $\mu_{\langle u\rangle}$, $\mu^c_{\langle u\rangle}$, $\mu^j_{\langle u\rangle}$ and $\mu^\kappa_{\langle u\rangle}$ be the smooth Revuz measures in the strict sense associated with the quadratic variational processes (or the sharp bracket PCAFs in the strict sense) $\langle M^u\rangle$, $\langle M^{u,c}\rangle$, $\langle M^{u,j}\rangle$ and $\langle M^{u,\kappa}\rangle$ respectively. Then

$$\mu_{\langle u\rangle}(\mathrm{d}x)=\mu^c_{\langle u\rangle}(\mathrm{d}x)+\mu^j_{\langle u\rangle}(\mathrm{d}x)+\mu^\kappa_{\langle u\rangle}(\mathrm{d}x).$$

Note that $\mathcal{E}(f,f)=\frac{1}{2}\nu_{\langle f\rangle}(E)$ with $\nu_{\langle f\rangle}:=\mu^c_{\langle f\rangle}+\mu^j_{\langle f\rangle}+2\mu^\kappa_{\langle f\rangle}$ provided $f\in\mathcal{F}_e$.

Let F be a bounded symmetric function on $E \times E$, which is extended to a function F defined on $E_\partial \times E_\partial$ by setting $F(x,\partial) = F(\partial,x) = F(x,x) = 0$ for $x \in E_\partial$ (actually there is no need to define the value $F(\partial,y)$ for $y \in E$). We say that F in the *class* $J_1(\mathbf{X})$ if $N(|F|)\mu_H$ belongs to $S_1(\mathbf{X})$. For a bounded finely continuous (nearly) Borel function $u \in \dot{\mathcal{F}}_{\text{loc}} \cap QC(E_\partial)$ satisfying $\mu_{\langle u\rangle} \in S_1(\mathbf{X})$, we set

$$U(x,y) := u(x) - u(y).$$

Since

$$|U(x,y)|^2 \leq 2\left(u(x)-u(y)\right)^2,$$

one can see that the relation $N(|U(x,y)|^2)\mu_H \leq \mu_{\langle u\rangle}$ implies $U^2 \in J_1(\mathbf{X})$. On the other hand, since $|e^U - 1 - U| \leq \frac{1}{2}e^{||U||_\infty}|U(x,y)|^2$ and

$$\begin{aligned}|e^U-1|^2 &\leq \left(\frac{||U||_\infty e^{|U(x,y)|}}{2}|U(x,y)| + |U(x,y)|\right)^2\\ &\leq \left(\frac{||U||_\infty e^{|U(x,y)|}}{2} + 1\right)^2 |U(x,y)|^2\end{aligned}$$

imply $e^U - 1 - U \in J_1(\mathbf{X})$ and $(e^U-1)^2 \in J_1(\mathbf{X})$ respectively. Therefore, there exists a purely discontinuous locally square integrable local martingale additive functional M^{e^U-1} on $[\![0,\zeta[\![$ such that $\Delta M_t^{e^U-1} = (e^U-1)(X_{t-},X_t)$, $t \in [0,\zeta[$ $\mathbf{P}_x$-a.s. for all $x \in E$ (see the proof of Lemma 3.2(i) in [2]). $M_t^{e^U-1}$ is given by

$$M_t^{e^U-1} = M_t^U + \sum_{0<s\leq t}(e^U-1-U)(X_{s-},X_s) - \int_0^t N(e^U-1-U)(X_s)\mathrm{d}H_s$$

where $M_t^U = M_t^{-u,j} + M_t^{-u,\kappa}$.

Let $\mathbf{U} = (\Omega, \widetilde{\mathcal{F}}_\infty, \widetilde{\mathcal{F}}_t, X_t, \mathbf{P}_x^U, \zeta)$ be the Girsanov transformed process of $\mathbf{X}$ by

$$U_t := \mathrm{Exp}(M^{e^U-1} + M^{-u,c})_t.$$

U_t is the Doléans-Dade exponential of $(M^{e^U-1} + M^{-u,c})_t$. It is easy to see that

$$U_t = \exp\left(-M_t^u - \int_0^t N\left(e^U - U - 1\right)(X_s)\mathrm{d}s - \frac{1}{2}\langle M^{u,c}\rangle_t\right). \tag{3.4}$$

The relation between $\mathbf{X}$ and $\mathbf{U}$ is given by $\mathbf{E}_x^U[f(X_t)] = \mathbf{E}_x[U_t f(X_t)]$ for $f \in \mathcal{B}_b(E)$.

The following has been proved in [5].

Lemma 3.1 (cf. [5, Lemma 3.3]) *Assume $\mu_{\langle u\rangle} \in S_K^1(\mathbf{X})$. Then the following hold:*

(1) *For* $\nu \in S^1_K(\mathbf{X})$, $e^{-2u}\nu \in S^1_K(\mathbf{U})$.

(2) *For* $\nu \in S^1_{EK}(\mathbf{X})$, $e^{-2u}\nu \in S^1_{EK}(\mathbf{U})$.

(3) *For* $\nu \in S^1_{LK}(\mathbf{X})$, $e^{-2u}\nu \in S^1_{LK}(\mathbf{U})$.

4 Main Result

Consider the non-local Feynman-Kac transforms by the additive functionals $A := N^u + A^\mu + A^F$ of the form (1.1). By (3.4), we see that

$$e_A(t) = U_t e^{u(X_t)-u(X_0)} \exp\left(A^{\bar\nu}_t + A^F_t\right),$$

where $\bar\nu = \bar\nu_1 - \bar\nu_2$ and $\bar\nu_1 := \mu_1 + \mu_u := \mu_1 + N(e^U - U - 1)\mu_H + \frac{1}{2}\mu^c_{\langle u\rangle}$ and $\bar\nu_2 := \mu_2$. In this section we define

$$\begin{aligned} Z_t &= e_A(t) \\ &= e^{u(X_t)-u(X_0)} U_t \exp\left(A^{\bar\nu}_t + A^F_t\right) \\ &= e^{u(X_t)-u(X_0)} U_t \exp\left(A^{\bar\nu_1}_t + A^{F_1}_t\right) \exp\left(-A^{\mu_2}_t - A^{F_2}_t\right) \\ &= e^{u(X_t)-u(X_0)} Z^{(1)}_t Z^{(2)}_t Z^{(3)}_t \end{aligned}$$

where $Z^{(1)}_t = \exp\left(-A^{\mu_2}_t - A^{F_2}_t\right), Z^{(2)}_t = U_t = \exp\left(-M^u_t - A^{\mu_u}_t\right)$ and $Z^{(3)}_t = \exp\left(A^{\bar\nu_1}_t + A^{F_1}_t\right)$. Let

$$R^A_\alpha f(x) := \mathbf{E}_x\left[\int_0^\infty e^{-\alpha t} Z_t f(X_t)\mathrm{d}t\right], \quad f \in \mathcal{B}_b(E).$$

Then $(R^A_\alpha)_{\alpha>\alpha_0}$ the resolvent of the transformed process from $\mathbf{X}$ by the multiplicative functional $(Z_t)_{t\geq 0}$.

The main result of this paper is the following.

Theorem 4.1 *Suppose that* $\mathbf{X}$ *has the doubly resolvent Feller property. Assume* $\mu_{\langle u\rangle} \in S^1_K(\mathbf{X})$ *for* $u \in \mathcal{F}_{loc} \cap C_b(E)$, $\mu = \mu_1 - \mu_2$ *and* $F = F_1 - F_2$ *with* $\mu_1 + N(e^{F_1} - 1)\mu_H \in S^1_{EK}(\mathbf{X}) \cap S^1_{LK}(\mathbf{X})$, $\mu_2 + N(F_2)\mu_H \in S^1_{LK}(\mathbf{X})$. *Then there exists* $\alpha_0 > 0$ *such that the resolvent* $(R^A_\alpha)_{\alpha>\alpha_0}$ *has the doubly resolvent Feller property. Here* α_0 *is the positive constant defined in the Section 2.*

Recall that $\mathbf{X}^* = (\Omega, X_t, \mathbf{P}^*_x)$ is the subprocess killed by $e^{-A^{\mu_2}_t - A^{F_2}_t}$. Note that if $\mu_2 + N(F_2)\mu_H \in S^1_{LK}(\mathbf{X})$ and $\mathbf{X}$ has the doubly resolvent Feller property, we can see by Corollary 3.1 that the subprocess $\mathbf{X}^*$ also has the doubly resolvent Feller property. Note that the Lévy system (N^*, H) of $\mathbf{X}^*$ is given by

$$N^*(x, \mathrm{d}y) = e^{-F_2(x,y)} N(x, \mathrm{d}y).$$

For the proof of Theorem 4.1, we need the following two lemmas.

Lemma 4.1 *Suppose* $\mathbf{X}^*$ *has the doubly resolvent Feller property and suppose* $\mu_{\langle u \rangle} \in S_K^1(\mathbf{X}^*)$ *for* $u \in \mathcal{F}_{loc} \cap C_b(E)$*. Then there exists* $\alpha_0 > 0$ *such that* $\{R_\alpha^{*,U_t}\}_{\alpha > \alpha_0}$ *defined by*

$$R_\alpha^{*,U_t} f(x) = \mathbf{E}_x^* \left[\int_0^\infty e^{-\alpha t} Z_t^{(2)} f(X_t) \mathrm{d}t \right]$$

has the doubly resolvent Feller property.

Proof. By Theorems 2.1 and 2.2, it suffices to check the conditions $(a)^*, (b)^s$ and $(c)^s$ for $Z_t^{(2)}$ hold under $\mathbf{X}^*$.

First we check $(a)^*$. Since $e^U - U - 1 \le \frac{1}{2} e^{\|U\|_\infty} U^2$, $\mu_u := N^* \left(e^U - U - 1 \right) \mu_H + \frac{1}{2} \mu_{\langle u \rangle}^c \in S_K^1(\mathbf{X}^*)$. Take $p > 1$ and $q > 1$ with $pq \in]1, 2]$. From the inequality $(1 + x)^r - 1 \le rx + (r-1)x^2$ for $x > -1$ and $r \in [1, 2]$, we see

$$\begin{aligned} N^* \left(e^{pqU} - 1 - pqU \right) \mu_H &= N^* \left(\left(1 + (e^U - 1) \right)^{pq} - 1 - pqU \right) \mu_H \\ &\le N^* \left(pq \left(e^U - 1 \right) + (pq - 1) \left(e^U - 1 \right)^2 - pqU \right) \mu_H \\ &\le C_{p,q,u} N^*(U^2) \mu_H \end{aligned}$$

which yields that $N^* \left(e^{pqU} - 1 - pqU \right) \mu_H \in S_K^1 \left(\mathbf{X}^* \right)$. Here $C_{p,q,u}$ is the positive constant depending on p, q and u. Let $M^{e^{pqU} - 1}$ be a locally square integrable martingale additive functional of $\mathbf{X}^*$ defined by

$$M_t^{e^{pqU} - 1} := \sum_{s \le t} \left(e^{pqU(X_{s-}, X_s)} - 1 \right) - \int_0^t N^* \left(e^{pqU} - 1 \right) (X_s) \mathrm{d}H_s, \quad t < \zeta.$$

Set $M_t^{(pq)} := M_t^{e^{pqU}-1} + pqM_t^{-u,c},\ t<\zeta$. Then

$$\begin{aligned}
Z_t^{(pq)} &:= \mathrm{Exp}\left(M^{(pq)}\right)_t \\
&= \exp\left(M_t^{e^{pqU}-1} + pqM_t^{-u,c} - \frac{p^2q^2}{2}\langle M^{u,c}\rangle_t\right) \\
&\qquad \times \prod_{s\le t}\left(1+\left(e^{pqU(X_{s-},X_s)}-1\right)\right)\exp\left(-\left(e^{pqU(X_{s-},X_s)}-1\right)\right) \\
&= \exp\left(M_t^{e^{pqU}-1} + pqM_t^{-u,c} - \frac{p^2q^2}{2}\langle M^{u,c}\rangle_t\right) \\
&\qquad \times \exp\left(pqM_t^{-u,j} + pq\int_0^t N^*(U)(X_s)\mathrm{d}H_s - \sum_{s\le t}\left(e^{pqU(X_{s-},X_s)}-1\right)\right) \\
&= \exp\left(pqM_t^{-u} - \int_0^t N^*\left(e^{pqU}-1-pqU\right)(X_s)\mathrm{d}H_s - \frac{p^2q^2}{2}\langle M^{u,c}\rangle_t\right) \\
&= \exp\left(pqM_t^{-u} - A_t^{\mu_u^{(pq)}}\right),
\end{aligned}$$

where $A_t^{\mu_u^{(pq)}} := \int_0^t N^*\left(e^{pqU}-1-pqU\right)(X_s)\mathrm{d}H_s + \frac{p^2q^2}{2}\langle M^{u,c}\rangle_t$. We then see by Hölder inequality that

$$\begin{aligned}
\mathbf{E}_x^*\left[\left(Z_t^{(2)}\right)^p\right] &= \mathbf{E}_x^*\left[\exp\left(pM_t^{-u} - pA_t^{\mu_u}\right)\right] \\
&= \mathbf{E}_x^*\left[\left(Z_t^{(pq)}\right)^{1/q}\exp\left(\frac{1}{q}A_t^{\mu_u^{(pq)}} - pA_t^{\mu_u}\right)\right] \\
&\le \mathbf{E}_x^*\left[Z_t^{(pq)}\right]^{1/q}\mathbf{E}_x^*\left[\exp\left(\frac{1}{q-1}A_t^{\mu_u^{(pq)}} - \frac{pq}{q-1}A_t^{\mu_u}\right)\right]^{(q-1)/q} \\
&\le \mathbf{E}_x^*\left[\exp\left(\frac{1}{q-1}\int_0^t N^*\left(\left(e^{pqU}-1\right)-pq\left(e^U-1\right)\right)(X_s)\mathrm{d}H_s\right.\right. \\
&\qquad\qquad \left.\left. + \frac{pq}{2(q-1)}(pq-1)\langle M^{u,c}\rangle_t\right)\right]. \qquad (4.1)
\end{aligned}$$

By applying the inequality $(1+x)^r - 1 - rx \le (r-1)x^2$ for $x>-1$ and $r\in[1,2]$ again, one can see that $\left(e^{pqU}-1\right) - pq\left(e^U-1\right) = \left(1+\left(e^U-1\right)\right)^{pq} - 1 - pq\left(e^U-1\right) \le (pq-1)\left(e^U-1\right)^2$. Then the right hand side of (4.1) is dominated by

$$\mathbf{E}_x^*\left[\exp\left(\frac{pq-1}{q-1}\left(\frac{1}{pq}\int_0^t N^*\left(\left(e^U-1\right)^2\right)(X_s)\mathrm{d}H_s + \frac{pq}{2}\langle M^{u,c}\rangle_t\right)\right)\right]^{(q-1)/q}.$$

Put $l_{t_0} := \sup_{x\in E}\mathbf{E}_x^*\left[\int_0^{t_0} N^*\left(\left(e^U-1\right)^2\right)(X_s)\mathrm{d}H_s + \frac{1}{2}\langle M^{u,c}\rangle_{t_0}\right] < 1$ for sufficiently small $t_0 > 0$. Now, in view of Khas'miskii's lemma,

$$\sup_{x\in E}\mathbf{E}_x^*\left[\left(Z_{t_0}^{(2)}\right)^p\right] \le \sup_{x\in E}\mathbf{E}_x^*\left[\exp\left(\frac{pq-1}{q-1}pq\left(\frac{1}{pq}\int_0^{t_0}N^*\left(\left(e^U-1\right)^2\right)(X_s)\mathrm{d}H_s + \frac{1}{2}\langle M^{u,c}\rangle_{t_0}\right)\right)\right]^{(q-1)/q}$$
$$\le \left\{\frac{1}{1-\frac{pq-1}{q-1}pql_{t_0}}\right\}^{(q-1)/q}.$$

Letting p and q be sufficiently close to 1, we can see $\frac{pq-1}{q-1}pql_{t_0} < 1$ and which implies that $\sup_{x\in E}\sup_{s\in[0,t]}\mathbf{E}_x^*\left[\left(Z_s^{(2)}\right)^p\right] < \infty$ for some (hence for all) $t > 0$ and for some $p > 1$. It is clear that $(a)^*$ implies $(b)^s$.

Next, we check $(c)^s$. By using Remark 2.1(3), it suffices to check $(c)^w$. In view of the proof for $(a)^*$, we can see that for relatively compact $D \subset E$,

$$\begin{aligned}\mathbf{E}_x^*\left[|Z_t^{(2)}-1| : t<\tau_D\right]^2 &\le \mathbf{E}_x^*\left[|Z_{t\wedge\tau_D}^{(2)}-1|\right]^2\\ &\le \mathbf{E}_x^*\left[|Z_{t\wedge\tau_D}^{(2)}-1|^2\right]\\ &= \mathbf{E}_x^*\left[\left(Z_{t\wedge\tau_D}^{(2)}\right)^2\right]-1.\end{aligned}$$

By way of the method for obtaining the inequality (4.1), we have for $q > 1$

$$\begin{aligned}\mathbf{E}_x^*\left[\left(Z_{t\wedge\tau_D}^{(2)}\right)^2\right] &\le \mathbf{E}_x^*\left[\exp\left(\frac{1}{q-1}\int_0^{t\wedge\tau_D}N^*\left(\left(e^{2qU}-1\right)-2q\left(e^U-1\right)\right)(X_s)\mathrm{d}H_s + \frac{2q}{2(q-1)}(2q-1)\langle M^{u,c}\rangle_{t\wedge\tau_D}\right)\right]\\ &\le \mathbf{E}_x^*\left[\exp\left(\frac{2q-1}{q-1}\int_0^{t\wedge\tau_D}N^*\left(\left(e^U-1\right)^2\right)(X_s)\mathrm{d}H_s + \frac{2q}{2(q-1)}(2q-1)\langle M^{u,c}\rangle_{t\wedge\tau_D}\right)\right]^{(q-1)/q}\\ &\le \mathbf{E}_x^*\left[\exp\left(C_q\left(\int_0^{t\wedge\tau_D}N^*\left(\left(e^U-1\right)^2\right)(X_s)\mathrm{d}H_s + \frac{1}{2}\langle M^{u,c}\rangle_{t\wedge\tau_D}\right)\right)\right]^{(q-1)/q}\end{aligned}$$

where $C_q := \frac{q}{q-1}(2q-1)$.

Since $C_t^D := \int_0^{t\wedge\tau_D}N^*\left(\left(e^U-1\right)^2\right)(X_s)\mathrm{d}H_s + \frac{1}{2}\langle M^{u,c}\rangle_{t\wedge\tau_D}$ is a PCAF of Kato class, we conclude that

$$
\begin{aligned}
\sup_{x\in D} \mathbf{E}_x^*\left[|Z_t^{(2)}-1| : t<\tau_D\right]^2 &\le \sup_{x\in E} \mathbf{E}_x^*\left[\left(Z_{t\wedge\tau_D}^{(2)}\right)^2\right]-1 \\
&\le \sup_{x\in E} \mathbf{E}_x^*\left[\exp(C_q C_t^D)\right]^{(q-1)/q}-1 \\
&\le \left(\frac{1}{1-C_q \sup_{x\in E}\mathbf{E}_x^*\left[C_t^D\right]}\right)^{(q-1)/q}-1 \\
&\longrightarrow 0 \qquad \text{as } t\to 0.
\end{aligned}
$$

The proof is complete.

Let $\mathbf{U}^* = \left(X_t, \mathbf{P}_x^{U^*}\right)$ be the transformed process of the subprocess $\mathbf{X}^*$ by U_t. We note that the Lévy system $\left(N^{U^*}, H\right)$ of $\mathbf{U}^*$ is given by

$$
N^{U^*}(x,\mathrm{d}y) = e^{U(x,y)}N^*(x,\mathrm{d}y) = e^{U(x,y)-F_2(x,y)}N(x,\mathrm{d}y).
$$

Lemma 4.2 *Suppose $\mathbf{X}^*$ has the doubly resolvent Feller property. Assume $\mu_{\langle u\rangle} \in S_K^1(\mathbf{X}^*)$ for $u\in\mathcal{F}_{loc}\cap C_b(E)$, $\mu_1 + N^*(e^{F_1}-1)\mu_H \in S_{EK}^1(\mathbf{X}^*)\cap S_{LK}^1(\mathbf{X}^*)$. Then there exists $\alpha_0>0$ such that $\{R_\alpha^{U^*,\bar\nu_1+F_1}\}_{\alpha>\alpha_0}$ defined by*

$$
R_\alpha^{U^*,\bar\nu_1+F_1}f(x) = \mathbf{E}_x^{U^*}\left[\int_0^\infty e^{-\alpha t}Z_t^{(3)}f(X_t)\mathrm{d}t\right]
$$

has the doubly resolvent Feller property.

Proof. Similar to the previous lemma, it suffices to check the conditions $(a)^*, (b)^s$ and $(c)^w$ hold.

First, we check $(a)^*$. Note that we see by Lemma 3.1(3) that $e^{-2u}\mu_{\langle u\rangle} \in S_K^1(\mathbf{U}^*)$ and $e^{-2u}\mu_1 + N\left(e^{U-F_2}(e^{F_1}-1)\right)\mu_H = e^{-2u}\mu_1 + N^{U^*}\left(e^{F_1}-1\right)\mu_H \in S_K^1(\mathbf{U}^*)$. Since $\left(Z_t^{(3)}\right)^p = e^{A_t^{p\bar\nu_1}+A_t^{pF_1}} = \mathrm{Exp}\left[A^{p\bar\nu_1}+A^{e^{pF_1}-1}\right]_t$, we can show that there exists $p>1$ such that

$$
e^{-2u}p\mu_1 + N^{U^*}\left(e^U-U-1)\right)\mu_H + \frac{p}{2}e^{-2u}\mu_{\langle u\rangle}^c + N^{U^*}\left(e^{pF_1}-1\right)\mu_H \in S_{EK}^1(\mathbf{U}^*).
$$

Indeed, recall $(1+x)^p-1 \le (p-1)x^2+px$ for $p\in]1,2]$ and for $x>-1$. By assumption, there exists $T>0$ such that

$$
\lambda := \sup_{x\in E}\mathbf{E}_x^{U^*}\left(A_T^{\bar\nu_1}+A_T^{e^{F_1}-1}\right) < 1.
$$

For such $T>0$, we set $l=\sup_{x\in E}\mathbf{E}_x^{U^*}\left[(A_T^{e^{F_1}-1})^2\right]<\infty$ and take $p\in\left]1,2\wedge\frac{1+l}{\lambda+l}\right[$. Then

$$
\begin{aligned}
\sup_{x\in E}\ &\mathbf{E}_x^{U^*}\left[A_T^{p\bar\nu_1}+A_T^{e^{pF_1}-1}\right]\\
&=\sup_{x\in E}\mathbf{E}_x^{U^*}\left[A_T^{p\bar\nu_1}+\sum_{s\le T}\left(e^{pF_1(X_{s-},X_s)}-1\right)\right]\\
&=\sup_{x\in E}\mathbf{E}_x^{U^*}\left[A_T^{p\bar\nu_1}+\sum_{s\le T}\left(\left(1+(e^{F_1(X_{s-},X_s)}-1)\right)^p-1\right)\right]\\
&\le\sup_{x\in E}\mathbf{E}_x^{U^*}\left[A_T^{p\bar\nu_1}+\sum_{s\le T}\left\{(p-1)\left(e^{F_1(X_{s-},X_s)}-1\right)^2+p\left(e^{F_1(X_{s-},X_s)}-1\right)\right\}\right]\\
&\le(p-1)\sup_{x\in E}\mathbf{E}_x^{U^*}\left[\sum_{s\le T}\left(e^{F_1(X_{s-},X_s)}-1\right)^2\right]\\
&\qquad+p\sup_{x\in E}\mathbf{E}_x^{U^*}\left[A_T^{\bar\nu_1}+\sum_{s\le T}\left(e^{F_1(X_{s-},X_s)}-1\right)\right]\\
&=(p-1)l+p\lambda<1.
\end{aligned}\tag{4.2}
$$

By Khas'minskii's lemma,

$$
\begin{aligned}
\sup_{x\in E}\mathbf{E}_x^{U^*}\left[\left(Z_T^{(3)}\right)^p\right]&=\sup_{x\in E}\mathbf{E}_x^{U^*}\left[\mathrm{Exp}\left(A^{p\bar\nu_1}+A^{e^{pF_1}-1}\right)_T\right]\\
&\le\frac{1}{1-\sup_{x\in E}\mathbf{E}_x^{U^*}\left[A_T^{p\bar\nu_1}+A_T^{e^{pF_1}-1}\right]}\\
&<\infty.
\end{aligned}
$$

Hence we have $\sup_{s\in[0,T]}\sup_{x\in E}\mathbf{E}_x^{U^*}\left[\left(Z_s^{(3)}\right)^p\right]<\infty$, that is, we obtain $(a)^*$ under $\mathbf{U}^*$. From (4.2) we already checked that $A_T^{p\bar\nu_1}+A_T^{e^{pF_1}-1}$ is of extended Kato class for some $T>0$. Thus by the Markov property we obtain $(b)^s$. Next, we check $(c)^w$. For relatively compact D of E,

$$
\begin{aligned}
\mathbf{E}_x^{U^*}\left[|Z_t^{(3)}-1|:t<\tau_D\right]&=\mathbf{E}_x^{U^*}\left[|\mathrm{Exp}\left(A^{\bar\nu_1}+A^{e^{F_1}-1}\right)_t-1|:t<\tau_D\right]\\
&=\mathbf{E}_x^{U^*}\left[|\mathrm{Exp}\left(\mathbf{1}_D*\left(A^{\bar\nu_1}+A^{e^{F_1}-1}\right)\right)_t-1|:t<\tau_D\right]\\
&\le\mathbf{E}_x^{U^*}\left[|\mathrm{Exp}\left(\mathbf{1}_D*\left(A^{\bar\nu_1}+A^{e^{F_1}-1}\right)\right)_t-1|\right].
\end{aligned}
$$

Since $e^{-2u}\mu_1\in S^1_{LK}(\mathbf{U}^*)$ and $N^{U^*}\left(e^U-U-1\right)\mu_H+\frac12 e^{-2u}\mu^c_{\langle u\rangle}\in S^1_K(\mathbf{U}^*)$, $e^{-2u}\bar\nu_1$ $:=e^{-2u}\mu_1+N^{U^*}\left(e^U-U-1\right)\mu_H+\frac12 e^{-2u}\mu^c_{\langle u\rangle}\in S^1_{LK}(\mathbf{U}^*)$ and consequently

$e^{-2u}\left(\bar{\nu}_1 + N^{U^*}\left(e^{F_1}-1\right)\right)\mu_H \in S^1_{LK}(\mathbf{U}^*)$. By Khas'miskii's lemma,

$$\begin{aligned}\limsup_{t\downarrow 0}\sup_{x\in D}\mathbf{E}^{U^*}_x\left[|Z^{(3)}_t-1| : t<\tau_D\right] &\le \lim_{t\downarrow 0}\frac{\sup_{x\in D}\mathbf{E}^{U^*}_x\left[\mathbf{1}_D * \left(A^{\bar{\nu}_1}_t + A^{e^{F_1}-1}_t\right)\right]}{1-\sup_{x\in D}\mathbf{E}^{U^*}_x\left[\mathbf{1}_D * \left(A^{\bar{\nu}_1}_t + A^{e^{F_1}-1}_t\right)\right]}\\ &= 0\end{aligned}$$

which implies $(c)^w$ for $Z^{(3)}_t$ under $\mathbf{U}^*$. Hence we obtain the assertion.

Proof of Theorem 4.1. By combining the remark just after Theorem 4.1, Lemmas 4.1 and 4.2, we obtain that there exists $\alpha_0 > 0$ such that the resolvent $(R^{U,\bar{\nu}+F}_\alpha)_{\alpha>\alpha_0}$ defined by

$$R^{U,\bar{\nu}+F}_\alpha f(x) := \int_0^\infty e^{-\alpha t} P^{U,\bar{\nu}+F}_t f(x)\mathrm{d}t,$$

has the doubly resolvent Feller property, where $P^{U,\bar{\nu}+F}_t f(x) := \mathbf{E}^U_x\left[\exp(A^{\bar{\nu}}_t + A^F_t)\right.$ $\left.f(X_t)\right] = \mathbf{E}^{U^*}_x\left[\exp\left(A^{\bar{\nu}_1}_t + A^{F_1}_t\right) f(X_t)\right]$ for $f \in \mathcal{B}_b(E)$. Since u is continuous and

$$R^A_\alpha f(x) = \int_0^\infty e^{-\alpha t} e^{-u(x)} P^{U,\bar{\nu}+F}_t (fe^u)(x)\mathrm{d}t, \qquad x \in E,$$

$(R^A_\alpha)_{\alpha>\alpha_0}$ also has the doubly resolvent Feller property.

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Department of Mathematics and Engineering
Graduate School of Science and Technology
Kumamoto University
Kumamoto, 860-8555
Japan

Department of Mathematics
Faculty of Mathematics and Natural Sciences
Universitas Brawijaya
Malang, 65145
Indonesia
e-mail: mila_n12@ub.ac.id