

# On a map on a finite group

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(Received July 13, 2019)

(Accepted July 31, 2019)

**Abstract.** We consider a special map on a finite group. We give a congruence of the cardinality of the inverse image of an element by the map. Also if the cardinality of the inverse image of the identity element by the map is greater than the order of the group, then the group is of even order. Also we give a condition the group to be solvable using the map.

## 1 Introduction

In [1, 2], Bandman et al. constructed a sequence of words on a finite group with two variables which gives a criterion for the group to be solvable. Similar sequences are studied in [5]. With easy modification of some words in [1, 2], we will find the following map over a finite group  $G$ :

$$\begin{array}{ccc} f_1 : G \times G & \longrightarrow & G \\ & \Downarrow & \Downarrow \\ & (x, y) & \longmapsto y^{-1}[xy^{-1}x^{-1}, x^{-1}yx] = y^{-1}[(y^{-1})^{x^{-1}}, y^x] \end{array}$$

where  $[a, b] = a^{-1}b^{-1}ab$  and  $a^b = b^{-1}ab$  for  $a, b \in G$ .

In this note, we will study a relation between the set of solutions of an equation

$$f_1(x, y) = a$$

for  $a \in G$  and structures of the finite group.

For a map  $f : G \times G \rightarrow G$  and  $a \in G$ , denote the set of solutions of the equation  $f(x, y) = a$ , or the inverse image of  $a \in G$  by  $f$ , by

$$\mathcal{S}_G^f(a) = \{(x, y) \in G \times G \mid f(x, y) = a\}.$$

Since  $f_1(b, a^{-1}) = a$  for  $a \in G$  and  $b \in C_G(a)$ , we see that  $f_1$  is surjective,  $\mathcal{S}_G^{f_1}(a) \supseteq \{(b, a^{-1}) \mid b \in C_G(a)\}$  and  $|\mathcal{S}_G^{f_1}(a)| \geq |C_G(a)|$ . First we will give a

**Mathematical Subject Classification (2010):** Primary 20D10; Secondary 20D05

**Key words:** finite group, simple group, equation over groups

\*This work is supported by KAKENHI 19K03405.

congruence of the number of solutions like a theorem of Frobenius as an application of the method of Brauer [4]:

**Theorem 1.1.** *Let  $G$  be a finite group. For any  $a \in G$ ,*

$$|\mathcal{S}_G^{f_1}(a)| \equiv 0 \pmod{|C_G(a)|}.$$

Next we will focus on  $|\mathcal{S}_G^{f_1}(1)|$ . As mentioned above,  $\mathcal{S}_G^{f_1}(1) \supseteq \{(x, 1) \mid x \in G\}$  and  $|\mathcal{S}_G^{f_1}(1)| \geq |G|$  for any finite group  $G$ . We will show the following:

**Theorem 1.2.** *If  $|\mathcal{S}_G^{f_1}(1)| > |G|$ , then  $G$  is of even order.*

After explaining how to get  $f_1$  from [1, 2], we will consider a decomposition of  $G \times G$  by  $\mathcal{S}_G^f(a)$  for  $a \in G$ . We will prove the following as an application of a result of [1, 2].

**Theorem 1.3.** *Suppose that  $|\mathcal{S}_G^f(a)| = |G|$  for any  $a \in G$ , then  $G$  is solvable.*

We mention that Theorem 1.3 is proved by using the classification theorem of finite simple groups.

## 2 Congruence of the number of solutions

For a subgroup  $H$  of a group  $G$ , we say that  $g_1, g_2 \in G$  are weakly equivalent with respect to  $H$  if there exists  $h \in H$  such that  $g_1^{-r} h g_2^r \in H$  for any  $r \in \mathbb{Z}$ . We denote by  $g_1 \underset{H}{\approx} g_2$  if  $g_1$  is weakly equivalent to  $g_2$  with respect to  $H$ . For  $x \in G$ , set  $H_x = \cap_{r \in \mathbb{Z}} H^{x^r}$ .

In [4], Brauer proved the following:

**Proposition 2.1** ([4]). *For a subgroup  $H$  of a group  $G$  and  $x \in G$ ,*

- (1)  $\{g \in G \mid g \underset{H}{\approx} x\} = \{h^{-1} x k h \mid h \in H, k \in H_x\}$ , and
- (2)  $|\{g \in G \mid g \underset{H}{\approx} x\}| = |H|$ .

*Proof.* See the paragraph after Proposition 3 of [4] for (1) and Proposition 5 of [4] for (2).

For  $a, y \in G$ , set  $\mathcal{S}_G^{f_1}(a, y) = \{x \in G \mid f_1(x, y) = a\}$ .

**Proposition 2.2.**  $|\mathcal{S}_G^{f_1}(a, y)| \equiv 0 \pmod{|C_G(a) \cap C_G(y)|}$  for  $a, y \in G$ .

*Proof.* Suppose that  $\mathcal{S}_G^{f_1}(a, y) \neq \emptyset$ . Take  $x \in \mathcal{S}_G^{f_1}(a, y)$ . Set  $H = C_G(a) \cap C_G(y)$ . For  $b \in G$  with  $b \underset{H}{\approx} x$ , there exists  $h \in H$  and  $k \in H_x$  such that  $b = h^{-1} x k h$  by Proposition 2.1 (1). Then

$$\begin{aligned} b y^{-1} b^{-1} &= (h^{-1} x k h) y^{-1} (h^{-1} k^{-1} x^{-1} h) = (x y^{-1} x^{-1})^h, \text{ and} \\ b^{-1} y b &= (h^{-1} k^{-1} x^{-1} h) y (h^{-1} x k h) = h^{-1} x^{-1} (x k^{-1} x^{-1}) y (x k x^{-1}) x h \\ &= (x^{-1} y x)^h \end{aligned}$$

since  $h, k \in C_G(y)$  and  $xk^{-1}x^{-1} \in H_x \subseteq C_G(y)$ . We have

$$[by^{-1}b^{-1}, b^{-1}yb] = [xy^{-1}x^{-1}, x^{-1}yx]^h = (ya)^h = ya.$$

Thus  $b \in \mathcal{S}_G^{f_1}(a, y)$ . This implies that  $\mathcal{S}_G^{f_1}(a, y)$  is a union of some  $\approx_H$ -classes of  $G$ . By Proposition 2.1 (2), we have the result.

Now we give a proof of Theorem 1.1.

*Proof of Theorem 1.1.* Take a set  $\{y_i \mid i = 1, \dots, r\}$  of representatives of  $C_G(a)$ -conjugacy classes of  $G$ . Since  $|\mathcal{S}_G^{f_1}(a, y)| = |\mathcal{S}_G^{f_1}(a, g^{-1}yg)|$  for  $g \in C_G(a)$ , we have

$$\begin{aligned} |\mathcal{S}_G^{f_1}(a)| &= \sum_{y \in G} |\mathcal{S}_G^{f_1}(a, y)| = \sum_{i=1}^r \frac{|C_G(a)|}{|C_G(a) \cap C_G(y_i)|} |\mathcal{S}_G^{f_1}(a, y_i)| \\ &= |C_G(a)| \sum_{i=1}^r \frac{|\mathcal{S}_G^{f_1}(a, y_i)|}{|C_G(a) \cap C_G(y_i)|}. \end{aligned}$$

By Proposition 2.2, we have the result.  $\square$

### 3 Solutions of $f_1(x, y) = 1$

In this section, we focus on  $|\mathcal{S}_G^{f_1}(1)|$ . The following lemma is easy but fundamental.

**Lemma 3.1.** *The following hold:*

- (1)  $f_1(x, y) = ((y^{-1})^x [y, x]^{[y, x^{-1}]})^{[x^{-1}, y]}$ .
- (2)  $f_1(x, y) = (y^{-1} [x, y]^{[x^2, y]})^{y^{-1} [y, x^2] x^{-1}}$ .

*Proof.* Although it is just a direct calculation, it is somewhat tricky. So we give a detailed calculation.

(1) We have

$$\begin{aligned} f_1(x, y) &= y^{-1} [xy^{-1}x^{-1}, x^{-1}yx] \\ &= y^{-1} (xyx^{-1})(x^{-1}y^{-1}x)(xy^{-1}x^{-1})(x^{-1}yx) \\ &= (y^{-1}xyx^{-2})y^{-1}(x^2y^{-1}x^{-1}y)(y^{-1}x^{-1}yx) \\ &= (y^{-1})^{x[x^{-1}, y]} [y, x] = ((y^{-1})^x [y, x]^{[y, x^{-1}]})^{[x^{-1}, y]}. \end{aligned}$$

(2) We have

$$\begin{aligned} f_1(x, y) &= y^{-1} [xy^{-1}x^{-1}, x^{-1}yx] \\ &= y^{-1} (xyx^{-1})(x^{-1}y^{-1}x)(xy^{-1}x^{-1})(x^{-1}yx) \\ &= x(x^{-1}y^{-1}xy)(x^{-2}y^{-1}x^2y)y^{-1}(y^{-1}x^{-2}yx^2)x^{-1} \\ &= ([x, y](y^{-1})^{[y, x^2]})^{x^{-1}} = ([x, y]^{[x^2, y]}y^{-1})^{[y, x^2]x^{-1}} \\ &= (y^{-1} [x, y]^{[x^2, y]})^{y^{-1} [y, x^2] x^{-1}}, \end{aligned}$$

which is required.

We prove Theorem 1.2.

*Proof of Theorem 1.2.* Suppose that  $|\mathcal{S}_G^f(1)| > |G|$ . There exists  $(x, y) \in G \times G$  with  $y \neq 1$  such that  $f_1(x, y) = 1$ . We have  $[y, x] = y^{x[x^{-1}, y]}$  by Lemma 3.1 (1), and we have  $[y, x] = (y^{-1})^{[y, x^2]}$  by Lemma 3.1 (2). This implies that  $y^{x[x^{-1}, y]} = (y^{-1})^{[y, x^2]}$ , or equivalently,

$$y^{x[x^{-1}, y][x^2, y]} = y^{-1}.$$

Set  $t = x[x^{-1}, y][x^2, y]$ . Then  $t \in N_G(\langle y \rangle) \setminus C_G(y)$  and  $t^2 \in C_G(y)$ . Thus  $|N_G(\langle y \rangle)/C_G(y)|$  is even and the result follows.  $\square$

The following is a restatement of Theorem 1.2

**Corollary 3.2.** *If  $G$  is of odd order, then  $|\mathcal{S}_G^{f_1}(1)| = |G|$ .*

Easy observation gives the following:

**Proposition 3.3.** *If  $G$  is solvable, then  $|\mathcal{S}_G^{f_1}(1)| = |G|$ .*

*Proof.* Suppose that  $G$  is solvable and that there exists  $(x, y) \in \mathcal{S}_G^{f_1}(1)$  with  $y \neq 1$ . Then  $y = [xy^{-1}x^{-1}, x^{-1}yx]$ . Consider the derived series  $G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_r = 1$ , where  $G_{i+1} = [G_i, G_i]$  for  $i = 0, \dots, r-1$ . There is  $j < r$  such that  $y \in G_j$  and  $y \notin G_{j+1}$ . Then  $xy^{-1}x^{-1}$  and  $x^{-1}yx \in G_j$  and therefore  $y = [xy^{-1}x^{-1}, x^{-1}yx] \in G_{j+1}$ , a contradiction. This yields that  $y = 1$  and  $\mathcal{S}_G^{f_1}(1) = \{(x, 1) \mid x \in G\}$ .

The following is a restatement of Proposition 3.3.

**Corollary 3.4.** *If  $|\mathcal{S}_G^{f_1}(1)| > |G|$ , then  $G$  is non-solvable.*

**Remark 3.5.** There is a non-solvable group  $G$  such that  $|\mathcal{S}_G^{f_1}(1)| = |G|$ . For example, let  $G = 2^4.SL(2, 5)$ , a non-split extension. A direct calculation shows that  $|\mathcal{S}_G^{f_1}(1)| = |G|$ . Thus the converse of Proposition 3.3 does not hold in general.

## 4 Some variations of $f_1$

We define  $f_j : G \times G \rightarrow G$  ( $j = 2, 3, 4$ ) by

$$\begin{aligned} f_2(x, y) &= (y^{-1})^x [y, x]^{[y, x^{-1}]}, \\ f_3(x, y) &= (yx)^{-1} [y, x]^{[y, x^{-1}]}, \quad \text{and} \\ f_4(x, y) &= y^{-1} [x, y]^{[x^2, y]}. \end{aligned}$$

**Proposition 4.1.** *For any finite group  $G$  and  $a \in G$ ,  $|\mathcal{S}_G^{f_i}(a)| = |\mathcal{S}_G^{f_1}(a)|$  for  $i = 2, 3, 4$ .*

*Proof.* By Lemma 3.1, we have  $|\mathcal{S}_G^{f_1}(a)| = |\mathcal{S}_G^{f_2}(a)| = |\mathcal{S}_G^{f_4}(a)|$ . Since

$$((xy)^{-1})^x = (yx)^{-1}, \quad [xy, x] = [y, x], \quad \text{and} \quad [xy, x^{-1}] = [y, x^{-1}],$$

we have  $f_2(x, xy) = f_3(x, y)$ . Thus we have  $|\mathcal{S}_G^{f_2}(a)| = |\mathcal{S}_G^{f_3}(a)|$ .

Consider two maps  $f : G \times G \rightarrow G$  and  $f' : G \times G \rightarrow G$ . We say  $f \sim f'$  if  $|\mathcal{S}_G^f(a)| = |\mathcal{S}_G^{f'}(a)|$  for each finite group  $G$  and for any  $a \in G$ . We have proved  $f_1 \sim f_2 \sim f_3 \sim f_4$  by Proposition 4.1. Of course, there are many maps  $f$  with  $f \sim f_1$ . We have listed some of them which looks simple.

## 5 A theorem of Bandman et. al.

Define  $u_1(x, y) = x^{-2}y^{-1}x$  and inductively

$$u_{n+1}(x, y) = [xu_n(x, y)^{-1}x^{-1}, yu_n(x, y)^{-1}y^{-1}].$$

The following is proved in [1, 2]:

**Theorem 5.1** ([1, 2, Theorem 1.1]). *A finite group  $G$  is solvable if and only if for some  $n$  the identity  $u_n(x, y) \equiv 1$  holds in  $G$ .*

Note that a commutator is defined by  $[a, b] = aba^{-1}b^{-1}$  in [1, 2]. We rewrite the definition of the sequence  $u_n$  by using our notation  $[a, b] = a^{-1}b^{-1}ab$ .

A non-abelian simple group is called minimal simple if all of its proper subgroups are solvable. Minimal simple groups are classified by Thompson [6].

**Theorem 5.2** ([2, Theorem 1.2]). *Let  $G$  be a minimal simple group. Then there are  $x, y \in G$  such that  $u_1(x, y) \neq 1$  and  $u_1(x, y) = u_2(x, y)$ .*

The following theorem is proved by using the classification theorem of finite simple groups.

**Theorem 5.3** ([3]). *If  $G$  is a non-abelian simple group, then  $G$  contains a subgroup which is a minimal simple group.*

Combining Theorems 5.2 and 5.3, the following holds:

**Corollary 5.4** ([2, Corollary 1.3]). *Let  $G$  be a non-abelian simple group. Then there are  $x, y \in G$  such that  $u_1(x, y) \neq 1$  and  $u_1(x, y) = u_2(x, y)$ .*

Set  $z = yx$ . Then

$$\begin{aligned} u_1(x, y) &= x^{-1}z^{-1}x, \quad \text{and} \\ u_2(x, y) &= [x(x^{-1}zx)x^{-1}, (zx^{-1})(x^{-1}z)x(zx^{-1})^{-1}] = [z, zx^{-2}zx^2z^{-1}] \\ &= z^{-1}(zx^{-2}z^{-1}x^2z^{-1})z(zx^{-2}zx^2z^{-1}) = [x^{-2}zx^2, z^{-1}]. \end{aligned}$$

Thus the condition  $u_1(x, y) \neq 1$  is equivalent to  $z \neq 1$  and the condition  $u_1(x, y) = u_2(x, y)$  is equivalent to

$$z = [xz^{-1}x^{-1}, x^{-1}zx],$$

or equivalently  $f_1(x, z) = 1$ .

In our situation, we can rewrite Corollary 5.4 as follows:

**Corollary 5.5.** *If  $G$  is a non-abelian simple group, then  $|\mathcal{S}_G^{f_1}(1)| > |G|$ .*

## 6 A decomposition of $G \times G$

We consider the decomposition of  $G \times G$  by  $\mathcal{S}_G^{f_1}(a)$  for  $a \in G$ . By using the classification theorem of finite simple groups, we have Theorem 1.3

*Proof of Theorem 1.3.* Suppose that  $G$  is a counterexample of minimal possible order. By Corollary 5.5,  $G$  is not a non-abelian simple group since  $|\mathcal{S}_G^{f_1}(1)| = |G|$ . Take a minimal normal subgroup  $N$  of  $G$ . If  $N$  is a direct product of  $r$ -copies of non-abelian simple group  $S$  for some  $r$ , then there exists  $(x, y) \in \mathcal{S}_S^{f_1}(1) \subseteq \mathcal{S}_G^{f_1}(1)$  with  $y \neq 1$ . This is not our case because  $|\mathcal{S}_G^{f_1}(1)| = |G|$ . Thus  $N$  is an elementary abelian subgroup.

For any  $a \in G$ ,  $|\{(x, y) \mid f_1(x, y) \in aN\}| = |G| \times |N|$  because  $|\mathcal{S}_G^{f_1}(an)| = |G|$  for  $n \in N$ . For  $(x, y)$  with  $f_1(x, y) \in aN$  and for any  $n, m \in N$ , we see that  $f_1(xn, ym) \in aN$ . This yields that

$$|\{(xN, yN) \mid f_1(xN, yN) = aN\}| = \frac{|G| \times |N|}{|N| \times |N|} = |G/N|.$$

Thus  $|\mathcal{S}_{G/N}^{f_1}(aN)| = |G/N|$  for any  $aN \in G/N$ . Since  $G$  is a minimal counterexample, we have  $G/N$  is solvable. This implies that  $G$  is also solvable, a contradiction.

**Remark 6.1.** There is a solvable group  $G$  such that  $|\mathcal{S}_G^{f_1}(a)| \neq |G|$  for some  $a \in G$ . The group  $3^2 : SL(2, 3)$  with trivial center is an example of smallest order. The second smallest one is  $A_4 \wr Z_2$ . Thus the converse of Theorem 1.3 does not hold in general.

**Acknowledgments** The author would like to thank the referee for valuable comments.

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