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Toric Fano manifolds of dimension at most eight with positive second Chern characters

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Abstract. We show that any toric Fano manifold of dimension at most eight with the positive second Chern character is isomorphic to the projective space by using `polymake`.

Contents

1 Introduction

Smooth Fano varieties are very important objects in algebraic geometry, though the definition is very simple, that is, a smooth projective variety with ample anti-canonical divisor. By Nakai-Moishezon criterion, this condition implies that the intersection $\mathrm{ch}_1(X) \cdot C$ is positive for any curve C on X, where $\mathrm{ch}_1(X)$ is the first Chern character of X. Replacing the first Chern character (resp. a curve C) by the second Chern character (resp. a surface S), the following notion is introduced.

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Definition 1.1. A smooth projective variety X over an algebraically closed field $k = \overline{k}$ is said to be ch_2-*positive* (resp. ch_2-*nef*) if

$$\mathrm{ch}_2(X) \cdot S > 0 \quad (\text{resp. } \mathrm{ch}_2(X) \cdot S \geq 0)$$

for any subsurface $S \subset X$, where $\mathrm{ch}_2(X) = \frac{1}{2}(\mathrm{c}_1^2 - 2\mathrm{c}_2)$ is the second Chern character of X.

ch_2-nef Fano manifolds are first studied by de Jong and Starr [5] in connection with the existence of rational surfaces on Fano manifolds. However, only few examples of ch_2-positive manifolds are known. For instance, the known examples of ch_2-positive smooth projective toric varieties (not necessarily Fano) are only projective spaces (see [15] and [17]) at the moment. In this paper, we restrict X to be a toric *Fano* manifold. Nobili [9] and the second author [14] proved that any ch_2-positive smooth toric Fano 4-fold is isomorphic to $\mathbb{P}^4$. The main result of this paper is to classify ch_2-positive smooth toric Fano d-folds for $5 \leq d \leq 8$. The result is similar as the known results.

Theorem 1.2. *Let X be a smooth toric Fano d-fold. If X is ch_2-positive and $d \leq 8$, then X is isomorphic to the d-dimensional projective space $\mathbb{P}^d$.*

Our classification is owed to the database of smooth reflexive polytopes given by Øbro [10] and Paffenholz [12], and the software called `polymake` [1] for computations relevant to polytopes.

This paper is organized as follows: In Section 2, we recall the formula to compute the intersection number of $\mathrm{ch}_2(X)$ and a torus-invariant subsurface S on X whose Picard number is equal to two. This formula is implemented as a script in `polymake` in Section 4. Section 3 is devoted to the calculations of the intersection numbers on so-called pseudo-symmetric toric Fano varieties $\widetilde{V}^d$ and V^d. One of them is the exceptional case we cannot apply the script to. In Section 4, we conclude the main result of this paper with the script.

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2 The intersection $\mathrm{ch}_2(X) \cdot S$ for a subsurface S of Picard number two

First, we collect some basic facts of toric geometry which we need. For details, see [3], [6], [7], [8], [11] and [13].

Let $X = X_\Sigma$ be the smooth projective toric d-fold over an algebraically closed field $k = \overline{k}$ associated to a fan Σ in $N := \mathbb{Z}^d$. For $\{v_1, \ldots, v_l\} \subset N$, $\langle v_1, \ldots, v_l \rangle$ stands for the cone in $N_\mathbb{R} := N \otimes \mathbb{R}$ generated by $v_1, \ldots, v_l$. Let $\mathrm{G}(\Sigma)$ be the set of primitive generators of one-dimensional cones in Σ. It is well-known that

$$\mathrm{ch}_2(X) = \frac{1}{2} \sum_{x \in \mathrm{G}(\Sigma)} D_x^2,$$

where D_x is the torus-invariant prime divisor corresponding to $x \in \mathrm{G}(\Sigma)$.

For a smooth projective toric variety X and a torus-invariant subsurface $S \subset X$ of Picard number one, it is well-known that the inequality $\mathrm{ch}_2(X) \cdot S > 0$ always holds. On the other hand, the intersection number of $\mathrm{ch}_2(X)$ and any torus-invariant subsurface of Picard number two can be easily calculated as follows: Let $X = X_\Sigma$ be a smooth projective toric d-fold, and $S \subset X$ a torus-invariant subsurface of Picard number two. Let $\tau \in \Sigma$ be the $(d-2)$-dimensional cone associated to S and $\tau \cap \mathrm{G}(\Sigma) = \{x_1, \ldots, x_{d-2}\}$. There exist exactly four maximal cones

$$\tau + \langle y_1, y_3\rangle,\ \tau + \langle y_2, y_3\rangle,\ \tau + \langle y_1, y_4\rangle \text{ and } \tau + \langle y_2, y_4\rangle$$

in Σ, where $\{y_1, y_2, y_3, y_4\} \subset \mathrm{G}(\Sigma)$. Let

$$y_1 + y_2 + c_3 y_3 + a_1 x_1 + \cdots + a_{d-2} x_{d-2} = 0 \text{ and}$$

$$y_3 + y_4 + c_1 y_1 + e_1 x_1 + \cdots + e_{d-2} x_{d-2} = 0$$

be the wall relations corresponding to $(d-1)$-dimensional cones $\tau + \langle y_3\rangle$ and $\tau + \langle y_1\rangle$, respectively, where $a_1, \ldots, a_{d-2}, c_1, c_3, e_1, \ldots, e_{d-2} \in \mathbb{Z}$. Then the following holds:

Proposition 2.1 ([16, Proposition 3.6])**.**

$$2\mathrm{ch}_2(X) \cdot S = -c_1\left(2 + c_3^2 + a_1^2 + \cdots + a_{d-2}^2\right)$$

$$+2\left(c_1 + c_3 + a_1 e_1 + \cdots + a_{d-2} e_{d-2}\right) - c_3\left(2 + c_1^2 + e_1^2 + \cdots + e_{d-2}^2\right).$$

This formula is implemented as a script explained in Section 4.

3 Pseudo-symmetric toric Fano manifolds

In this section, we show the non-positivity of $\mathrm{ch}_2(\widetilde{V}^d)$ and $\mathrm{ch}_2(V^d)$, where $\widetilde{V}^d$ and V^d are so-called *pseudo-symmetric* toric Fano varieties studied in [4] and [18]. This result complements our script in Section 4.

For $d = 2n \in 2\mathbb{N}$, we define the d-dimensional smooth toric Fano varieties $\widetilde{V}^d$ and V^d as follows (for the precise description of these varieties, please see [2]): Let $\{e_1, \ldots, e_{2n}\} \subset N_\mathbb{R}$ be the standard basis, and put

$$x_1 := e_1, \ldots, x_{2n} := e_{2n}, x_{2n+1} := -(e_1 + \cdots + e_{2n}),$$

$$y_1 := -e_1, \ldots, y_{2n} := -e_{2n}, y_{2n+1} := e_1 + \cdots + e_{2n}.$$

Then $\widetilde{V}^d$ is the smooth toric Fano d-fold $X_{\widetilde{\Sigma}}$ such that

$$\mathrm{G}(\widetilde{\Sigma}) = \{x_1, \ldots, x_{2n+1}, y_1, \ldots, y_{2n}\},$$

while V^d is the smooth toric Fano d-fold X_Σ such that

$$\mathrm{G}(\Sigma) = \{x_1, \ldots, x_{2n+1}, y_1, \ldots, y_{2n+1}\}.$$

$\widetilde{V}^2$ and V^2 are isomorphic to the del Pezzo surfaces of degree 7 and 6, respectively, which are not ch_2-positive. Hence we may assume $d \geq 4$.

Theorem 3.1. *$\widetilde{V}^d$ and V^d are* not *ch_2-positive for any $d = 2n \in 2\mathbb{N}$.*

Proof. For $\widetilde{V}^d$, the Picard number of the torus-invariant surface S_τ associated to the $(d-2)$-dimensional cone

$$\tau := \langle x_1, \ldots, x_{d-2}\rangle \in \widetilde{\Sigma}$$

is two, because there exist exactly four maximal cones

$$\tau + \langle x_{d-1}, x_d\rangle,\ \tau + \langle x_{d-1}, y_d\rangle,\ \tau + \langle y_{d-1}, x_d\rangle \text{ and } \tau + \langle y_{d-1}, y_d\rangle$$

which contain τ as a face. The relations

$$x_{d-1} + y_{d-1} = 0 \text{ and } x_d + y_d = 0$$

tell us that $S_\tau \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathrm{ch}_2(\widetilde{V}^d) \cdot S_\tau = 0$ by Proposition 2.1. Therefore, $\widetilde{V}^d$ is *not* ch_2-positive.

For V^d, there are no torus-invariant subsurfaces of Picard number two in V^d. So, we cannot apply Proposition 2.1. In this case, we can show the non-positivity of $\mathrm{ch}_2(V^d)$ by using the typical method of the calculation of intersection numbers: It is well-known that the maximal cones of Σ are

$$\langle x_{i_1}, \ldots, x_{i_n}, y_{j_1}, \ldots, y_{j_n}\rangle,$$

where $1 \le i_1 < \cdots < i_n \le 2n+1$, $1 \le j_1 < \cdots < j_n \le 2n+1$ and $\{i_1, \ldots, i_n\} \cap \{j_1, \ldots, j_n\} = \emptyset$. Let $S_\tau \subset V^d$ be the torus-invariant subsurface associated to the $(2n-2)$-dimensional cone

$$\tau := \langle x_1, \ldots, x_{n-1}, y_n, \ldots, y_{2n-2}\rangle.$$

There exist exactly six maximal cones

$$\tau + \langle x_{2n-1}, y_{2n}\rangle,\ \tau + \langle x_{2n-1}, y_{2n+1}\rangle,\ \tau + \langle y_{2n-1}, x_{2n}\rangle,\ \tau + \langle y_{2n-1}, x_{2n+1}\rangle,$$
$$\tau + \langle x_{2n}, y_{2n+1}\rangle \text{ and } \tau + \langle y_{2n}, x_{2n+1}\rangle$$

which contain τ. Namely, S_τ is isomorphic to the del Pezzo surface S_6 of degree 6. For $1 \le i \le 2n+1$, let D_i and E_i be the torus-invariant prime divisors corresponding to x_i and y_i, respectively. Then, we have relations

$$D_i - E_i - D_{2n+1} + E_{2n+1} = 0 \quad (1 \le i \le 2n)$$

in $\mathrm{Pic}(V^d)$. Obviously,

$$E_1^2 \cdot S_\tau = \cdots = E_{n-1}^2 \cdot S_\tau = D_n^2 \cdot S_\tau = \cdots = D_{2n-2}^2 \cdot S_\tau = 0 \text{ and}$$
$$D_{2n-1}^2 \cdot S_\tau = D_{2n}^2 \cdot S_\tau = D_{2n+1}^2 \cdot S_\tau = E_{2n-1}^2 \cdot S_\tau = E_{2n}^2 \cdot S_\tau = E_{2n+1}^2 \cdot S_\tau = -1.$$

On the other hand,

$$D_1^2 \cdot S_\tau = (E_1 + D_{2n+1} - E_{2n+1}) \cdot (E_1 + D_{2n+1} - E_{2n+1}) \cdot S_\tau$$

$$= (E_1^2 + D_{2n+1}^2 + E_{2n+1}^2 + 2E_1D_{2n+1} - 2D_{2n+1}E_{2n+1} - 2E_1E_{2n+1}) \cdot S_\tau$$
$$= 0 + (-1) + (-1) + 2 - 2 - 2 \times 0 = -2.$$

By symmetry, we have

$$D_i^2 \cdot S_\tau = -2 \text{ for } 1 \leq i \leq n-1, \text{ while } E_j^2 \cdot S_\tau = -2 \text{ for } n \leq j \leq 2n-2.$$

Therefore, since

$$2\mathrm{ch}_2(V^d) \cdot S_\tau = -6 + (2n-2) \times (-2) < 0,$$

V^d is *not* ch_2-positive.

4 Main results

In this section, we give a proof of Theorem 1.2. Our proof consists of three ingredients; the database of smooth reflexive polytopes, a script to compute the intersection $\mathrm{ch}_2(X) \cdot S$ for a subsurface S with Picard number two, and Theorem 3.1.

4.1 The database of smooth reflexive polytopes

Our classification is owed to the database of smooth reflexive lattice polytopes. Øbro [10] provided an algorithm to determine all smooth toric Fano d-folds for any $d \in \mathbb{N}$. By using his algorithm, Øbro classified all smooth toric Fano d-folds for $d \leq 8$. As for $d = 9$, the classification was done by an improved implementation of the algorithm by B. Lorentz and A. Paffenholz [12]. As a result, the numbers of the isomorphism classes of smooth toric Fano d-folds for $d \leq 9$ are given as follows.

d	1	2	3	4	5	6	7	8	9
# of toric Fano d-folds	1	5	18	124	866	7622	72256	749892	8229721

The data of smooth toric Fano varieties for dimensions 3 to 9 is given in `polymake` format on the web:

`https://polymake.org/polytopes/paffenholz/www/fano.html`

We use the files named `fano-v`k`d.tgz` ($3 \leq k \leq 6$), `fano-v7d-`ℓ`.tgz` ($0 \leq \ell \leq 7$) and `fano-v8d-`ℓ`.tgz` ($0 \leq \ell \leq 74$) on the above webpage.

4.2 Implementation

We implement Proposition 2.1 as follows.

1. Obtain a list of all primitive generators of the fan Σ of each smooth toric Fano d-fold from the files of the database.

2. Obtain a list of primitive generators consisting of each $(d-2)$-dimensional cone τ in Σ, then enumerate maximal cones containing τ.

3. If the number of maximal cones containing τ is equal to four, then take one of them as σ_1. In addition, obtain the generators of τ as $x_1, \ldots, x_{d-2}$, and the generators of σ_1 except $x_1, \ldots, x_{d-2}$ as y_1, y_3.

4. Obtain a maximal cone σ_2 such that σ_2 contains the $(d-1)$-dimensional cone $\tau + \langle y_3 \rangle$ but does not contain y_1 as a generator. Then, we get the generator of σ_2 except $\{x_1, \ldots, x_{d-2}, y_3\}$ as y_2.

5. Obtain a maximal cone σ_3 such that σ_3 contains the $(d-1)$-dimensional cone $\tau + \langle y_1 \rangle$ but does not contain y_3 as a generator. Then, we get the generator of σ_3 except $\{x_1, \ldots, x_{d-2}, y_1\}$ as y_4.

6. Compute the coefficients $a_1, \ldots, a_{d-2}, c_1, c_3, e_1, \ldots, e_{d-2}$ in the wall relations. Then, compute the intersection $\mathrm{ch}_2(X) \cdot S$ where S is the subsurface corresponding to the cone τ by substituting $a_1, \ldots, a_{d-2}, c_1, c_3, e_1, \ldots, e_{d-2}$ into the formula in Proposition 2.1.

Let us see the above implementation in each step. One may consult the website (`https://polymake.org/doku.php`) for the installation of `Polymake`. Download the files named `fano-v*d.tgz` of the database of smooth reflexive polytopes from the website as noted before, then put them on any directory. Our script is written in Perl which is an interface language of `Polymake`.

Step (1) We use the application `fan` to compute calculations on a fan. The function `unpack_tarball` in the script `tarball` restores the files `fano-v*d.tgz`. We substitute it into the array `@a`. We extract a data of a smooth reflexive polytope from `@a` and substitute it into `@Q`. The function `polarize` induces the polar dual polytope `$P` to `$Q`. The function `face_fan` converts the polytope `$P` into the data of the fan (named `$fan`). We extract the set of generators of `$fan` by the function `RAYS` as an array `$rays`.

Step (2) The function `MAXIMAL_CONES` is applied to `$rays` and returns the family of the labelled set of indices of generators in `$rays` generating a maximal cone. The function `N_MAXIMAL_CONES` returns the number of maximal cones in `$fan`. The function `CONES->[k]` returns the family of the labelled set of indices of generators in `$rays` generating a (k+1)-cone. The function `incl` is to analyze the inclusion relation of given two sets. The value `incl (A,B)` is equal to one if `A` contains `B`. Hence, the value `$link` is equal to the number of maximal cones containing a (`$d`−2)-cone `$fan->CONES->[$d-3]->row($c0)` where `$c0` is a loop counter to indicate a (`$d`−2)-cone in `$fan->CONES->[$d-3]`.

Step (3) If `$link` is equal to four at `$c0`, the corresponding subsurface S has the Picard number two. Then, we substitute their generators into an array `@X`. Here an element in `@X` denotes the vectors x_i as in Proposition 2.1. In addition, since generators in `$rays` are not necessarily primitive, we need to convert them to be primitive by the function `primitive`. By using `incl` as in

Step (2), we obtain a maximal cone `$fan->MAXIMAL_CONES->[$c1]` containing `$fan->CONES->[$d-3]->row($c0)`. Taking the difference between `$fan->MAXIMAL_CONES->[$c1]` and `$fan->CONES->[$d-3]->row($c0)`, we obtain the set `$u` of the indices of the generator rays corresponding to y_1, y_3. Then, we obtain the vectors `$Y[0], $Y[1]` corresponding to y_1, y_3 and their indices `$y1, $y3`.

Step (4) and (5) Taking the set of generators of $\tau + \langle y_3 \rangle$ by `$fan->CONES->[$d-3]->row($c0) +$y3`, we repeat a similar procedure as Step (3).

Step (6) First, we compute the coefficients $c_3, a_1, \ldots, a_{d-2}$ in the former of the two wall relations in Proposition 2.1. Substituting `$Y[0]`, `$Y[1]` and `@X` into an array `@M`, we convert `@M` into a $d \times d$-matrix `$mat`. Then, we compute the coefficients as `$coef1` by using the function `cramer (A,b)` which gives the solution of the system $A\mathbf{x} = \mathbf{b}$ by Cramer's rule. Remark that `$coef1->[0]` is always equal to one, which corresponds to the coefficient of y_1 in the former of the two wall relations. Moreover `$coef1->[1]` corresponds to c_3, and `$coef1->[k]` (2 ≤`k`≤`$d-1`) corresponds to a_{k-1} respectively. Similarly, we obtain the coefficients $c_1, e_1, \ldots, e_{d-2}$ in the latter of the two wall relations in Proposition 2.1 as `$coef2`. Substituting `$coef1` and `$coef2` into the formula in Proposition 2.1, we obtain the intersection $2\mathrm{ch}_2(X) \cdot S$ as `$intersection`.

See a practical script to determine whether X is ch_2-positive or not in the last of this section.

4.3 Results and conjectures

By using our script, we find the following results.

Proposition 4.1. *For any smooth toric Fano d-fold X of $d = 5$, 7 and $\rho(X) \geq 2$, there exists a torus-invariant surface $S \subset X$ such that $\rho(S) = 2$ and $\mathrm{ch}_2(X) \cdot S \leq 0$. In particular, X is not ch_2-positive.*

As for $d = 4$, 6, 8, there exist the exceptional cases we cannot apply our script to.

Proposition 4.2. *For any smooth toric Fano d-fold X of $d = 4$, 6, 8 and $\rho(X) \geq 2$ except for V^d, there exists a torus-invariant surface $S \subset X$ such that $\rho(S) = 2$ and $\mathrm{ch}_2(X) \cdot S \leq 0$.*

Combining this proposition with Theorem 3.1, it is proved that any smooth toric Fano d-fold X of $d = 4$, 6, 8 and $\rho(X) \geq 2$ is not ch_2-positive. With Proposition 4.1, Proposition 4.2 and the known results for $d = 1, 2, 3$, we can conclude Theorem 1.2.

The lists of our main results are available on the web:

`https://sites.google.com/a/fukuoka-u.ac.jp/satoric/toricfano_ch2`

In the lists, we explicitly describe a surface $S \subset X$ such that $\mathrm{ch}_2(X) \cdot S \leq 0$ for any smooth toric Fano d-fold X of $5 \leq d \leq 7$ except for $\mathbb{P}^5$, $\mathbb{P}^6$, V^6 and $\mathbb{P}^7$.

Thus, we end this subsection by proposing the following two conjectures:

Conjecture 4.3. *Let X be a smooth toric Fano d-fold. If X is* ch_2*-positive, then X is isomorphic to the d-dimensional projective space $\mathbb{P}^d$.*

Conjecture 4.4. *Let X be a smooth toric Fano d-fold. If X is isomorphic to neither $\mathbb{P}^d$ nor V^d, then there exists a torus invariant subsurface $S \subset X$ of Picard number two such that* $\mathrm{ch}_2(X) \cdot S \leq 0$.

Remark 4.5. Obviously, Conjecture 4.4 implies Conjecture 4.3 by Theorem 3.1.

4.4 Script

In this subsection, we build the scripts explained in Subsection 4.2 into a practical script to determine whether X is ch_2-positive or not. The following script returns a message "`not ch_2-positive`" if X admits a surface $S \subset X$ such that $\rho(S) = 2$ and $\mathrm{ch}_2(X) \cdot S \leq 0$.

```
use warnings;
use utf8;
use application "fan";
binmode STDIN, ':encoding(cp932)';
binmode STDOUT, ':encoding(cp932)';
binmode STDERR, ':encoding(cp932)';

&ch2positive("fano-v*d.tgz"); #enter the full path of the file.

sub ch2positive {
    script("tarballs");
    my @a=unpack_tarball($_[0]);
    my $d=*;    #input the dimension of manifolds
    my $c0=0;   #loop counter
    my $c1=0;   #loop counter
    my $poly_c=0;       #loop counter for polytopes

    my $link=0; #number of maximal cones
                                containing a cone of codimension two
    my @X = cols(zero_matrix($d-2, $d)); #array for the vectors x_i
    my @Y = cols(zero_matrix(4, $d));   #array for the vector y_i
    my $y1; #index of vector y_1 in $v1
    my $y2; #index of vector y_2 in $v1
    my $y3; #index of vector y_3 in $v1
    my $y4; #index of vector y_4 in $v1
    my $coef1; #coefficients in the first wall relation
    my $coef2; #coefficients in the second wall relation
    my $square1=0;
        #square sum of x_i in the first wall relation
    my $square2=0;
        #square sum of the coefficients in the second wall relation
    my $cross=0; #inner product of $coef1 and $coef2
    my $intersection=0; #intersection number of ch_2(X) and S
    my $pc0=0; #counter for the number of surfaces
                        with non-positive intersection number

    my $v0; #vector whose elements are indices of 1-cones
    my $v1; #vector whose elements are indices of 1-cones
    my $u0; #set of indices of 1-cones
    my $set; #set of indices of 1-cones
```

```
    my @M = cols(zero_matrix($d,$d)); #matrix consisting of @X and @Y

#Step (1)

    while($poly_c < $#a+1){
           print $_[0];
           print "_";
           print $poly_c;
           print "\n";
           my $Q = $a[$poly_c];
           my $P = polarize($Q);
           my $fan = face_fan($P);
           my $rays = $fan->RAYS;
           my $max_cones = $fan->MAXIMAL_CONES;
           my $N_max_cones = $fan->N_MAXIMAL_CONES;

#Step (2)

          $c0=0;
          $pc0=0;
          while ($c0 < $fan->F_VECTOR->[$d-3]){
                $c1=0;
                $link=0;
                my $ind_surface = $fan->CONES->[$d-3]->row($c0);
                while ($c1 < $N_max_cones){
                        if (incl($max_cones->[$c1], $ind_surface)==1){
                                $link++;
                        }
                        $c1++;
                }

        if ($link==4){
            $square1 =0;
            $square2 =0;
            $cross =0;

            $c1=0;
            $v0 = new Vector<Int>($ind_surface);
            while ($c1<$d-2){
                $X[$c1]= new Vector (primitive($rays->[$v0->[$c1]]));
                $c1++;
            }

#Step (3)

            $c1=0;
            while ($c1<$N_max_cones){
                if (incl($max_cones->[$c1], $ind_surface)==1){
                    $u0 = $max_cones->[$c1] - $ind_surface;
                    $v1 = new Vector<Int>($u0);
                    $Y[0] = new Vector(primitive($rays->[$v1->[0]]));
                    $Y[1] = new Vector(primitive($rays->[$v1->[1]]));
                    $y1 = $v1->[0];
                    $y3 = $v1->[1];
                    $c1=$N_max_cones;
                } else{
                        $c1++;
                }
```

```
            }

#Step (4)

            $c1=0;
            $set = $ind_surface +$y3;
            while ($c1<$N_max_cones){
                if (incl($max_cones->[$c1], $set)==1){
                    $u0 = $max_cones->[$c1] - $set;
                    $v1 = new Vector<Int>($u0);
                    if ($v1->[0]!=$y1){
                        $Y[2] = new Vector(primitive($rays->[$v1->[0]]));
                        $y2 = $v1->[0];
                        $c1=$N_max_cones;
                    } else {
                        $c1++;
                    }
                } else {
                    $c1++;
                }
            }

#Step (5)

           $c1=0;
            $set = $ind_surface +$y1;
            while ($c1<$N_max_cones){
                if (incl($max_cones->[$c1], $set)==1){
                    $u0 = $max_cones->[$c1] - $set;
                    $v1 = new Vector<Int>($u0);
                    if ($v1->[0]!=$y3){
                        $Y[3] = new Vector(primitive($rays->[$v1->[0]]));
                        $y4 = $v1->[0];
                        $c1=$N_max_cones;
                    } else {
                        $c1++;
                    }
                } else {
                    $c1++;
                }
            }

#Step(6)

            $M[0]=$Y[0];
            $M[1]=$Y[1];
            $c1=2;
            while($c1<$d){
                $M[$c1] = $X[$c1-2];
                $c1++;
            }
            my $mat= transpose(new Matrix (@M));
            $coef1=cramer($mat,(-1)*$Y[2]);

            $M[0]=$Y[1];
            $M[1]=$Y[0];
            $mat= transpose(new Matrix (@M));
            $coef2=cramer($mat,(-1)*$Y[3]);
```

```
            $c1=2;
            while ($c1<$d){
                $square1 += ($coef1->[$c1])*($coef1->[$c1]);
                $square2 += ($coef2->[$c1])*($coef2->[$c1]);
                $cross += ($coef1->[$c1])*($coef2->[$c1]);
                $c1++;
            }

            $intersection =
            - $coef2->[1]*(2+$coef1->[1]*$coef1->[1]+$square1)
            +2*($coef1->[1]+$coef2->[1]+$cross)
            -$coef1->[1]*(2+$coef2->[1]*$coef2->[1]+$square2);

            if ($intersection<=0){
                $pc0++;
                $c0 = $fan->F_VECTOR->[$d-3];
                last;
            }
        }
        $c0++;
    }
    if ($pc0!=0){
        print "not ch_2-positive";
        print "\n\n";
    } else {
        print "cannot determine via surfaces of Picard number two";
        print "\n\n";
    }
    $poly_c++;
    }
}
```

Remark 4.6. The function `F_VECTOR->[k]` returns the number of (`k`+1)-dimensional cones in a given fan.

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