Kumamoto J. Math.
Vol.**35** (2022), 1–60

Complete list of connection relations for Gauss hypergeometric differential equation

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(Received November 10, 2021)
(Accepted November 15, 2021)

Abstract. The complete list of the connection relations for the Gauss hypergeometric differential equation is given. In generic case, the connection relation is well known and can be found in many books. In this paper, the connection relations for non-generic – logarithmic and apparent – cases are obtained.

1 Introduction

The Gauss hypergeometric differential equation

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0 \tag{1.1}$$

$(a, b, c \in \mathbb{C})$ appears in various fields in mathematics and physics, and plays an important role in each field. Among numbers of formulas for the Gauss hypergeometric differential equation, the connection relations are of particular importance. The purpose of this paper is to give the complete list of the connection relations for all $(a, b, c) \in \mathbb{C}^3$ without any exception. The connection relation for generic (a, b, c) can be found in many books on special functions or differential equations (cf. [1, 2, 3]). Therefore our purpose is to give the connection relations for non-generic cases.

After the preliminary given in Section 2, we specify local solutions of (1.1) in Section 3. Using these local solutions, in Section 4 we describe the connection relations for (1.1) between fundamental systems of solutions at $x = 0$ and $x = 1$. The symmetry of (1.1) with respect to the positions of the singular points (the S_4 symmetry) makes it possible to derive the connection relations between $x = \infty$ and $x = 0$ from ones between $x = 0$ and $x = 1$. Section 5 is devoted to this derivation. In the last section, we explain how to derive the monodromy representation for (1.1) from the connection relations given in Section 4. The connection relation for the Legendre differential equation is also derived. In the Appendix, we give two tables of the determinant or the inverse matrix for the coefficient matrix of each connection relation. These will be useful to derive the inverse relations.

Mathematical Subject Classification (2020): 33C05, 34M35
Key words: connection coefficient, hypergeometric function, Legendre equation

2 Digamma function, Pochhammer symbol and notation

In this section, we introduce the digamma function, the Pochhammer symbol and notation which will be used in describing the connection relations.

The digamma function $\psi(x)$ is defined by

$$\psi(x) = (\log \Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The digamma function is holomorphic on $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, meromorphic on $\mathbb{C}$, and has the partial fractional expansion

$$\psi(x) = -\gamma - \sum_{n=0}^{\infty} \left(\frac{1}{x+n} - \frac{1}{1+n} \right), \tag{2.1}$$

where γ is Euler's constant defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

The formula (2.1) is derived from Weierstrass's formula for the gamma function. Also from the formulas

$$\Gamma(x+1) = x\Gamma(x),\ \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \tag{2.2}$$

for the gamma function, we can derive the formulas for the digamma function

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{2.3}$$

$$\psi(x) - \psi(1-x) = -\pi \cot \pi x. \tag{2.4}$$

Several special values are known:

$$\begin{aligned} \psi(1) &= -\gamma, \\ \psi(n+1) &= -\gamma + \sum_{k=1}^{n} \frac{1}{k} \quad (n \in \mathbb{Z}_{>0}), \\ \psi\left(\frac{1}{2}\right) &= -\gamma - 2\log 2. \end{aligned} \tag{2.5}$$

For any $a \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$, we define the Pochhammer symbol (a, n) by

$$(a,n) = \begin{cases} 1 & (n=0), \\ a(a+1)\cdots(a+n-1) & (n \geq 1). \end{cases} \tag{2.6}$$

If $a \notin \mathbb{Z}_{\leq 0}$, we have

$$(a,n) = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{2.7}$$

The following formula, although it can be easily derived from (2.7) by taking $n \mapsto n+1$, will be used many times in this paper:

$$(a+n)\Gamma(a) = \frac{\Gamma(a+n+1)}{(a,n)} \quad (a \notin \mathbb{Z}_{\leq 0}, n \in \mathbb{Z}_{>0}). \tag{2.8}$$

Note that we have

$$(-n,n) = (-1)^n n!, \ (-n, n+k) = 0 \quad (n,k \in \mathbb{Z}_{>0}). \tag{2.9}$$

The Laurent series expansion of the gamma function at $x = -n \in \mathbb{Z}_{\leq 0}$ can be represented as

$$\Gamma(x) = \frac{1}{(-n,n)} \cdot \frac{1}{x+n} + O(1). \tag{2.10}$$

As notation, in this paper we understand

$$\begin{aligned} \{k, k+1, \dots, l\} &= \emptyset \quad (l < k), \\ \{k, k-1, \dots, l\} &= \emptyset \quad (l > k), \\ \sum_{j=k}^{l} a_j &= 0 \quad (l < k). \end{aligned}$$

3 Hypergeometric series and local solutions

Basically the local solutions of the hypergeometric differential equation (1.1) are expressed by the hypergeometric series $F(a,b,c;x)$ which is defined by

$$F(a,b,c;x) = \sum_{k=0}^{\infty} \frac{(a,k)(b,k)}{(c,k)k!} x^k. \tag{3.1}$$

If $a,b,c \notin \mathbb{Z}_{\leq 0}$, the hypergeometric series $F(a,b,c;x)$ is a power series of radius of convergence 1. If $a \in \mathbb{Z}_{\leq 0}$ (resp. $b \in \mathbb{Z}_{\leq 0}$) and $c \notin \{0, -1, \dots, a+1\}$ (resp. $c \notin \{0, -1, \dots, b+1\}$), the hypergeometric series $F(a,b,c;x)$ is a polynomial of degree $|a|$ (resp. $|b|$). If $a \in \mathbb{Z}_{\leq 0}$ (resp. $b \in \mathbb{Z}_{\leq 0}$) and $c = a$ (resp. $b = c$), we have

$$F(a,b,c;x) = \sum_{k=0}^{\infty} \frac{(b,k)}{k!} x^k = (1-x)^{-b}$$

(resp. the result obtained by $b \mapsto a$). For the other cases, the hypergeometric series $F(a,b,c;x)$ is not defined.

The table of the local exponents of the hypergeometric differential equation (1.1) is given by

$$\begin{Bmatrix} x=0 & x=1 & x=\infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{Bmatrix},$$

which we call the Riemann scheme of (1.1). According to this Riemann scheme, we specify local solutions at the singular points $x = 0, 1, \infty$.

First we specify the local solutions at each singular point when the difference of the local exponents is not an integer. We call these cases the *generic cases*. At $x = 0$, if $c \notin \mathbb{Z}$, there exists a fundamental system of solutions $(y_1(x), y_2(x))$ defined by

$$y_1(x) = F(a, b, c; x), \tag{3.2}$$

$$y_2(x) = x^{1-c} F(a - c + 1, b - c + 1, 2 - c; x). \tag{3.3}$$

At $x = 1$, if $c - a - b \notin \mathbb{Z}$, there exists a fundamental system of solutions $(y_3(x), y_4(x))$ defined by

$$y_3(x) = F(a, b, a + b - c + 1; 1 - x), \tag{3.4}$$

$$y_4(x) = (1 - x)^{c-a-b} F(c - a, c - b, c - a - b + 1; 1 - x). \tag{3.5}$$

At $x = \infty$, if $a-b \notin \mathbb{Z}$, there exists a fundamental system of solutions $(y_5(x), y_6(x))$ defined by

$$y_5(x) = x^{-a} F(a, a - c + 1, a - b + 1; x^{-1}), \tag{3.6}$$

$$y_6(x) = x^{-b} F(b - c + 1, b, b - a + 1; x^{-1}). \tag{3.7}$$

Next we give the local solutions at each singular point when the difference of the local exponents is an integer.

We assume $c \in \mathbb{Z}$. The difference of the local exponents at $x = 0$ becomes an integer. If $c = 1$, both $y_1(x)$ and $y_2(x)$ are defined, however, coincide. In this case, in order to get another solution independent of $y_1(x)$, we define

$$\hat{y}_2(x) = \lim_{c \to 1} \frac{1}{1 - c} (y_2(x) - y_1(x)). \tag{3.8}$$

Since a linear combination of solutions is also a solution and c is independent of x, $\hat{y}_2(x)$ becomes a solution if the limit converges. It actually converges. We regard $y_1(x), y_2(x)$ as functions in (x, c), and denote them by $y_1(x, c), y_2(x, c)$. Since $y_1(x, 1) = y_2(x, 1)$ holds, we have

$$\lim_{c \to 1} \frac{1}{1 - c} (y_2(x, c) - y_1(x, c)) = -\frac{\partial y_2}{\partial c}(x, 1) + \frac{\partial y_1}{\partial c}(x, 1).$$

Thus we obtain

$$\hat{y}_2(x) = \log x \sum_{k=0}^{\infty} \frac{(a, k)(b, k)}{(k!)^2} x^k + \sum_{k=1}^{\infty} \frac{(a, k)(b, k)}{(k!)^2} \sum_{j=1}^{k} \left(\frac{1}{a + j - 1} + \frac{1}{b + j - 1} - \frac{2}{j} \right) x^k. \tag{3.9}$$

If $c \geq 2$, we have $\mathrm{Re}(1 - c) < \mathrm{Re}(0)$, and then $y_1(x)$ remains a solution. If $c \leq 0$, we have $\mathrm{Re}(1 - c) > \mathrm{Re}(0)$, and then $y_2(x)$ remains a solution. If

$$c = m \in \mathbb{Z}_{\geq 2}, \;\; a - c, b - c \notin \{-1, -2, \ldots, -m + 1\},$$

as explained above, $y_2(x)$ is not defined. We extract the divergent factor in $y_2(x)$ by rewriting as

$$\begin{aligned} y_2(x) = x^{1-c}\sum_{k=0}^{m-2}\frac{(a-c+1,k)(b-c+1,k)}{(2-c,k)k!}x^k \\ + \frac{1}{m-c}\cdot\frac{(a-c+1,m-1)(b-c+1,m-1)}{(2-c,m-2)(m-1)!} \\ \times x^{m-c}\sum_{k=0}^{\infty}\frac{(a-c+m,k)(b-c+m,k)}{(m-c+1,k)(m,k)}x^k. \end{aligned}$$

Then we define another solution $\hat{y}_2(x)$ by

$$\hat{y}_2(x) = \lim_{c\to m}\left(\frac{(2-c,m-2)(m-1)!}{(a-c+1,m-1)(b-c+1,m-1)}y_2(x) - \frac{1}{m-c}y_1(x)\right). \quad (3.10)$$

We shall show the convergence of the limit. We have

$$\begin{aligned} &\frac{(2-c,m-2)(m-1)!}{(a-c+1,m-1)(b-c+1,m-1)}y_2(x) - \frac{1}{m-c}y_1(x) \\ =&\frac{(2-c,m-2)(m-1)!}{(a-c+1,m-1)(b-c+1,m-1)}x^{1-c}\sum_{k=0}^{m-2}\frac{(a-c+1,k)(b-c+1,k)}{(2-c,k)k!}x^k \\ &+ \frac{1}{m-c}\left(x^{m-c}\sum_{k=0}^{\infty}\frac{(a-c+m,k)(b-c+m,k)}{(m-c+1,k)(m,k)}x^k - \sum_{k=0}^{\infty}\frac{(a,k)(b,k)}{(c,k)k!}x^k\right). \end{aligned}$$

In the right hand side, the limit of the first term is obtained only by putting m into c. We set

$$\begin{aligned} f(c) &= x^{m-c}\sum_{k=0}^{\infty}\frac{(a-c+m,k)(b-c+m,k)}{(m-c+1,k)(m,k)}x^k, \\ g(c) &= \sum_{k=0}^{\infty}\frac{(a,k)(b,k)}{(c,k)k!}x^k. \end{aligned}$$

Then we see $f(m) = g(m)$, and hence the limit of the second term becomes $-f'(m) + g'(m)$. Therefore we get

$$\begin{aligned} \hat{y}_2(x) = \log x\sum_{k=0}^{\infty}\frac{(a,k)(b,k)}{(m,k)k!}x^k \\ + x^{1-m}\frac{(2-m,m-2)(m-1)!}{(a-m+1,m-1)(b-m+1,m-1)} \\ \times\sum_{k=0}^{m-2}\frac{(a-m+1,k)(b-m+1,k)}{(2-m,k)k!}x^k \\ + \sum_{k=1}^{\infty}\frac{(a,k)(b,k)}{(m,k)k!}\sum_{j=1}^{k}\left(\frac{1}{a+j-1} + \frac{1}{b+j-1} - \frac{1}{j} - \frac{1}{m+j-1}\right)x^k. \end{aligned} \quad (3.11)$$

If

$$c=-m\ (m\in\mathbb{Z}_{\geq 0}),\ a,b\notin\{0,-1,\dots,-m\},$$

$y_1(x)$ is not defined. In this case, we define another solution $\hat{y}_1(x)$ by

$$\hat{y}_1(x)=\lim_{c\to -m}\left(\frac{(c,m)(m+1)!}{(a,m+1)(b,m+1)}y_1(x)-\frac{1}{c+m}y_2(x)\right). \tag{3.12}$$

In a similar way as $\hat{y}_2(x)$, we get

$$\begin{aligned}\hat{y}_1(x)&=x^{1+m}\log x\sum_{k=0}^{\infty}\frac{(a+m+1,k)(b+m+1,k)}{(2+m,k)k!}x^k\\&\quad+\frac{(-m,m)(m+1)!}{(a,m+1)(b,m+1)}\sum_{k=0}^{m}\frac{(a,k)(b,k)}{(-m,k)k!}x^k\\&\quad+x^{1+m}\sum_{k=1}^{\infty}\frac{(a+m+1,k)(b+m+1,k)}{(2+m,k)k!}\\&\qquad\times\sum_{j=1}^{k}\left(\frac{1}{a+m+j}+\frac{1}{b+m+j}-\frac{1}{j}-\frac{1}{m+1+j}\right)x^k.\end{aligned} \tag{3.13}$$

We call these cases the *logarithmic cases*. On the other hand, if

$$c=m\in\mathbb{Z}_{>1},\ a-c \text{ or } b-c\in\{-1,-2,\dots,-m+1\},$$

$y_2(x)$ is defined and remains a solution which is independent of $y_1(x)$. Or if

$$c=-m\ (m\in\mathbb{Z}_{\geq 0}),\ a \text{ or } b\in\{0,-1,\dots,-m\},$$

$y_1(x)$ is defined and remains a solution which is independent of $y_2(x)$. We call these two cases the *apparent cases*.

Similarly, if $c-a-b\in\mathbb{Z}$, we have the logarithmic cases and apparent cases at $x=1$. If $c-a-b=0$, $y_3(x)$ and $y_4(x)$ coincide. In order to get another independent solution, we define

$$\begin{aligned}\hat{y}_4(x)&=\lim_{c\to a+b}\frac{1}{c-a-b}(y_4(x)-y_3(x))\\&=\log(1-x)\sum_{k=0}^{\infty}\frac{(a,k)(b,k)}{(k!)^2}(1-x)^k\\&\quad+\sum_{k=1}^{\infty}\frac{(a,k)(b,k)}{(k!)^2}\sum_{j=1}^{k}\left(\frac{1}{a+j-1}+\frac{1}{b+j-1}-\frac{2}{j}\right)(1-x)^k.\end{aligned} \tag{3.14}$$

If

$$c-a-b=-n\ (n\in\mathbb{Z}_{\geq 1}),\ c-a,c-b\notin\{0,-1,\dots,-n+1\},$$

$y_3(x)$ remains a solution, and we get another solution $\hat{y}_4(x)$ defined by

$$\begin{aligned}\hat{y}_4(x) &= \lim_{c\to a+b-n}\left(\frac{(c-a-b+1,n-1)n!}{(c-a,n)(c-b,n)}y_4(x)-\frac{1}{c-a-b+n}y_3(x)\right)\\ &= \log(1-x)\sum_{k=0}^{\infty}\frac{(a,k)(b,k)}{(n+1,k)k!}(1-x)^k\\ &\quad+(1-x)^{-n}\frac{(1-n,n-1)n!}{(a-n,n)(b-n,n)}\sum_{k=0}^{n-1}\frac{(a-n,k)(b-n,k)}{(1-n,k)k!}(1-x)^k\\ &\quad+\sum_{k=1}^{\infty}\frac{(a,k)(b,k)}{(n+1,k)k!}\sum_{j=1}^{k}\left(\frac{1}{a+j-1}+\frac{1}{b+j-1}-\frac{1}{j}-\frac{1}{n+j}\right)(1-x)^k.\end{aligned} \tag{3.15}$$

If

$$c-a-b=n\in\mathbb{Z}_{\geq 1},\ a,b\notin\{0,-1,\ldots,-n+1\},$$

$y_4(x)$ remains a solution, and we get another solution $\hat{y}_3(x)$ defined by

$$\begin{aligned}\hat{y}_3(x) &= \lim_{c\to a+b+n}\left(\frac{(a+b-c+1,n-1)n!}{(a,n)(b,n)}y_3(x)-\frac{1}{a+b-c+n}y_4(x)\right)\\ &= -(1-x)^n\log(1-x)\sum_{k=0}^{\infty}\frac{(a+n,k)(b+n,k)}{(n+1,k)k!}(1-x)^k\\ &\quad+\frac{(1-n,n-1)n!}{(a,n)(b,n)}\sum_{k=0}^{n-1}\frac{(a,k)(b,k)}{(1-n,k)k!}(1-x)^k\\ &\quad+(1-x)^n\sum_{k=1}^{\infty}\frac{(a+n,k)(b+n,k)}{(n+1,k)k!}\\ &\qquad\times\sum_{j=1}^{k}\left(\frac{1}{j}+\frac{1}{n+j}-\frac{1}{a+n+j-1}-\frac{1}{b+n+j-1}\right)(1-x)^k.\end{aligned} \tag{3.16}$$

These are the logarithmic cases. The apparent cases occur if

$$c-a-b=-n\ (n\in\mathbb{Z}_{>0}),\ c-a \text{ or } c-b\in\{0,-1,\ldots,-n+1\},$$

or if

$$c-a-b=n\in\mathbb{Z}_{\geq 1},\ a \text{ or } b\in\{0,-1,\ldots,-n+1\},$$

and in these cases $(y_3(x),y_4(x))$ is a fundamental system of solutions at $x=1$.

Also at $x=\infty$, we have the logarithmic cases and the apparent cases if $a-b\in\mathbb{Z}$.

If $a-b=0$, $y_5(x)$ and $y_6(x)$ coincide, and then we define

$$
\begin{aligned}
\hat{y}_6(x) &= \lim_{b\to a} \frac{1}{a-b}(y_6(x)-y_5(x)) \\
&= x^{-a}\log x \sum_{k=0}^{\infty} \frac{(a,k)(a-c+1,k)}{(k!)^2} x^{-k} \\
&\quad + x^{-a}\sum_{k=1}^{\infty} \frac{(a,k)(a-c+1,k)}{(k!)^2} \sum_{j=1}^{k}\left(\frac{1}{a+j-1}+\frac{1}{a-c+j}-\frac{2}{j}\right) x^{-k}.
\end{aligned}
\tag{3.17}
$$

If

$$
a-b=m\in\mathbb{Z}_{\geq 1},\ b, b-c+1 \notin \{0,-1,\ldots,-m+1\},
$$

we define

$$
\begin{aligned}
\hat{y}_6(x) &= \lim_{b\to a-m}\left(\frac{(b-a+1,m-1)m!}{(b,m)(b-c+1,m)}y_6(x)-\frac{1}{b-a+m}y_5(x)\right) \\
&= x^{-a}\log x\sum_{k=0}^{\infty}\frac{(a,k)(a-c+1,k)}{(m+1,k)k!}x^{-k} \\
&\quad + \frac{(1-m,m-1)m!}{(a-m,m)(a-c-m+1,m)}x^{m-a}\sum_{k=0}^{m-1}\frac{(a-m,k)(a-c-m+1,k)}{(1-m,k)k!}x^{-k} \\
&\quad + x^{-a}\sum_{k=1}^{\infty}\frac{(a,k)(a-c+1,k)}{(m+1,k)k!}\sum_{j=1}^{k}\left(\frac{1}{a+j-1}+\frac{1}{a-c+j}-\frac{1}{j}-\frac{1}{m+j}\right)x^{-k}.
\end{aligned}
\tag{3.18}
$$

If

$$
b-a=m\in\mathbb{Z}_{\geq 1},\ a, a-c+1 \notin \{0,-1,\ldots,-m+1\},
$$

we define

$$
\begin{aligned}
\hat{y}_5(x) &= \lim_{a\to b-m}\left(\frac{(a-b+1,m-1)m!}{(a,m)(a-c+1,m)}y_5(x)-\frac{1}{a-b+m}y_6(x)\right) \\
&= x^{-b}\log x\sum_{k=0}^{\infty}\frac{(b,k)(b-c+1,k)}{(m+1,k)k!}x^{-k} \\
&\quad + \frac{(1-m,m-1)m!}{(b-m,m)(b-c-m+1,m)}x^{m-b}\sum_{k=0}^{m-1}\frac{(b-m,k)(b-c-m+1,k)}{(1-m,k)k!}x^{-k} \\
&\quad + x^{-b}\sum_{k=1}^{\infty}\frac{(b,k)(b-c+1,k)}{(m+1,k)k!}\sum_{j=1}^{k}\left(\frac{1}{b+j-1}+\frac{1}{b-c+j}-\frac{1}{j}-\frac{1}{m+j}\right)x^{-k}.
\end{aligned}
\tag{3.19}
$$

In Section 5 we study the connection relations between $x=0$ and $x=\infty$. There we will use another set of slightly different solutions, and $y_5(x), y_6(x), \hat{y}_5(x), \hat{y}_6(x)$ will not be used in the following of this paper.

4 Connection relations between $x = 0$ and $x = 1$

Connection relations for the hypergeometric differential equation (1.1) mean linear relations between two fundamental systems of local solutions at two singular points. As will be explained in the next section, the hypergeometric differential equation has a highly symmetric nature, and then we can obtain all connection relations from connection relations for any pair of singular points. Therefore, in this section, we restrict ourselves to show the connection relations between $x = 0$ and $x = 1$.

We consider the solutions $y_j(x)$ and $\hat{y}_j(x)$ $(1 \leq j \leq 4)$ defined in the previous section. By the analytic continuation, we may regard these as functions defined on the complex plane with cuts

$$D_{01} = \mathbb{C} \setminus ((-\infty, 0] \cup [1, +\infty)),$$

where $(-\infty, 0]$ and $[1, \infty)$ denote intervals on $\mathbb{R}$. We fix the branches of $y_j(x)$ and $\hat{y}_j(x)$ by assigning

$$\arg x = \arg(1 - x) = 0$$

on the interval $(0, 1)$.

Theorem 4.1. *Let $y_1(x), y_2(x), y_3(x), y_4(x)$ be defined by (3.2), (3.3), (3.4), (3.5), respectively, $\hat{y}_2(x)$ by (3.9) and (3.11), $\hat{y}_1(x)$ by (3.13), $\hat{y}_4(x)$ by (3.14) and (3.15), and $\hat{y}_3(x)$ by (3.16). The following relations hold on the domain D_{01}.*
(i) (generic:generic) *If*

$$c \notin \mathbb{Z}, \ \ c - a - b \notin \mathbb{Z},$$

we have

$$\begin{aligned} y_1(x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} y_3(x) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} y_4(x), \\ y_2(x) &= \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} y_3(x) + \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} y_4(x). \end{aligned} \tag{4.1}$$

(ii) (logarithmic:generic)
(ii-1) *If*

$$c = m \in \mathbb{Z}_{\geq 1}, \ \ a - c, b - c \notin \{-1, -2, \ldots, -m+1\}, \ \ c - a - b \notin \mathbb{Z},$$

we have

$$\begin{aligned} y_1(x) &= \frac{\Gamma(m)\Gamma(m-a-b)}{\Gamma(m-a)\Gamma(m-b)} y_3(x) + \frac{\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)} y_4(x), \\ \hat{y}_2(x) &= \frac{\Gamma(m)\Gamma(m-a-b)}{\Gamma(m-a)\Gamma(m-b)} (\psi(1) + \psi(m) - \psi(1-a) - \psi(1-b)) y_3(x) \\ &\quad + \frac{\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)} (\psi(1) + \psi(m) - \psi(a) - \psi(b)) y_4(x). \end{aligned} \tag{4.2}$$

(ii-2) *If*

$$c=-m\ (m\in\mathbb{Z}_{\geq 0}),\ a,b\notin\{0,-1,\dots,-m\},\ c-a-b\notin\mathbb{Z},$$

we have

$$\begin{aligned}\hat{y}_1(x)=&\frac{\Gamma(2+m)\Gamma(-m-a-b)}{\Gamma(1-a)\Gamma(1-b)}(\psi(1)+\psi(m+2)-\psi(-m-a)-\psi(-m-b))y_3(x)\\&+\frac{\Gamma(2+m)\Gamma(a+b+m)}{\Gamma(a+m+1)\Gamma(b+m+1)}\\&\quad\times(\psi(1)+\psi(m+2)-\psi(a+m+1)-\psi(b+m+1))y_4(x),\\ y_2(x)=&\frac{\Gamma(2+m)\Gamma(-m-a-b)}{\Gamma(1-a)\Gamma(1-b)}y_3(x)+\frac{\Gamma(2+m)\Gamma(a+b+m)}{\Gamma(a+m+1)\Gamma(b+m+1)}y_4(x).\end{aligned}\tag{4.3}$$

(iii) (apparent:generic)
(iii-1) *If*

$$c=m\in\mathbb{Z}_{\geq 2},\ a-c=-l\ (l\in\{1,2,\dots,m-1\}),\ c-a-b\notin\mathbb{Z},$$

we have

$$\begin{aligned}y_1(x)&=\frac{(l,m-l)}{(l-b,m-l)}y_3(x)+\frac{(m-l,l)}{(b-l,l)}y_4(x),\\ y_2(x)&=\begin{cases}\dfrac{(b-l+1,l-1)}{(m-l,l-1)}y_3(x) & (l<m-1),\\ y_4(x) & (l=m-1).\end{cases}\end{aligned}\tag{4.4}$$

(iii-2) *If*

$$c=-m\ (m\in\mathbb{Z}_{\geq 0}),\ a=-l\ (l\in\{0,1,\dots,m\}),\ c-a-b\notin\mathbb{Z},$$

we have

$$\begin{aligned}y_1(x)&=\begin{cases}\dfrac{(m+b-l+1,l)}{(m-l+1,l)}y_3(x) & (l<m),\\ y_4(x) & (l=m),\end{cases}\\ y_2(x)&=\frac{(l+1,m-l+1)}{(l-m-b,m-l+1)}y_3(x)+\frac{(m-l+1,l+1)}{(m-l+b,l+1)}y_4(x).\end{aligned}\tag{4.5}$$

(iv) (generic:logarithmic)
(iv-1) *If*

$$c\notin\mathbb{Z},\ c-a-b=-n\ (n\in\mathbb{Z}_{\geq 0}),\ c-a,c-b\notin\{0,-1,\dots,-n+1\},$$

we have

$$\begin{aligned} y_1(x) =& \frac{\Gamma(2+n-a-b)}{(-n,n)\Gamma(1-a)\Gamma(1-b)}(\psi(1)+\psi(n+1)-\psi(1-a)-\psi(1-b))y_3(x) \\ &- \frac{\Gamma(2+n-a-b)}{(-n,n)\Gamma(1-a)\Gamma(1-b)}\hat{y}_4(x), \\ y_2(x) =& \frac{\Gamma(a+b-n)}{(-n,n)\Gamma(a-n)\Gamma(b-n)}(\psi(1)+\psi(n+1)-\psi(a)-\psi(b))y_3(x) \\ &- \frac{\Gamma(a+b-n)}{(-n,n)\Gamma(a-n)\Gamma(b-n)}\hat{y}_4(x). \end{aligned} \tag{4.6}$$

(iv-2) *If*

$$c \notin \mathbb{Z},\ c-a-b=n\in\mathbb{Z}_{\geq 1},\ a,b\notin\{0,-1,\ldots,-n+1\},$$

we have

$$\begin{aligned} y_1(x) =& -\frac{\Gamma(2-a-b-n)}{(-n,n)\Gamma(1-a-n)\Gamma(1-b-n)}\hat{y}_3(x) \\ &+ \frac{\Gamma(2-a-b-n)}{(-n,n)\Gamma(1-a-n)\Gamma(1-b-n)} \\ &\quad \times (\psi(1)+\psi(n+1)-\psi(1-a-n)-\psi(1-b-n))y_4(x), \\ y_2(x) =& -\frac{\Gamma(a+b+n)}{(-n,n)\Gamma(a)\Gamma(b)}\hat{y}_3(x) \\ &+ \frac{\Gamma(a+b+n)}{(-n,n)\Gamma(a)\Gamma(b)}(\psi(1)+\psi(n+1)-\psi(a+n)-\psi(b+n))y_4(x). \end{aligned} \tag{4.7}$$

(v) (logarithmic:logarithmic)
(v-1) *If*

$$\begin{aligned} &c=m\in\mathbb{Z}_{\geq 1},\ a-c,b-c\notin\{-1,-2,\ldots,-m+1\}, \\ &c-a-b=-n\ (n\in\mathbb{Z}_{\geq 0}),\ c-a,c-b\notin\{0,-1,\ldots,-n+1\}, \end{aligned}$$

we have

$$\begin{aligned} y_1(x) =& \frac{\Gamma(m)}{(-n,n)\Gamma(m-a)\Gamma(a-n)}(\psi(1)+\psi(n+1)-\psi(a)-\psi(m+n-a))y_3(x) \\ &- \frac{\Gamma(m)}{(-n,n)\Gamma(m-a)\Gamma(a-n)}\hat{y}_4(x), \\ \hat{y}_2(x) =& \frac{\Gamma(m)}{(-n,n)\Gamma(m-a)\Gamma(a-n)}\Big((\psi(1)+\psi(m)-\psi(a)-\psi(m+n-a)) \\ &\quad \times (\psi(1)+\psi(n+1)-\psi(a)-\psi(m+n-a)) - \frac{\pi^2}{\sin^2\pi a}\Big)y_3(x) \\ &- \frac{\Gamma(m)}{(-n,n)\Gamma(m-a)\Gamma(a-n)}(\psi(1)+\psi(m)-\psi(a)-\psi(m+n-a))\hat{y}_4(x). \end{aligned} \tag{4.8}$$

(v-2) *If*

$$c = m \in \mathbb{Z}_{\geq 1},\ a-c, b-c \notin \{-1,-2,\ldots,-m+1\},$$
$$c-a-b = n \in \mathbb{Z}_{\geq 1},\ a, b \notin \{0,-1,\ldots,-n+1\},$$

we have

$$\begin{aligned}
y_1(x) &= -\frac{\Gamma(m)}{(-n,n)\Gamma(a)\Gamma(m-n-a)}\hat{y}_3(x) \\
&\quad + \frac{\Gamma(m)}{(-n,n)\Gamma(a)\Gamma(m-n-a)}(\psi(1)+\psi(n+1)-\psi(m-a)-\psi(a+n))y_4(x), \\
\hat{y}_2(x) &= -\frac{\Gamma(m)}{(-n,n)\Gamma(a)\Gamma(m-n-a)}(\psi(1)+\psi(m)-\psi(a)-\psi(m-n-a))\hat{y}_3(x) \\
&\quad + \frac{\Gamma(m)}{(-n,n)\Gamma(a)\Gamma(m-n-a)}\Big((\psi(1)+\psi(m)-\psi(a)-\psi(m-n-a)) \\
&\quad \times (\psi(1)+\psi(n+1)-\psi(m-a)-\psi(a+n)) - \frac{\pi^2}{\sin^2 \pi a}\Big)y_4(x).
\end{aligned} \tag{4.9}$$

(v-3) *If*

$$c = -m\ (m \in \mathbb{Z}_{\geq 0}),\ a, b \notin \{0,-1,\ldots,-m\},$$
$$c-a-b = -n\ (n \in \mathbb{Z}_{\geq 0}),\ c-a, c-b \notin \{0,-1,\ldots,-n+1\},$$

we have

$$\begin{aligned}
\hat{y}_1(x) &= \frac{\Gamma(m+2)}{(-n,n)\Gamma(1-a)\Gamma(a+m-n+1)} \\
&\quad \times \Big((\psi(1)+\psi(m+2)-\psi(a+m+1)-\psi(1-a+n)) \\
&\qquad \times (\psi(1)+\psi(n+1)-\psi(1-a)-\psi(a+m-n+1)) - \frac{\pi^2}{\sin^2 \pi a}\Big)y_3(x) \\
&\quad - \frac{\Gamma(m+2)}{(-n,n)\Gamma(1-a)\Gamma(a+m-n+1)} \\
&\quad \times (\psi(1)+\psi(m+2)-\psi(a+m+1)-\psi(1-a+n))\hat{y}_4(x), \\
y_2(x) &= \frac{\Gamma(m+2)}{(-n,n)\Gamma(1-a)\Gamma(a+m-n+1)} \\
&\quad \times (\psi(1)+\psi(n+1)-\psi(1-a)-\psi(a+m-n+1))y_3(x) \\
&\quad - \frac{\Gamma(m+2)}{(-n,n)\Gamma(1-a)\Gamma(a+m-n+1)}\hat{y}_4(x).
\end{aligned} \tag{4.10}$$

(v-4) *If*

$$c = -m\ (m \in \mathbb{Z}_{\geq 0}),\ a, b \notin \{0,-1,\ldots,-m\},$$
$$c-a-b = n \in \mathbb{Z}_{\geq 1},\ a, b \notin \{0,-1,\ldots,-n+1\},$$

we have

$$\begin{aligned}
\hat{y}_1(x) = &-\frac{\Gamma(m+2)}{(-n,n)\Gamma(a+m+1)\Gamma(1-a-n)} \\
&\times(\psi(1)+\psi(m+2)-\psi(a+m+1)-\psi(1-a-n))\hat{y}_3(x) \\
&+\frac{\Gamma(m+2)}{(-n,n)\Gamma(a+m+1)\Gamma(1-a-n)} \\
&\times\Big((\psi(1)+\psi(m+2)-\psi(a+m+1)-\psi(1-a-n)) \\
&\quad\times(\psi(1)+\psi(n+1)-\psi(a+m+1)-\psi(1-a-n))-\frac{\pi^2}{\sin^2\pi a}\Big)y_4(x), \\
y_2(x) = &-\frac{\Gamma(m+2)}{(-n,n)\Gamma(a+m+1)\Gamma(1-a-n)}\hat{y}_3(x) \\
&+\frac{\Gamma(m+2)}{(-n,n)\Gamma(a+m+1)\Gamma(1-a-n)} \\
&\times(\psi(1)+\psi(n+1)-\psi(a+m+1)-\psi(1-a-n))y_4(x).
\end{aligned} \tag{4.11}$$

(vi) (apparent:logarithmic)
(vi-1) *If*

$$\begin{aligned}
&c=m\in\mathbb{Z}_{\geq 2},\ a-c=-l\ (l\in\{1,2,\dots,m-1\}), \\
&c-a-b=-n\ (n\in\mathbb{Z}_{\geq 0}),\ c-a,c-b\notin\{0,-1,\dots,-n+1\},
\end{aligned}$$

we have

$$\begin{aligned}
y_1(x) &= \frac{(m-l-n,l+n)}{(-n,n)(l-1)!}(\psi(1)+\psi(n+1)-\psi(m-l)-\psi(n+l))y_3(x) \\
&\quad-\frac{(m-l-n,l+n)}{(-n,n)(l-1)!}\hat{y}_4(x), \\
y_2(x) &= \frac{(n+1,l-1)}{(m-l,l-1)}y_3(x).
\end{aligned} \tag{4.12}$$

(vi-2) *If*

$$\begin{aligned}
&c=m\in\mathbb{Z}_{\geq 2},\ a-c=-l\ (l\in\{1,2,\dots,m-1\}), \\
&c-a-b=n\in\mathbb{Z}_{\geq 1},\ a,b\notin\{0,-1,\dots,-n+1\},
\end{aligned}$$

we have

$$\begin{aligned}
y_1(x) &= -\frac{(m-l,l)}{(-n,n)(l-n-1)!}\hat{y}_3(x) \\
&\quad+\frac{(m-l,l)}{(-n,n)(l-n-1)!}(\psi(1)+\psi(n+1)-\psi(l)-\psi(m-l+n))y_4(x), \\
y_2(x) &= \frac{(n+1,l-n-1)}{(m-l+n,l-n-1)}y_4(x).
\end{aligned} \tag{4.13}$$

(vi-3) *If*

$$c=-m\ (m\in\mathbb{Z}_{\geq 0}),\ a=-l\ (l\in\{0,1,\ldots,m\}),$$
$$c-a-b=-n\ (n\in\mathbb{Z}_{\geq 0}),\ c-a,c-b\notin\{0,-1,\ldots,-n+1\},$$

we have

$$\begin{aligned} y_1(x)&=\frac{(n+1,l)}{(m-l+1,l)}y_3(x),\\ y_2(x)&=\frac{(m-l-n+1,l+n+1)}{(-n,n)l!}\\ &\quad\times(\psi(1)+\psi(n+1)-\psi(l+1)-\psi(m-l-n+1))y_3(x)\\ &\quad-\frac{(m-l-n+1,l+n+1)}{(-n,n)l!}\hat{y}_4(x). \end{aligned} \tag{4.14}$$

(vi-4) *If*

$$c=-m\ (m\in\mathbb{Z}_{\geq 0}),\ a=-l\ (l\in\{0,1,\ldots,m\}),$$
$$c-a-b=n\in\mathbb{Z}_{\geq 1},\ a,b\notin\{0,-1,\ldots,-n+1\},$$

we have

$$\begin{aligned} y_1(x)&=\frac{(n+1,l-n)}{(m-l+n+1,l-n)}y_4(x),\\ y_2(x)&=-\frac{(m-l+1,l+1)}{(-n,n)(l-n)!}\hat{y}_3(x)\\ &\quad+\frac{(m-l+1,l+1)}{(-n,n)(l-n)!}\\ &\quad\times(\psi(1)+\psi(n+1)-\psi(m-l+1)-\psi(l-n+1))y_4(x). \end{aligned} \tag{4.15}$$

(vii) (generic:apparent)
(vii-1) *If*

$$c\notin\mathbb{Z},\ c-a-b=-n\ (n\in\mathbb{Z}_{\geq 1}),\ c-a=-l\ (l\in\{0,1,\ldots,n-1\}),$$

we have

$$\begin{aligned} y_1(x)&=\begin{cases}\dfrac{(n-l,l)}{(c,l)}y_4(x) & (l<n-1),\\ \dfrac{1-c}{n}y_3(x)+\dfrac{(1,n-1)}{(c,n-1)}y_4(x) & (l=n-1),\end{cases}\\ y_2(x)&=\begin{cases}\dfrac{(c-1,l+1)}{(n-l,l+1)}y_3(x)+\dfrac{(l+1,n-l-1)}{(2-c,n-l-1)}y_4(x) & (l<n-1),\\ y_4(x) & (l=n-1).\end{cases} \end{aligned} \tag{4.16}$$

(vii-2) *If*

$$c \notin \mathbb{Z},\ c-a-b=n \in \mathbb{Z}_{\geq 1},\ a=-l\ (l \in \{0,1,\ldots,n-1\}),$$

we have

$$\begin{aligned} y_1(x) &= \begin{cases} \dfrac{(n-l,l)}{(c,l)} y_3(x) & (l<n-1), \\ \dfrac{(1,n-1)}{(c,n-1)} y_3(x) + \dfrac{1-c}{n} y_4(x) & (l=n-1), \end{cases} \\ y_2(x) &= \begin{cases} \dfrac{(l+1,n-l-1)}{(2-c,n-l-1)} y_3(x) + \dfrac{(c-1,l+1)}{(n-l,l+1)} y_4(x) & (l<n-1), \\ y_3(x) & (l=n-1). \end{cases} \end{aligned} \tag{4.17}$$

(viii) (logarithmic:apparent)
(viii-1) *If*

$$\begin{aligned} &c=m \in \mathbb{Z}_{\geq 1},\ a-c, b-c \notin \{-1,-2,\ldots,-m+1\}, \\ &c-a-b=-n\ (n \in \mathbb{Z}_{\geq 1}),\ c-a=-l\ (l \in \{0,1,\ldots,n-1\}), \end{aligned}$$

we have

$$\begin{aligned} y_1(x) &= \frac{(n-l,l)}{(m,l)} y_4(x), \\ \hat{y}_2(x) &= -\frac{(1-m,m-1)(1,l)}{(n-l-m+1,l+m)} y_3(x) \\ &\quad + \frac{(n-l,l)}{(m,l)} (\psi(1)+\psi(m)-\psi(m+l)-\psi(n-l)) y_4(x). \end{aligned} \tag{4.18}$$

(viii-2) *If*

$$\begin{aligned} &c=m \in \mathbb{Z}_{\geq 1},\ a-c, b-c \notin \{-1,-2,\ldots,-m+1\}, \\ &c-a-b=n \in \mathbb{Z}_{\geq 1},\ a=-l\ (l \in \{0,1,\ldots,n-1\}), \end{aligned}$$

we have

$$\begin{aligned} y_1(x) &= \frac{(n-l,l)}{(m,l)} y_3(x), \\ \hat{y}_2(x) &= \frac{(n-l,l)}{(m,l)} (\psi(1)+\psi(m)-\psi(l+1)-\psi(n-l-m+1)) y_3(x) \\ &\quad - \frac{(1-m,m-1)(1,l)}{(n-l-m+1,l+m)} y_4(x). \end{aligned} \tag{4.19}$$

(viii-3) *If*

$$\begin{aligned} &c=-m\ (m \in \mathbb{Z}_{\geq 0}),\ a,b \notin \{0,-1,\ldots,-m\}, \\ &c-a-b=-n\ (n \in \mathbb{Z}_{\geq 1}),\ c-a=-l\ (l \in \{0,1,\ldots,n-1\}), \end{aligned}$$

we have

$$\begin{aligned}
\hat{y}_1(x) = & -\frac{(-m-1,m+1)(1,l-m-1)}{(n-l,l+1)}y_3(x) \\
& + \frac{(l+1,n-l-1)}{(m+2,n-l-1)} \\
& \quad \times (\psi(1)+\psi(m+2)-\psi(l+1)-\psi(n-l+m+1))y_4(x), \\
y_2(x) = & \frac{(l+1,n-l-1)}{(m+2,n-l-1)}y_4(x).
\end{aligned} \tag{4.20}$$

(viii-4) *If*

$$\begin{aligned}
& c=-m \ (m\in\mathbb{Z}_{\geq 0}), \ a,b\notin\{0,-1,\dots,-m\}, \\
& c-a-b=n\in\mathbb{Z}_{\geq 1}, \ a=-l \ (l\in\{0,1,\dots,n-1\}),
\end{aligned}$$

we have

$$\begin{aligned}
\hat{y}_1(x) = & \frac{(l+1,n-l-1)}{(m+2,n-l-1)}(\psi(1)+\psi(m+2)-\psi(l-m)-\psi(n-l))y_3(x) \\
& - \frac{(-m-1,m+1)(1,l-m-1)}{(n-l,l+1)}y_4(x), \\
y_2(x) = & \frac{(l+1,n-l-1)}{(m+2,n-l-1)}y_3(x).
\end{aligned} \tag{4.21}$$

(ix) (apparent:apparent)
(ix-1) *If*

$$\begin{aligned}
& c=m\in\mathbb{Z}_{\geq 2}, \ a-c=-l \ (l\in\{1,2,\dots,m-1\}), \\
& c-a-b=-n \ (n\in\mathbb{Z}_{\geq 1}), \ c-b\in\{0,-1,\dots,-n+1\},
\end{aligned}$$

we have

$$\begin{aligned}
y_1(x) &= \begin{cases} \dfrac{(m-l,n+l-m)}{(m,n+l-m)}y_4(x) & (l<m-1), \\ -\dfrac{m-1}{n}y_3(x)+\dfrac{(1,n-1)}{(m,n-1)}y_4(x) & (l=m-1), \end{cases} \\
y_2(x) &= \begin{cases} \dfrac{(n+1,l-1)}{(m-l,l-1)}y_3(x) & (l<m-1), \\ y_4(x) & (l=m-1). \end{cases}
\end{aligned} \tag{4.22}$$

(ix-2) *If*

$$\begin{aligned}
& c=m\in\mathbb{Z}_{\geq 2}, \ a-c=-l \ (l\in\{1,2,\dots,m-1\}), \\
& c-a-b=n\in\mathbb{Z}_{\geq 1}, \ b\in\{0,-1,\dots,-n+1\},
\end{aligned}$$

we have

$$y_1(x) = \begin{cases} \dfrac{(1,n-1)}{(m,n-1)} y_3(x) - \dfrac{m-1}{n} y_4(x) & (l=1), \\ \dfrac{(l,n-l)}{(m,n-l)} y_3(x) & (l>1), \end{cases}$$

$$y_2(x) = \begin{cases} y_3(x) & (l=1), \\ \dfrac{(n-l+1,l-1)}{(2-m,l-1)} y_3(x) + \dfrac{(n+1,m-l-1)}{(l,m-l-1)} y_4(x) & (1<l<m-1), \\ y_4(x) & (1<l=m-1). \end{cases} \tag{4.23}$$

(ix-3) *If*

$$c=-m\ (m\in\mathbb{Z}_{\geq 0}),\ a=-l\ (l\in\{0,1,\dots,m\}),$$
$$c-a-b=-n\ (n\in\mathbb{Z}_{\geq 1}),\ c-a\in\{0,-1,\dots,-n+1\},$$

we have

$$y_1(x) = \begin{cases} (m+1)y_3(x) + y_4(x) & (m-l=n-1=0), \\ y_4(x) & (0=m-l<n-1), \\ \dfrac{(n+1,l)}{(m-l+1,l)} y_3(x) & (0<m-l\leq n-1), \end{cases}$$

$$y_2(x) = \begin{cases} \dfrac{(l+1,m-l+1)}{(-n,m-l+1)} y_3(x) + \dfrac{(m-l+1,n-m+l-1)}{(m+2,n-m+l-1)} y_4(x) & (m-l<n-1), \\ y_4(x) & (m-l=n-1). \end{cases} \tag{4.24}$$

(ix-4) *If*

$$c=-m\ (m\in\mathbb{Z}_{\geq 0}),\ a=-l\ (l\in\{0,1,\dots,m\}),$$
$$c-a-b=n\in\mathbb{Z}_{\geq 1},\ a\in\{0,-1,\dots,-n+1\},$$

we have

$$y_1(x) = \begin{cases} \dfrac{(n-l,l)}{(-m,l)} y_3(x) & (l<m, l<n-1), \\ \dfrac{(1,n-1)}{(-m,n-1)} y_3(x) + \dfrac{m+1}{n} y_4(x) & (l<m, l=n-1), \\ y_4(x) & (l=m), \end{cases}$$

$$y_2(x) = \begin{cases} \dfrac{(l+1,n-l-1)}{(m+2,n-l-1)} y_3(x) + \dfrac{(m-l+1,l+1)}{(-n,l+1)} y_4(x) & (l<n-1), \\ y_3(x) & (l=n-1). \end{cases} \tag{4.25}$$

Remark 4.1. (i) The hypergeometric differential equation (1.1) is symmetric in a and b. Then we omitted the assertions obtained by the action $a \leftrightarrow b$. For example, in the case (iii-1), the case $b-c=-l$ is omitted.

(ii) On the other hand, the hypergeometric differential equation (1.1) has a deeper symmetry in the positions of the singular points. This symmetry is S_4 symmetry, and will be explained in the next section. By using this symmetry, the assertions (iv), (vii) and (viii) are derived from the assertions (ii), (iii) and (vi), respectively. However, we have not omitted these assertions for reader's convenience.
(iii) Some of the connection coefficients given in the theorem have isolated singular points. For example, in the right hand side of (4.2), one finds $\psi(a)$ which has a pole of order 1 at $a \in \mathbb{Z}_{\le 0}$. We can show that, in every case, the isolated singular point is removable. For the above example, $a \in \mathbb{Z}_{\le 0}$ is a removable singular point for $\psi(a)/\Gamma(a)$.

Our proof of Theorem 4.1 is based on the famous Gauss-Kummer identity.

Lemma 4.2. (Gauss-Kummer identity) *If* $c \notin \mathbb{Z}_{\le 0}$ *and* $\mathrm{Re}(c-a-b) > 0$*, we have*

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{4.26}$$

A proof of Lemma 4.2 is found in [2, Theorem 3.7.1].

Corollary 4.3. (i) *For* $p \in \mathbb{Z}_{\ge 0}$*,* $c \notin \{0, -1, \dots, -p+1\}$ *and any* b*, the identity*

$$\sum_{k=0}^{p} \frac{(-p, k)(b, k)}{(c, k)k!} = \frac{(c-b, p)}{(c, p)}$$

holds.
(ii) *For* $a \in \mathbb{C}$ *and* $p \in \mathbb{Z}_{\ge 0}$*, the identity*

$$\sum_{k=0}^{p} \frac{(a, k)}{k!} = \frac{(a+1, p)}{p!}$$

holds.

Proof. (i) Take an integer $p \ge 0$, and take b, c so that $c \notin \mathbb{Z}_{\le 0}$ and $\mathrm{Re}(c+p-b) > 0$ hold. Then, from (4.26) we obtain

$$F(-p, b, c; 1) = \frac{\Gamma(c)\Gamma(c+p-b)}{\Gamma(c+p)\Gamma(c-b)} = \frac{(c-b, p)}{(c, p)}.$$

The left hand side is the finite sum in the left hand side of the formula in the assertion (i), and then we get the formula. Since the formula is an identity of rational functions in (b, c), we may put any value (b, c) in the domains of definition of the rational functions. Therefore the formula holds as long as $c \notin \{0, -1, \dots, -p+1\}$.

(ii) We put $c = -p$ into the identity in (i) to obtain

$$\sum_{k=0}^{p} \frac{(b, k)}{k!} = \frac{(-p-b, p)}{(-p, p)} = \frac{(-p-b)(-p-b+1)\cdots(-b-1)}{(-1)^p p!} = \frac{(b+1, p)}{p!}.$$

□

Proof of Theorem 4.1.
(i) In this case, we can take $(y_1(x), y_2(x))$ as a fundamental system of solutions at $x = 0$, and $(y_3(x), y_4(x))$ as a fundamental system of solutions at $x = 1$. Then a linear relation

$$y_1(x) = Ay_3(x) + By_4(x) \tag{4.27}$$

holds on D_{01} with constants A, B. We assume $0 < \mathrm{Re}(c-a-b) < 1$, and take the limit $x \to 1$ along the real axis in $|x| < 1$. Then, thanks to Lemma 4.2, we get

$$A = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Put this into (4.27) and take the limit $x \to 0$ along the real axis in $|x-1| < 1$. Then, again by the help of Lemma 4.2, we have

$$1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \cdot \frac{\Gamma(a+b-c+1)\Gamma(1-c)}{\Gamma(b-c+1)\Gamma(a-c+1)} + B\frac{\Gamma(1-a-b+c)\Gamma(1-c)}{\Gamma(1-b)\Gamma(1-a)}.$$

Solving this, we get

$$B = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}.$$

Thus we get the first relation in (4.1). Once the relation is obtained, we can relax the assumption $0 < \mathrm{Re}(c-a-b) < 1$ by the analytic continuation. The second relation can be obtained similarly.
(ii-1) In this case, $(y_1(x), \hat{y}_2(x))$ and $(y_3(x), y_4(x))$ are fundamental systems of solutions at $x = 0$ and $x = 1$, respectively. We shall obtain the relation (4.2) from (4.1) by taking the limit $c \to m$.

We assume that c is in a neighborhood of $m \in \mathbb{Z}_{\geq 1}$ with $c \neq m$. Since the both sides of the first relation in (4.1) are defined at $c = m \in \mathbb{Z}_{\geq 1}$, we can put $c = m$ to get the first relation in (4.2).

We shall obtain the second relation. We first assume $m = 1$. In this case, $\hat{y}_2(x)$ is defined by (3.9). By using the relations in (4.1), we have

$$\begin{aligned}
&\frac{1}{1-c}(y_2(x) - y_1(x)) \\
&\quad = \frac{\Gamma(c-a-b)}{1-c}\left(\frac{\Gamma(2-c)}{\Gamma(1-a)\Gamma(1-b)} - \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\right) y_3(x) \\
&\qquad + \frac{\Gamma(a+b-c)}{1-c}\left(\frac{\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} - \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\right) y_4(x) \\
&\quad = \Gamma(c-a-b)\,\frac{f_3(c) - g_3(c)}{1-c} y_3(x) + \Gamma(a+b-c)\,\frac{f_4(c) - g_4(c)}{1-c} y_4(x),
\end{aligned}$$

where we set

$$\begin{aligned}
f_3(c) &= \frac{\Gamma(2-c)}{\Gamma(1-a)\Gamma(1-b)}, \quad g_3(c) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \\
f_4(c) &= \frac{\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}, \quad g_4(c) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}.
\end{aligned}$$

Since $f_3(1) = g_3(1)$ and $f_4(1) = g_4(1)$ hold, we get

$$\begin{aligned}\hat{y}_2(x) &= \lim_{c\to 1}\frac{1}{1-c}(y_2(x) - y_1(x)) \\ &= \Gamma(1-a-b)(-f_3'(1) + g_3'(1))y_3(x) + \Gamma(a+b-1)(-f_4'(1) + g_4'(1))y_4(x).\end{aligned}$$

The derivatives of the right hand side are given by

$$\begin{aligned}&f_3'(1) = f_3(1)(-\psi(1)),\ g_3'(1) = g_3(1)(\psi(1) - \psi(1-a) - \psi(1-b)), \\ &f_4'(1) = f_4(1)(\psi(1) + \psi(a) + \psi(b)),\ g_4'(1) = g_4(1)\psi(1),\end{aligned}$$

and then we obtain the second relation in (4.2) with $m = 1$.

Next we assume $m \geq 2$. By the definition (3.10) of $\hat{y}_2(x)$, we compute

$$\frac{(2-c, m-2)(m-1)!}{(a-c+1, m-1)(b-c+1, m-1)}y_2(x) - \frac{1}{m-c}y_1(x).$$

Replace $y_1(x), y_2(x)$ by the right hand sides of (4.1). Then the coefficients of $y_3(x)$ becomes

$$\begin{aligned}&\frac{(2-c, m-2)(m-1)!\Gamma(2-c)\Gamma(c-a-b)}{(a-c+1, m-1)(b-c+1, m-1)\Gamma(1-a)\Gamma(1-b)} - \frac{1}{m-c}\cdot\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\ =&\frac{\Gamma(m-c)\Gamma(m)\Gamma(c-a-b)\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(a-c+m)\Gamma(b-c+m)\Gamma(1-a)\Gamma(1-b)} - \frac{1}{m-c}\cdot\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\end{aligned}$$

where we used the identity (2.7). Since $\Gamma(m-c) = \Gamma(m-c+1)/(m-c)$, the coefficients of $y_3(x)$ can be written as

$$\frac{\Gamma(c-a-b)}{m-c}(f(c) - g(c)),$$

where

$$\begin{aligned}f(c) &= \frac{\Gamma(m-c+1)\Gamma(m)\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(a-c+m)\Gamma(b-c+m)\Gamma(1-a)\Gamma(1-b)}, \\ g(c) &= \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}.\end{aligned}$$

We note that

$$\begin{aligned}f(m) &= \frac{\Gamma(m)\Gamma(a-m+1)\Gamma(b-m+1)}{\Gamma(a)\Gamma(b)\Gamma(1-a)\Gamma(1-b)} \\ &= \frac{\Gamma(m)\Gamma(a-m+1)\Gamma(b-m+1)\sin\pi a\sin\pi b}{\pi^2} \\ &= \frac{\Gamma(m)}{\Gamma(m-a)\Gamma(m-b)} \\ &= g(m).\end{aligned}$$

Therefore we have

$$\lim_{c\to m}\frac{1}{m-c}(f(c) - g(c)) = -f'(m) + g'(m).$$

Thus the limit of the coefficient of $y_3(x)$ becomes

$$\begin{aligned}&\frac{\Gamma(m-a-b)\Gamma(m)}{\Gamma(m-a)\Gamma(m-b)}\\&\quad\times\big(\psi(1)+\psi(a-m+1)+\psi(b-m+1)-\psi(a)-\psi(b)\\&\qquad+\psi(m)-\psi(m-a)-\psi(m-b)\big).\end{aligned}$$

By applying the formula (2.4), we have

$$\psi(a-m+1)-\psi(m-a)-\psi(a)=\psi(1-a).$$

This gives the coefficient of $y_3(x)$ in the second relation in (4.2). The coefficient of $y_4(x)$ can be computed similarly.

(ii-2) In this case, $(\hat{y}_1(x), y_2(x))$ is a fundamental system of solutions at $x=0$. The second relation in (4.1) is defined at $c=-m\in\mathbb{Z}_{\leq 0}$, and then we get the second relation in (4.3) by putting $c=-m$. To get the relation for $\hat{y}_1(x)$, we compute

$$\frac{(c,m)(m+1)!}{(a,m+1)(b,m+1)}y_1(x)-\frac{1}{c+m}y_2(x)$$

according to the definition (3.12). Put the right hand sides of (4.1) into $y_1(x)$ and $y_2(x)$. Then the coefficient of $y_3(x)$ becomes

$$\frac{(c,m)(m+1)!\Gamma(c)\Gamma(c-a-b)}{(a,m+1)(b,m+1)\Gamma(c-a)\Gamma(c-b)}-\frac{1}{c+m}\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(a-c+1)\Gamma(b-c+1)}.$$

We use (2.8) to get

$$(c,m)\Gamma(c)=\frac{\Gamma(c+m+1)}{c+m}.$$

Therefore the coefficient of $y_3(x)$ can be written as

$$\frac{\Gamma(c-a-b)}{c+m}\left(\frac{\Gamma(c+m+1)(m+1)!}{(a,m+1)(b,m+1)\Gamma(c-a)\Gamma(c-b)}-\frac{\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}\right).$$

Then, in a similar way as the proof of (ii-1), we get the coefficient of $y_3(x)$ in the first relation in (4.3) by taking the limit $c\to -m$. The coefficient of $y_4(x)$ can be obtained similarly.

(iii-1) In this case, $(y_1(x), y_2(x))$ remains a fundamental system of solutions at $x=0$. For the previous logarithmic case, we obtained the connection relations by taking limits. For the present apparent case, however, this method does not work. If we want to obtain the apparent case from the generic case, we should take the double limit

$$c\to m\in\mathbb{Z}_{\geq 2},\ a-c\to -l\in\{-1,-2,\ldots,-m+1\},$$

and in this case this double limit does not exist. This is similar to that we have no canonical boundary value of the rational function z_1/z_2 at $(z_1,z_2)=(0,0)\in\mathbb{C}^2$. Therefore we look at the solutions directly.

We look at $y_2(x)$ given by (3.3). By putting $c = m$ and $a - c = -l$, we have

$$y_2(x) = x^{1-m} F(1 - l, b - m + 1, 2 - m; x).$$

Note that $1 - l, 2 - m \in \mathbb{Z}_{\leq 0}$ and $|1 - l| \leq |2 - m|$. If $l = m - 1$, we have

$$y_2(x) = x^{1-m} \sum_{k=0}^{\infty} \frac{(b - m + 1, k)}{k!} x^k = x^{1-m}(1 - x)^{m-1-b},$$

which coincides with $y_4(x)$. If $1 \leq l < m - 1$, we have

$$y_2(x) = x^{1-m} \sum_{k=0}^{l-1} \frac{(1 - l, k)(b - m + 1)}{(2 - m, k)k!} x^k,$$

and then it is holomorphic at $x = 1$. Hence it is a constant multiple of $y_3(x)$, and the constant is given by

$$y_2(1) = \sum_{k=0}^{l-1} \frac{(1 - l, k)(b - m + 1)}{(2 - m, k)k!} = \frac{(1 - b, l - 1)}{(2 - m, l - 1)}$$

thanks to Corollary 4.3 (i). Thus we get the second relation in (4.4). Since the first relation in (4.1) is defined at $c = m$ and $c - a = l$, we get the relation for $y_1(x)$ by putting these values. Using (2.7), we get the first relation in (4.4).
(iii-2) The result of this case is obtained in a similar way as (iii-1).
(iv-1) In this case, $(y_1(x), y_2(x))$ and $(y_3(x), \hat{y}_4(x))$ are fundamental systems of solutions at $x = 0$ and $x = 1$, respectively. First we consider the case $n = 0$. Then $\hat{y}_4(x)$ is defined by (3.14). We start with the relations in (4.1), and take the limit $c \to a + b$. We use

$$\Gamma(c - a - b) = \frac{\Gamma(c - a - b + 1)}{c - a - b}, \ \Gamma(a + b - c) = \frac{\Gamma(a + b - c + 1)}{a + b - c}.$$

Then the first relation in (4.1) can be written as

$$\begin{aligned} y_1(x) &= \frac{\Gamma(2 - c)}{c - a - b} \left(\frac{\Gamma(c - a - b + 1)}{\Gamma(1 - a)\Gamma(1 - b)} y_3(x) - \frac{\Gamma(a + b - c + 1)}{\Gamma(a - c + 1)\Gamma(b - c + 1)} y_4(x) \right) \\ &= -\frac{\Gamma(2 - c)\Gamma(a + b - c + 1)}{\Gamma(a - c + 1)\Gamma(b - c + 1)} \cdot \frac{1}{c - a - b} (y_4(x) - y_3(x)) \\ &\quad + \frac{\Gamma(2 - c)}{c - a - b} \left(\frac{\Gamma(c - a - b + 1)}{\Gamma(1 - a)\Gamma(1 - b)} - \frac{\Gamma(a + b - c + 1)}{\Gamma(a - c + 1)\Gamma(b - c + 1)} \right) y_3(x). \end{aligned}$$

The limit $c \to a + b$ can be obtained in a similar manner as (ii-1), and we get the first relation in (4.6) with $n = 0$. The relation for $y_2(x)$ is obtained similarly.

Next we consider the case $n \geq 1$. In this case, $\hat{y}_4(x)$ is defined by (3.15). Then we rewrite the first relation in (4.1) as

$$
\begin{aligned}
y_1(x) = {} & \frac{(c-a,n)(c-b,n)}{(c-a-b+1,n-1)n!} \cdot \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \\
& \times \left(\frac{(c-a-b+1,n-1)n!}{(c-a,n)(c-b,n)} y_4(x) - \frac{1}{c-a-b+n} y_3(x) \right) \\
& + \left(\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \right. \\
& \left. + \frac{1}{c-a-b+n} \frac{(c-a,n)(c-b,n)}{(c-a-b+1,n-1)n!} \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \right) y_3(x).
\end{aligned}
$$

The first term in the right hand side converges to

$$
\begin{aligned}
& \frac{(b-n,n)(a-n,n)}{(1-n,n-1)n!} \cdot \frac{\Gamma(2+n-a-b)\Gamma(n)}{\Gamma(n-b+1)\Gamma(n-a+1)} \hat{y}_4(x) \\
= & \frac{1}{(-1)^{n-1}(n-1)!n!} \cdot \frac{\Gamma(b)}{\Gamma(b-n)} \cdot \frac{\Gamma(a)}{\Gamma(a-n)} \cdot \frac{\Gamma(2+n-a-b)(n-1)!}{\Gamma(1-(b-n))\Gamma(1-(a-n))} \hat{y}_4(x) \\
= & \frac{1}{(-1)^{n-1}n!} \cdot \frac{\Gamma(a)\Gamma(b)\sin\pi(a-n)\sin\pi(b-n)}{\pi^2} \Gamma(2+n-a-b)\hat{y}_4(x) \\
= & - \frac{\Gamma(2+n-a-b)}{(-n,n)\Gamma(1-a)\Gamma(1-b)} \hat{y}_4(x),
\end{aligned}
$$

where we used (2.7) and (2.9). In order to compute the coefficient of $y_3(x)$ in the second term, we use (2.8) to have

$$
\Gamma(c-a-b) = \frac{\Gamma(c-a-b+n+1)}{(c-a-b,n)} \cdot \frac{1}{c-a-b+n}.
$$

Then the coefficient of $y_3(x)$ is written as

$$
\begin{aligned}
\frac{\Gamma(2-c)}{c-a-b+n} & \left(\frac{\Gamma(c-a-b+n+1)}{(c-a-b,n)\Gamma(1-a)\Gamma(1-b)} \right. \\
& \left. + \frac{(c-a,n)(c-b,n)\Gamma(a+b-c)}{(c-a-b+1,n-1)n!\Gamma(a-c+1)\Gamma(b-c+1)} \right).
\end{aligned}
$$

Here we have

$$
\frac{\Gamma(a+b-c)}{(c-a-b+1,n-1)} = \frac{(c-a-b)\Gamma(a+b-c)}{(c-a-b,n)} = -\frac{\Gamma(a+b-c+1)}{(c-a-b,n)}.
$$

Then the above coefficient is written as

$$
\begin{aligned}
& \frac{\Gamma(2-c)}{(c-a-b,n)} \cdot \frac{1}{c-a-b+n} \\
& \times \left(\frac{\Gamma(c-a-b+n+1)}{\Gamma(1-a)\Gamma(1-b)} - \frac{(c-a,n)(c-b,n)\Gamma(a+b-c+1)}{n!\Gamma(a-c+1)\Gamma(b-c+1)} \right).
\end{aligned}
$$

Now the limit $c \to a+b-n$ can be computed in a similar way as before, and we get the first relation in (4.6). The second relation in (4.6) is obtained similarly.
(iv-2) The result of this case is obtained in a similar way as (iv-1).
(v-1) In this case, $(y_1(x), \hat{y}_2(x))$ and $(y_3(x), \hat{y}_4(x))$ are fundamental systems of solutions at $x=0$ and $x=1$, respectively. We shall obtain the connection relations from the relations in (4.2) by taking the limit $c \to a+b-n$.

We can derive the first relation in (4.8) almost in the same way as the proof of (iv-1). We shall derive the second relation in (4.8) from the second relation in (4.2). First we consider the case $n=0$. Noting that $\hat{y}_4(x)$ is defined by (3.14), we compute the second relation in (4.2) as

$$\begin{aligned}\hat{y}_2(x) = &\frac{(m-a-b)\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)}(\psi(1)+\psi(m)-\psi(a)-\psi(b))\frac{y_4(x)-y_3(x)}{m-a-b}\\ &+\left(\frac{\Gamma(m)\Gamma(m-a-b)}{\Gamma(m-a)\Gamma(m-b)}(\psi(1)+\psi(m)-\psi(1-a)-\psi(1-b))\right.\\ &\quad\left.+\frac{\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)}(\psi(1)+\psi(m)-\psi(a)-\psi(b))\right)y_3(x),\end{aligned}$$

and take the limit $b \to m-a$. The first term in the right hand side converges to

$$-\frac{\Gamma(m)}{\Gamma(a)\Gamma(m-a)}(\psi(1)+\psi(m)-\psi(a)-\psi(m-a))\hat{y}_4(x).$$

The coefficient of $y_3(x)$ in the second term is written as

$$\begin{aligned}&\frac{\Gamma(m)}{m-a-b}\left(\frac{\Gamma(m-a-b+1)}{\Gamma(m-a)\Gamma(m-b)}(\psi(1)+\psi(m)-\psi(1-a)-\psi(1-b))\right.\\ &\qquad\left.-\frac{\Gamma(a+b-m+1)}{\Gamma(a)\Gamma(b)}(\psi(1)+\psi(m)-\psi(a)-\psi(b))\right)\\ =&\frac{\Gamma(m)}{m-a-b}(f(b)-g(b)),\end{aligned}$$

where we set

$$\begin{aligned}f(b) &= \frac{\Gamma(m-a-b+1)}{\Gamma(m-a)\Gamma(m-b)}(\psi(1)+\psi(m)-\psi(1-a)-\psi(1-b)),\\ g(b) &= \frac{\Gamma(a+b-m+1)}{\Gamma(a)\Gamma(b)}(\psi(1)+\psi(m)-\psi(a)-\psi(b)).\end{aligned}$$

By the help of (2.4), we have

$$\begin{aligned}f(m-a) &= \frac{1}{\Gamma(m-a)\Gamma(a)}(\psi(1)+\psi(m)-\psi(1-a)-\psi(1-m+a))\\ &= \frac{1}{\Gamma(m-a)\Gamma(a)}(\psi(1)+\psi(m)-(\psi(a)+\pi\cot\pi a)\\ &\qquad -(\psi(m-a)+\pi\cot\pi(m-a))\\ &= \frac{1}{\Gamma(m-a)\Gamma(a)}(\psi(1)+\psi(m)-\psi(a)-\psi(m-a))\\ &= g(m-a),\end{aligned}$$

and hence the limit of the coefficient of $y_3(x)$ becomes

$$\begin{aligned}
&\Gamma(m)(-f'(m-a)+g'(m-a))\\
=&\frac{\Gamma(m)}{\Gamma(a)\Gamma(m-a)}\Big((\psi(1)+\psi(m)-\psi(a)-\psi(m-a))(\psi(1)-\psi(a))-\psi'(1-m+a)\\
&\quad+(\psi(1)+\psi(m)-\psi(a)-\psi(m-a))(\psi(1)-\psi(m-a))-\psi'(m-a)\Big)\\
=&\frac{\Gamma(m)}{\Gamma(a)\Gamma(m-a)}\Big((\psi(1)+\psi(m)-\psi(a)-\psi(m-a))(2\psi(1)-\psi(a)-\psi(m-a))\\
&\quad-\psi'(1-m+a)-\psi'(m-a)\Big).
\end{aligned}$$

By differentiating (2.4), we get

$$\psi'(x)+\psi'(1-x)=\frac{\pi^2}{\sin^2\pi x},$$

and then we have

$$-\psi'(1-m+a)-\psi'(m-a)=-\frac{\pi^2}{\sin^2\pi a}.$$

Therefore the limit of the coefficient of $y_3(x)$ coincides with the coefficient of $y_3(x)$ of the second relation in (4.8) with $n=0$. Thus the second relation in (4.8) with $n=0$ is shown. The second relation in (4.8) with $n\geq 1$ can be shown by combining the method in the proof of (iv-1) and the above argument.

(v-2), (v-3), (v-4) can be shown similarly.

(vi-1) In this case, $(y_1(x),y_2(x))$ and $(y_3(x),\hat{y}_4(x))$ are fundamental systems of solutions at $x=0$ and $x=1$, respectively. From the assumptions $c=m, a-c=-l, c-a-b=-n$, where m,l,n are integers satisfying $m\geq 2, 1\leq l\leq m-1, n\geq 0$, we obtain

$$c-a=l\geq 1,\ c-b=m-l-n.$$

Then the condition $c-a\notin\{0,-1,\dots,-n+1\}$ is satisfied, and $c-b\geq -n+1$ holds by $m-l\geq 1$. Therefore the condition $c-b\notin\{0,-1,\dots,-n+1\}$ implies

$$m-l\geq n+1. \tag{4.28}$$

We shall obtain the relations in (4.12) from the relations in (4.4) by taking the limit $b\to l+n$.

First we consider the relation for $y_1(x)$. When $n=0$, $\hat{y}_4(x)$ is defined by (3.14), and then we compute the first relation in (4.4) as

$$\begin{aligned}
y_1(x) &= \frac{(l-b)(m-l,l)}{(b-l,l)} \cdot \frac{1}{l-b}(y_4(x)-y_3(x)) \\
&\quad + \left(\frac{(l,m-l)}{(l-b,m-l)} + \frac{(m-l,l)}{(b-l,l)}\right) y_3(x) \\
&= -\frac{(m-l,l)}{(b-l+1,l-1)} \cdot \frac{1}{l-b}(y_4(x)-y_3(x)) \\
&\quad + \frac{1}{l-b}\left(\frac{(l,m-l)}{(l-b+1,m-l-1)} - \frac{(m-l,l)}{(b-l+1,l-1)}\right) y_3(x).
\end{aligned}$$

The first term in the right hand side converges to

$$-\frac{(m-l,l)}{(1,l-1)}\hat{y}_4(x)$$

as $b \to l$. We note that

$$\begin{aligned}
\frac{(l,m-l)}{(l-b+1,m-l-1)} &\to \frac{(l,m-l)}{(1,m-l-1)}, \\
\frac{(m-l,l)}{(b-l+1,l-1)} &\to \frac{(m-l,l)}{(1,l-1)}
\end{aligned}$$

as $b \to l$, and that

$$\frac{(l,m-l)}{(1,m-l-1)} - \frac{(m-l,l)}{(1,l-1)} = \frac{(l,m-l)(1,l-1)-(m-l,l)(1,m-l-1)}{(1,m-l-1)(1,l-1)} = 0.$$

Therefore the second term in the right hand side converges to

$$\frac{(m-l,l)}{(1,l-1)}(2\psi(1)-\psi(m-l)-\psi(l))y_3(x),$$

where we used the formula

$$\frac{d}{dx}(x+p,q) = (x+p,q)(\psi(x+p+q)-\psi(x+p)).$$

Thus the limit $b \to l$ gives the right hand side of the first relation in (4.12) with $n=0$. For $n \geq 1$, $\hat{y}_4(x)$ is defined by (3.15), and then we write the first relation in (4.4) as

$$\begin{aligned}
y_1(x) = {} & \frac{(l,n)(m-b,n)}{(l-b+1,n-1)n!} \cdot \frac{(m-l,l)}{(b-l,l)}\left(\frac{(l-b+1,n-1)n!}{(l,n)(m-b,n)}y_4(x) - \frac{1}{l+n-b}y_3(x)\right) \\
& + \left(\frac{(l,m-l)}{(l-b,m-l)} + \frac{1}{l+n-b} \cdot \frac{(l,n)(m-b,n)}{(l-b+1,n-1)n!} \cdot \frac{(m-l,l)}{(b-l,l)}\right) y_3(x).
\end{aligned}$$

We can see that the first term of the right hand side converges to

$$-\frac{(m-l-n,l+n)}{(-n,n)(l-1)!}\hat{y}_4(x)$$

as $b \to l+n$. By the inequality (4.28), we have $l \leq l+n \leq m-1$, and hence

$$\begin{aligned}(l-b, m-l) &= (l-b)\cdots(m-1-b)\\ &= (l-b)\cdots(l+n-b)\cdots(m-1-b).\end{aligned}$$

Then the coefficient of $y_3(x)$ in the second term can be written as

$$\begin{aligned}&\frac{1}{l+n-b}\left(\frac{(l,m-l)}{(l-b,n)(l+n-b+1,m-l-n-1)}+\frac{(l,n)(m-b,n)(m-l,l)}{(l-b+1,n-1)n!(b-l,l)}\right)\\ =&\frac{1}{(l-b,n)}\cdot\frac{1}{l+n-b}\left(\frac{(l,m-l)}{(l+n-b+1,m-l-n-1)}-\frac{(l,n)(m-b,n)(m-l,l)}{n!(b-l+1,l-1)}\right).\end{aligned}$$

Then, in a similar way as we have done, we can compute the limit as $b \to l+n$ to get

$$\frac{(m-l-n,l+n)}{(-n,n)(l-1)!}(\psi(1)-\psi(m-l)-\psi(n+l)+\psi(n+1)).$$

Therefore we obtain the first relation in (4.12).

Next we consider the second relation in (4.4). If $l = m-1$, we have $y_2(x) = y_4(x)$ by (4.4), and also have $n = 0$ by (4.28). Then, in this case, we have

$$y_2(x) = \lim_{b\to m-1} y_4(x) = y_3(x),$$

which coincides with the second relation in (4.12) with $l = m-1$ and $n = 0$. If $1 \leq l < m-1$, the second relation in (4.4) is defined at $b = l+n$, and we get the second relation in (4.12) by putting $b = l+n$.

The cases (vi-2), (vi-3), (vi-4) are obtained in a similar way. We only note the conditions for the integers m, l, n:

$$\begin{array}{ll}\text{(vi-2)} & 1 \leq n < l \leq m-1,\\ \text{(vi-3)} & 0 \leq l \leq m,\ 0 \leq n \leq m-l,\\ \text{(vi-4)} & 1 \leq n \leq l \leq m.\end{array}$$

(vii-1) In this case, $(y_1(x), y_2(x))$ and $(y_3(x), y_4(x))$ are fundamental systems of solutions at $x = 0$ and $x = 1$, respectively. We start with the relations in (4.36) in Corollary 4.4, below. By the conditions on parameters, we have

$$a = c+l,\ b = n-l \geq 1,\ c \notin \mathbb{Z}.$$

The first relation in (4.36) is defined in this case, and we get

$$y_3(x) = \frac{\Gamma(1-c)\Gamma(n+1)}{\Gamma(l+1)\Gamma(n-l-c+1)}y_1(x) + \frac{\Gamma(c-1)\Gamma(n+1)}{\Gamma(c+l)\Gamma(n-l)}y_2(x). \tag{4.29}$$

On the other hand, the second relation in (4.36) is not defined at the above (a, b, c), and then we look at $y_4(x)$ directly. If $l = n-1$, we have

$$y_4(x) = (1-x)^{-n}F(-n+1, c-1, -n+1; 1-x) = (1-x)^{-n}x^{1-c},$$

which should coincide with $y_2(x)$. If $0 \le l < n-1$, we have

$$\begin{aligned}
y_4(x) &= (1-x)^{-n} F(-l, c-n+l, -n+1; 1-x) \\
&= (1-x)^{-n} \sum_{k=0}^{l} \frac{(-l,k)(c-n+l,k)}{(1-n,k)k!}(1-x)^k,
\end{aligned}$$

which is holomorphic at $x = 0$. Then it is a constant multiple of $y_1(x)$, and the constant is given by

$$y_4(0) = \sum_{k=0}^{l} \frac{(-l,k)(c-n+l,k)}{(1-n,k)k!} = \frac{(1-c-l,l)}{(1-n,l)},$$

where we used Corollary 4.3 (i). Thus we get

$$y_4(x) = \begin{cases} y_2(x) & (l = n-1), \\ \dfrac{(1-c-l,l)}{(1-n,l)} y_1(x) & (0 \le l < n-1). \end{cases} \tag{4.30}$$

Solving (4.29), (4.30) in $y_1(x), y_2(x)$, we get the relation (4.16).

(vii-2) can be shown in a similar manner.

(viii-1) We shall obtain the relations in this case as a limit of the previous result (4.16). We note that, by the conditions on the parameters, we have

$$1 \le m \le n - l. \tag{4.31}$$

First we consider the case $l = n-1$. Then we have $m = 1$ by (4.31). We shall derive the relations by taking the limit $c \to 1$. The relation for $y_1(x)$ in (4.16) is defined at $c = 1$, and then we can put $c = 1$ to get

$$y_1(x) = y_4(x).$$

By using (4.16), the relation for $\hat{y}_2(x)$ is obtained as

$$\begin{aligned}
\hat{y}_2(x) &= \lim_{c\to 1} \frac{1}{1-c}(y_2(x) - y_1(x)) \\
&= \lim_{c\to 1} \frac{1}{1-c}\left(y_4(x) - \left(\frac{1-c}{n} y_3(x) + \frac{(1,n-1)}{(c,n-1)} y_4(x) \right) \right) \\
&= -\frac{1}{n} y_3(x) + \lim_{c\to 1} \frac{1}{1-c}\left(1 - \frac{(1,n-1)}{(c,n-1)} \right) y_4(x) \\
&= -\frac{1}{n} y_3(x) + (\psi(1) - \psi(n)) y_4(x).
\end{aligned}$$

Next we assume $0 \le l < n-1$. The relation for $y_1(x)$ is obtained from the first relation of (4.16) by putting $c = m$. The result is

$$y_1(x) = \frac{(1-n,l)}{(1-m-l,l)} y_4(x),$$

which contains the above result for $l = n - 1$. The relation for $\hat{y}_2(x)$ with $m = 1$ is obtained in a similar way as above, by using (4.16) with $0 \le l < n - 1$, and we get

$$\hat{y}_2(x) = -\frac{l!}{(n-l, l+1)} y_3(x) + \frac{(1-n, l)}{(-l, l)} (2\psi(1) - \psi(n-l) - \psi(l+1)) y_4(x).$$

For $m \ge 2$, the relation for $\hat{y}_2(x)$ is obtained as the limit

$$\hat{y}_2(x) = \lim_{c \to m} \left(\frac{(2-c, m-2)(m-1)!}{(a-c+1, m-1)(b-c+1, m-1)} y_2(x) - \frac{1}{m-c} y_1(x) \right),$$

where $y_1(x), y_2(x)$ in the right hand side are given by (4.16) in terms of $y_3(x), y_4(x)$. The limit can be computed as we have done so far, and we get the second relation in (4.18). Note that this result contains the cases $l = n - 1$ and $l < n - 1$ with $m = 1$ obtained above. Therefore we have (4.18) as the unified result.

(viii-2), (viii-3), (viii-4) can be shown in a similar manner. We need no new idea.

(ix-1) We consider the apparent case at both $x = 0$ and $x = 1$ with $1 - c < 0, c - a - b < 0$. Then we have

$$\begin{aligned} &c = m \in \mathbb{Z}_{\ge 2},\ a - c \ \text{ or } \ b - c \in \{-1, -2, \ldots, -m+1\}, \\ &c - a - b = -n \in \mathbb{Z}_{\le -1},\ c - a \ \text{ or } \ c - b \in \{0, -1, \ldots, -n+1\}. \end{aligned}$$

Without loss of generality, we may assume $a - c = -l$ with $1 \le l \le m - 1$. Then we have $c - a = l \ge 1$, which does not belong to $\{0, -1, \ldots, -n+1\}$. Then we should assume $c - b \in \{0, -1, \ldots, -n+1\}$. On the other hand, we have $c - b = m - l - n \ge -n + 1$, and then the condition implies $m - l - n \le 0$. Therefore the conditions on the parameters are given by

$$a = m - l,\ b = n + l,\ c = m$$

with integers m, l, n satisfying

$$m \ge 2,\ 1 \le l \le m - 1,\ n \ge m - l.$$

In this case, $y_1(x), y_2(x)$ make a fundamental system of solutions at $x = 0$ with exponents $0, 1 - m$, respectively, and then $y_1(x)$ is subdominant at $x = 0$. Also $y_3(x), y_4(x)$ make a fundamental system of solutions at $x = 1$ with exponents $0, -n$, respectively, and then $y_3(x)$ is subdominant at $x = 1$.

First we consider the case $l = m - 1$. In this case, we have

$$(a, b, c) = (1, m + n - 1, m).$$

Then we get

$$\begin{aligned} y_2(x) &= x^{1-m} F(2-m, n, 2-m; x) = x^{1-m}(1-x)^{-n}, \\ y_4(x) &= (1-x)^{-n} F(m-1, 1-n, 1-n; 1-x) = (1-x)^{-n} x^{1-m}, \end{aligned}$$

and hence

$$y_2(x) = y_4(x)$$

holds. In order to get the relation for $y_1(x)$, we look at $y_1(x)$ and $y_3(x)$:

$$\begin{aligned} y_1(x) &= F(1, n+m-1, m; x), \\ y_3(x) &= F(1, n+m-1, n+1; 1-x). \end{aligned}$$

Since $m-1-(n+m-1) = -n < 0$ (resp. $(n+1)-1-(n+m-1) = 1-m < 0$), $y_1(x)$ (resp. $y_3(x)$) diverges when $x \to 1$ (resp. $x \to 0$). Then, in order to evaluate the behavior at $x = 1$ (resp. $x = 0$), we operate gauge transformations. For $y_1(x)$, we operate the gauge transformation $y(x) \mapsto z(x) = (1-x)^n y(x)$. In Riemann's notation of P-function, we have

$$(1-x)^n \left\{ \begin{matrix} x=0 & x=1 & x=\infty & \\ 0 & 0 & 1 & ; x \\ 1-m & -n & m+n-1 & \end{matrix} \right\} = \left\{ \begin{matrix} x=0 & x=1 & x=\infty \\ 0 & 0 & 1-n \; ; x \\ 1-m & n & m-1 \end{matrix} \right\}.$$

Then

$$z_1(x) = F(1-n, m-1, m; x) = \sum_{k=0}^{n-1} \frac{(1-n,k)(m-1,k)}{(m,k),k!} x^k$$

is a solution of the transformed equation, and hence $(1-x)^{-n} z_1(x)$ is a solution of the original hypergeometric differential equation. Since this solution is holomorphic and takes value 1 at $x = 0$, and $y_1(x)$ is subdominant, we get

$$y_1(x) = (1-x)^{-n} z_1(x).$$

Similarly we operate the gauge transformation $y(x) \mapsto z(x) = x^{m-1} y(x)$, and obtain

$$x^{m-1} \left\{ \begin{matrix} x=0 & x=1 & x=\infty & \\ 0 & 0 & 1 & ; x \\ 1-m & -n & m+n-1 & \end{matrix} \right\} = \left\{ \begin{matrix} x=0 & x=1 & x=\infty \\ 0 & 0 & 2-m \, ; x \\ m-1 & -n & n \end{matrix} \right\}.$$

Then

$$z_3(x) = F(2-m, n, n+1; 1-x) = \sum_{k=0}^{m-2} \frac{(2-m,k)(n,k)}{(n+1,k)k!} (1-x)^k$$

is a solution of the transformed equation, and hence $x^{1-m} z_3(x)$ is a solution of the original hypergeometric differential equation. Since this solution is holomorphic and takes value 1 at $x = 1$, and $y_3(x)$ is subdominant, we get

$$y_3(x) = x^{1-m} z_3(x).$$

Let

$$y_1(x) = A y_3(x) + B y_4(x)$$

be the connection relation, where A, B are constants. By using the above explicit expressions, we write this relation as

$$(1-x)^{-n} z_1(x) = A x^{1-m} z_3(x) + B x^{1-m} (1-x)^{-n}. \tag{4.32}$$

Multiplying the both sides of (4.32) by $(1-x)^n$ and taking the limit $x \to 1$, we get

$$z_1(1) = B.$$

Also multiplying the both sides of (4.32) by x^{m-1} and taking the limit $x \to 0$, we get

$$0 = A z_3(0) + B.$$

The values $z_1(1)$ and $z_3(0)$ are given by Corollary 4.3 (i), and we obtain

$$\begin{aligned} B &= \frac{(1, n-1)}{(m, n-1)}, \\ A &= -\frac{(1, n-1)}{(m, n-1)} \frac{(n+1, m-2)}{1, m-2)} \\ &= -\frac{1 \cdots (n-1) \cdot (n+1) \cdots (m+n-1)}{1 \cdots (m-2) \cdot m \cdots (m+n-1)} \\ &= -\frac{m-1}{n}. \end{aligned}$$

This gives the first relation of (4.22) with $l = m-1$.

Next we consider the case $1 \le l < m-1$. For $y_2(x)$ we have

$$y_2(x) = x^{1-m} F(1-l, n+l-m+1, 2-m; x) = x^{1-m} \sum_{k=0}^{l-1} \frac{(1-l, k)(n+l-m+1, k)}{(2-m, k) k!} x^k,$$

which is holomorphic at $x = 1$. Since $y_3(x)$ is holomorphic and subdominant at $x = 1$, $y_2(x)$ is a constant multiple of $y_3(x)$. The constant $y_2(1)$ is evaluated by Corollary 4.3 (i), and then we get

$$y_2(x) = \frac{(1-n-l, l-1)}{(2-m, l-1)} y_3(x).$$

For $y_4(x)$, we have

$$\begin{aligned} y_4(x) &= (1-x)^{-n} F(l, m-l-n, 1-n : 1-x) \\ &= (1-x)^{-n} \sum_{k=0}^{n+l-m} \frac{(m-l-n, k)(l, k)}{(1-n, k) k!} (1-x)^k, \end{aligned}$$

which is holomorphic at $x = 0$. Since $y_1(x)$ is holomorphic and subdominant at $x = 0$, $y_4(x)$ is a constant multiple of $y_1(x)$. The constant $1/y_4(0)$ is evaluated by Corollary 4.3 (i), and we get

$$y_1(x) = \frac{(1-n, n+l-m)}{(1-n-l, n+l-m)} y_4(x).$$

(ix-2) In this case, the parameters are given by

$$a = k - l,\ b = l - n,\ c = m$$

with integers m, l, n satisfying

$$m \geq 2,\ 1 \leq l \leq m-1,\ n \geq l.$$

Note that $y_1(x)$ is subdominant at $x = 0$, and $y_4(x)$ is subdominant at $x = 1$. The explicit forms of $y_1(x), y_2(x), y_3(x)$ are given by

$$\begin{aligned}
y_1(x) &= F(m-l, l-n, m; x) = \sum_{k=0}^{n-l} \frac{(m-l,k)(l-n,k)}{(m,k)k!} x^k, \\
y_2(x) &= x^{1-m} F(1-l, l-m-n+1, 2-m; x) \\
&= \begin{cases} x^{1-m} \displaystyle\sum_{k=0}^{l-1} \frac{(1-l,k)(l-m-n+1,k)}{(2-m,k)k!} x^k =: x^{1-m} z_2(x) & (l < m-1), \\ x^{1-m}(1-x)^n & (l = m-1), \end{cases} \\
y_3(x) &= F(m-l, l-n, 1-n; 1-x) \\
&= \begin{cases} x^{1-m} & (l = 1), \\ \displaystyle\sum_{k=0}^{n-l} \frac{(m-l,k)(l-n,k)}{(1-n,k)k!} (1-x)^k & (l > 1). \end{cases}
\end{aligned}$$

Since $y_4(x) = (1-x)^n F(l, m-l+n, n+1; 1-x)$ diverges as $x \to 0$, we operate the gauge transformation

$$x^{m-1} \begin{Bmatrix} x=0 & x=1 & x=\infty & \\ 0 & 0 & m-l & ; x \\ 1-m & n & l-n & \end{Bmatrix} = \begin{Bmatrix} x=0 & x=1 & x=\infty & \\ 0 & 0 & 1-l & ; x \\ m-1 & n & l-m-n+1 & \end{Bmatrix}.$$

The transformed equation has a solution

$$\begin{aligned}
&(1-x)^n F(1-m+l, n-l+1, n+1; 1-x) \\
&= (1-x)^n \sum_{k=0}^{m-l-1} \frac{(1-m+l,k)(n-l+1,k)}{(n+1,k)k!} (1-x)^k \\
&=: (1-x)^n z_4(x)
\end{aligned}$$

and hence the original equation has a solution $x^{1-m}(1-x)^n z_4(x)$, which is of exponent n at $x = 1$. Since $y_4(x)$ is subdominant, the solution coincides with $y_4(x)$, and hence we have the expression

$$y_4(x) = x^{1-m}(1-x)^n z_4(x).$$

We note that

$$y_1(1) = \frac{(l, n-l)}{(m, n-l)},\ z_2(1) = \frac{(n-l+1, l-1)}{(2-m, l-1)},\ z_4(0) = \frac{(l, m-l-1)}{(n+1, m-l-1)}$$

and, when $l > 1$,

$$y_3(0) = \frac{(m, n-l)}{(l, n-l)}$$

hold thanks to Corollary 4.3 (i).

First we shall obtain the connection relation for $y_1(x)$:

$$y_1(x) = Ay_3(x) + By_4(x). \tag{4.33}$$

When $l = 1$, the relation (4.33) becomes

$$y_1(x) = Ax^{1-m} + Bx^{1-m}(1-x)^n z_4(x). \tag{4.34}$$

By putting $x = 1$ in (4.34), we get

$$A = y_1(1) = \frac{(l, n-l)}{(m, n-l)}.$$

Multiplying both sides of (4.34) by x^{m-1} and putting $x = 0$, we have

$$A + Bz_4(0) = 0,$$

from which we obtain

$$B = -\frac{(l, n-l)}{(m, n-l)} \cdot \frac{(n+1, m-l-1)}{(l, m-l-1)} = -\frac{m-1}{n}.$$

When $l > 1$, $y_3(x)$ becomes a polynomial. Then, in the relation (4.33), $y_1(x)$ and $y_3(x)$ are subdominant at $x = 0$, from which we derive $B = 0$. By putting $x = 0$, we get $1 = Ay_3(0)$, from which

$$A = \frac{(l, n-l)}{(m, n-l)}$$

is derived. Thus we get the relation for $y_1(x)$ in (4.23).

Next we shall study the connection relation for $y_2(x)$. We notice that we should study the four cases

$$1 = l < m-1, \ 1 < l < m-1, \ 1 < l = m-1, \ 1 = l = m-1$$

separately. When $1 = l < m-1$, we have $y_2(x) = x^{1-m}$, from which we obtain $y_2(x) = y_3(x)$. We assume $1 < l < m-1$. Set

$$y_2(x) = Cy_3(x) + Dy_4(x).$$

Then we have

$$x^{1-m} z_2(x) = Cy_3(x) + Dx^{1-m}(1-x)^n z_4(x). \tag{4.35}$$

Multiplying both sides by x^{m-1} and putting $x = 0$, we get $1 = Dz_4(0)$, from which

$$D = \frac{(n+1, m-l-1)}{(l, m-l-1)}$$

is derived. Putting $x=1$ into (4.35), we obtain

$$C = z_2(1) = \frac{(n-l+1, l-1)}{(2-m, l-1)}.$$

If we assume $1 < l = m-1$, we have $y_2(x) = x^{1-m}(1-x)^n$ and $y_4(x) = x^{1-m}(1-x)^n$. Thus we get $y_2(x) = y_4(x)$. Since this relation does not contain $y_3(x)$, it holds also for the case $1 = l = m-1$. Thus we obtain the connection relation for $y_2(x)$ in (4.23).

(ix-3) In this case, the parameters are given by

$$a = -l,\ b = n+l-m,\ c = -m$$

with integers m, l, n satisfying

$$0 \le l \le m,\ n \ge m-l+1.$$

In a similar way as we have done above, we get the following expressions of $y_j(x)$:

$$y_1(x) = \begin{cases} \displaystyle\sum_{k=0}^{l} \frac{(-l,k)(n+l-m,k)}{(-m,k)k!} x^k & (l < m), \\ (1-x)^{-n} & (l = m), \end{cases}$$

$$y_2(x) = x^{m+1}(1-x)^{-n} \sum_{k=0}^{n-m+l-1} \frac{(m-l-n+1,k)(l+1,k)}{(m+2,k)k!} x^k,$$

$$y_3(x) = \sum_{k=0}^{l} \frac{(-l,k)(n+l-m,k)}{(n+1,k)k!} (1-x)^k,$$

$$y_4(x) = \begin{cases} \displaystyle(1-x)^{-n} \sum_{k=0}^{m-l} \frac{(l-m,k)(-n-l,k)}{(1-n,k)k!} (1-x)^k & (m-l < n-1), \\ (1-x)^{-n} x^{m+1} & (m-l = n-1). \end{cases}$$

Then, applying the arguments we have done so far, we get the result.

(ix-4) In this case, the parameters are given by

$$a = -l,\ b = l-m-n,\ c = -m$$

with integers m, l, n satisfying

$$0 \le l \le m,\ n \ge l+1.$$

We have the following expressions of $y_j(x)$:

$$y_1(x) = \begin{cases} \displaystyle\sum_{k=0}^{l} \frac{(-l,k)(l-m-n,k)}{(-m,k)k!} x^k & (l < m), \\ (1-x)^n & (l = m), \end{cases}$$

$$y_2(x) = x^{m+1} \sum_{k=0}^{n-l-1} \frac{(l-n+1,k)(m-l+1,k)}{(m+2,k)k!} x^k,$$

$$y_3(x) = \begin{cases} \displaystyle\sum_{k=0}^{l} \frac{(-l,k)(l-m-n,k)}{(1-n,k)k!} (1-x)^k & (l < n-1), \\ x^{m+1} & (l = n-1), \end{cases}$$

$$y_4(x) = (1-x)^n \sum_{k=0}^{m-l} \frac{(l-m,k)(n-l,k)}{(n+1,k)k!} (1-x)^k.$$

All $y_j(x)$ are polynomials. The result is obtained in a similar way as above. □

Corollary 4.4. *If $c \notin \mathbb{Z}, c-a-b \notin \mathbb{Z}$, we have*

$$\begin{aligned} y_3(x) &= \frac{\Gamma(1-c)\Gamma(a+b-c+1)}{\Gamma(a-c+1)\Gamma(b-c+1)} y_1(x) + \frac{\Gamma(c-1)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)} y_2(x), \\ y_4(x) &= \frac{\Gamma(1-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)} y_1(x) + \frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)} y_2(x). \end{aligned} \tag{4.36}$$

Proof. The result is obtained by solving the linear relation (4.1) in Theorem 4.1 in $(y_3(x), y_4(x))$. Note that, by the help of the formulas (2.10), the determinant of the coefficient matrix of the linear relation (4.1) is reduced to

$$\frac{1-c}{a+b-c}.$$

Then we get the result by a simple calculation. □

5 S_4 symmetry and connection relations between $x = 0$ and $x = \infty$

The hypergeometric differential equation (1.1) can be regarded as a normal form of a differential equation on the space of mutually distinct four points in $\mathbb{P}^1$. If we normalize three points among the four points to $0, 1, \infty$ by a Möbius transformation (an automorphism of $\mathbb{P}^1$), the image of the remaining point becomes the independent variable x of (1.1). The symmetric group S_4 acts on the set of the four points as permutations. It turns out that the action of S_4 yields a symmetry of (1.1), which is known as Kummer's 24 solutions. We shall give Kummer's 24 solutions soon later. Choose three points among the four points, and fix the set of the three points. The permutations on the set of the three points make a subgroup S_3 of S_4. If we normalize the three points in the set to $0, 1, \infty$, the S_3 action yields Möbius transformations of x. In this way, we obtain a faithful representation of S_3 given by the transformations

$$x \mapsto x,\ 1-x,\ \frac{1}{x},\ \frac{1}{1-x},\ \frac{x-1}{x},\ \frac{x}{x-1}. \tag{5.1}$$

The action of an element in S_4 sends every local solution of (1.1) to a local solution. Thus we obtain 24 expressions of the local solutions, which are called Kummer's 24 solutions. Kummer's 24 solutions are given by

$$\begin{aligned}
y_1(x) &= F(a,b,c;x) \\
&= (1-x)^{c-a-b}F(c-a,c-b,c;x) \\
&= (1-x)^{-a}F\Big(c-b,a,c;\frac{x}{x-1}\Big) \\
&= (1-x)^{-b}F\Big(c-a,b,c;\frac{x}{x-1}\Big), \\
y_2(x) &= x^{1-c}F(a-c+1,b-c+1,2-c;x) \\
&= x^{1-c}(1-x)^{c-a-b}F(1-a,1-b,2-c;x) \\
&= x^{1-c}(1-x)^{c-a-1}F\Big(1-b,a-c+1,2-c;\frac{x}{x-1}\Big) \\
&= x^{1-c}(1-x)^{c-b-1}F\Big(1-a;b-c+1,2-c;\frac{x}{x-1}\Big), \\
y_3(x) &= F(a,b,a+b-c+1;1-x) \\
&= x^{1-c}F(b-c+1,a-c+1,a+b-c+1;1-x) \\
&= x^{-a}F\Big(a-c+1,a,a+b-c+1;\frac{x-1}{x}\Big) \\
&= x^{-b}F\Big(b-c+1,b,a+b-c+1;\frac{x-1}{x}\Big), \\
y_4(x) &= (1-x)^{c-a-b}F(c-a,c-b,c-a-b+1;1-x) \\
&= x^{1-c}(1-x)^{c-a-b}F(1-a,1-b,c-a-b+1;1-x) \\
&= x^{b-c}(1-x)^{c-a-b}F\Big(1-b,c-b,c-a-b+1;\frac{x-1}{x}\Big) \\
&= x^{a-c}(1-x)^{c-a-b}F\Big(1-a,c-a,c-a-b+1;\frac{x-1}{x}\Big), \\
y_5(x) &= x^{-a}F\Big(a,a-c+1,a-b+1;\frac{1}{x}\Big) \\
&= (-x)^{b-c}(1-x)^{c-a-b}F\Big(1-b,c-b,a-b+1;\frac{1}{x}\Big) \\
&= (1-x)^{-a}F\Big(a,c-b,a-b+1;\frac{1}{1-x}\Big) \\
&= (-x)^{1-c}(1-x)^{c-a-1}F\Big(a-c+1,1-b,a-b+1;\frac{1}{1-x}\Big),
\end{aligned} \tag{5.2}$$

$$
\begin{aligned}
y_6(x) &= x^{-b} F\Big(b, b-c+1, b-a+1; \frac{1}{x}\Big) \\
&= (-x)^{a-c}(1-x)^{c-a-b} F\Big(1-a, c-a, b-a+1; \frac{1}{x}\Big) \\
&= (1-x)^{-b} F\Big(b, c-a, b-a+1; \frac{1}{1-x}\Big) \\
&= (-x)^{1-c}(1-x)^{c-b-1} F\Big(b-c+1, 1-a, b-a+1; \frac{1}{1-x}\Big).
\end{aligned}
$$

An explicit derivation is found in [2, §1.3]. We can use the S_3 action (5.1) and Kummer's 24 solutions (5.2) to derive connection relations between $x = 1$ and $x = \infty$ and between $x = 0$ and $x = \infty$ from the connection relations between $x = 0$ and $x = 1$. The main purpose of this section is to derive the connection relations between $x = 0$ and $x = \infty$.

First we consider the transformation $x \mapsto 1 - x$. Set

$$x_1 = 1 - x.$$

By the transformation $x \to x_1$, the Riemann's P-function is changed as

$$
\begin{aligned}
\begin{Bmatrix} x=0 & x=1 & x=\infty & \\ 0 & 0 & a & ; x \\ 1-c & c-a-b & b & \end{Bmatrix}
&= \begin{Bmatrix} x_1=0 & x_1=1 & x_1=\infty & \\ 0 & 0 & a & ; x_1 \\ c-a-b & 1-c & b & \end{Bmatrix} \\
&= \begin{Bmatrix} x_1=0 & x_1=1 & x_1=\infty & \\ 0 & 0 & a_1 & ; x_1 \\ 1-c_1 & c_1-a_1-b_1 & b_1 & \end{Bmatrix},
\end{aligned}
$$

where we set

$$a_1 = a, \ b_1 = b, \ c_1 = a + b - c + 1.$$

Then we have

$$y_1(x) = F(a, b, c; x) = F(a_1, b_1, a_1 + b_1 - c_1 + 1; 1 - x_1).$$

Let $y_j^{[1]}(x_1)$ $(1 \le j \le 4)$ be obtained from $y_j(x)$ by formally replacing (a, b, c, x) by (a_1, b_1, c_1, x_1). Then the above equality implies

$$y_1(x) = y_3^{[1]}(x_1).$$

In a similar way, we get

$$y_2(x) = y_4^{[1]}(x_1), \ y_3(x) = y_1^{[1]}(x_1), \ y_4(x) = y_2^{[1]}(x_1).$$

The functions $y_j^{[1]}(x_1)$ are defined on D_{01} in x_1-space, and the branches of $x_1^{1-c_1}$, $\log x_1$, $(1 - x_1)^{c_1 - a_1 - b_1}$ and $\log(1 - x_1)$ are defined by

$$\arg x_1 = \arg(1 - x_1) = 0 \text{ on } (0, 1),$$

which are the consequences of $\arg x = \arg(1-x) = 0$ via $x_1 = 1-x$. For $1 \le j \le 4$, we define $\hat{y}_j^{[1]}(x_1)$ by replacing (x, a, b, c) by (x_1, a_1, b_1, c_1) of $\hat{y}_j(x)$, and it turns out that $\hat{y}_j^{[1]}(x_1)$ are obtained by the same formulas (3.12), (3.8), (3.10), (3.16), (3.14), (3.15) from $y_j^{[1]}(x_1)$. Therefore the results in Theorem 4.1 directly imply the connection relations for $y_j^{[1]}(x_1), \hat{y}_j^{[1]}(x_1)$, only by reading

$$y_1(x) = y_3^{[1]}(x_1),\ y_2(x) = y_4^{[1]}(x_1),\ y_3(x) = y_1^{[1]}(x_1),\ y_4(x) = y_2^{[1]}(x_1),$$
$$\hat{y}_1(x) = \hat{y}_3^{[1]}(x_1),\ \hat{y}_2(x) = \hat{y}_4^{[1]}(x_1),\ \hat{y}_3(x) = \hat{y}_1^{[1]}(x_1),\ \hat{y}_4(x) = \hat{y}_2^{[1]}(x_1)$$

and replacing (x, a, b, c) by (x_1, a_1, b_1, c_1). The results thus obtained give expressions of the local solutions at $x_1 = 1$ as linear combinations of the local solutions at $x_1 = 0$.

In order to obtain the connection relations between $x = 0$ and $x = \infty$, next we consider the transformation $x \mapsto (x-1)/x$. This transformation sends $0, 1$ to $\infty, 0$, respectively. Set

$$x_2 = \frac{x-1}{x}.$$

By this transformation, the domain D_{01} in x-space is sent to

$$D_{\infty 0} = \mathbb{C} \setminus [0, +\infty)$$

in x_2-space. In D_{01} in x-space, we defined $\arg x = \arg(1-x) = 0$ on $(0,1)$. We have

$$x = \frac{1}{1-x_2},\ 1 - x = \frac{-x_2}{1-x_2},$$

and then we may define

$$\arg(1-x_2) = 0,\ \arg(-x_2) = 0 \text{ on } (-\infty, 0).$$

Now we look at the change of Riemann's P-function:

$$\begin{aligned}
\left\{\begin{matrix} x=0 & x=1 & x=\infty & \\ 0 & 0 & a & ;x \\ 1-c & c-a-b & b & \end{matrix}\right\} &= \left\{\begin{matrix} x_2=0 & x_2=1 & x_2=\infty & \\ 0 & a & 0 & ;x_2 \\ c-a-b & b & 1-c & \end{matrix}\right\} \\
&= (1-x_2)^a \left\{\begin{matrix} x_2=0 & x_2=1 & x_2=\infty & \\ 0 & 0 & a & ;x_2 \\ c-a-b & b-a & a-c+1 & \end{matrix}\right\} \\
&= (1-x_2)^a \left\{\begin{matrix} x_2=0 & x_2=1 & x_2=\infty & \\ 0 & 0 & a_2 & ;x_2 \\ 1-c_2 & c_2-a_2-b_2 & b_2 & \end{matrix}\right\},
\end{aligned}$$

where the relations of (a, b, c) and (a_2, b_2, c_2) are given by

$$\begin{cases} a_2 = a, \\ b_2 = a - c + 1, \\ c_2 = a + b - c + 1, \end{cases} \qquad \begin{cases} a = a_2, \\ b = c_2 - b_2, \\ c = a_2 - b_2 + 1. \end{cases} \tag{5.3}$$

According to these relations, we can rewrite $y_j(x)$ $(1 \le j \le 4)$ in (a_2, b_2, c_2, x_2). Noting

$$\frac{x}{x-1} = \frac{1}{x_2},$$

we use the third expression of $y_1(x)$ in (5.2) to get

$$\begin{aligned}
y_1(x) &= (1-x)^{-a} F\Big(c-b, a, c; \frac{x}{x-1}\Big) \\
&= \Big(\frac{-x_2}{1-x_2}\Big)^{-a} F\Big(c-b, a, c; \frac{1}{x_2}\Big) \\
&= (1-x_2)^a (-x_2)^{-a_2} F\Big(a_2, a_2-c_2+1, a_2-b_2+1; \frac{1}{x_2}\Big).
\end{aligned}$$

For $y_2(x)$, we use the third expression in (5.2) to get

$$\begin{aligned}
y_2(x) &= x^{1-c}(1-x)^{c-a-1} F\Big(1-b, a-c+1, 2-c; \frac{x}{x-1}\Big) \\
&= \Big(\frac{1}{1-x_2}\Big)^{1-c} \Big(\frac{-x_2}{1-x_2}\Big)^{c-a-1} F\Big(1-b, a-c+1, 2-c; \frac{1}{x_2}\Big) \\
&= (1-x_2)^a (-x_2)^{-b_2} F\Big(b_2-c_2+1, b_2, b_2-a_2+1; \frac{1}{x_2}\Big).
\end{aligned}$$

Also by the help of the third expression of $y_1(x)$ in (5.2), we can rewrite $y_3(x)$ as

$$\begin{aligned}
y_3(x) &= F\Big(a, b, a+b-c+1; \frac{-x_2}{1-x_2}\Big) \\
&= F\Big(a_2, c_2-b_2, c_2; \frac{x_2}{x_2-1}\Big) \\
&= (1-x_2)^a F(a_2, b_2, c_2; x_2).
\end{aligned}$$

In a similar way, we can rewrite $y_4(x)$ as

$$\begin{aligned}
y_4(x) &= \Big(\frac{-x_2}{1-x_2}\Big)^{c-a-b} F\Big(c-a, c-b, c-a-b+1; \frac{-x_2}{1-x_2}\Big) \\
&= (1-x_2)^a (-x_2)^{1-c_2} (1-x_2)^{c_2-a_2-1} F\Big(1-b_2, a_2-c_2+1, 2-c_2; \frac{x_2}{x_2-1}\Big) \\
&= (1-x_2)^a (-x_2)^{1-c_2} F(a_2-c_2+1, b_2-c_2+1, 2-c_2; x_2).
\end{aligned}$$

Therefore, if we define

$$\begin{aligned}
z_5(x_2) &= (-x_2)^{-a_2} F\Big(a_2, a_2-c_2+1, a_2-b_2+1; \frac{1}{x_2}\Big), \\
z_6(x_2) &= (-x_2)^{-b_2} F\Big(b_2-c_2+1, b_2, b_2-a_2+1; \frac{1}{x_2}\Big), \\
z_1(x_2) &= F(a_2, b_2, c_2; x_2), \\
z_2(x_2) &= (-x_2)^{1-c_2} F(a_2-c_2+1, b_2-c_2+1, 2-c_2; x_2),
\end{aligned}$$

we have

$$\begin{aligned} y_1(x) &= (1-x_2)^a z_5(x_2),\ y_2(x) = (1-x_2)^a z_6(x_2), \\ y_3(x) &= (1-x_2)^a z_1(x_2),\ y_4(x) = (1-x_2)^a z_2(x_2), \end{aligned} \tag{5.4}$$

and $(z_5(x_2), z_6(x_2))$ (resp. $(z_1(x_2), z_2(x_2))$) makes a fundamental system of solutions at $x_2 = \infty$ (resp. $x_2 = 0$). Then we can derive from the connection relation (4.1) the connection relation between these two sets of fundamental systems of solutions in generic case. We rewrite (4.1) in terms of (a_2, b_2, c_2) by using the dictionary (5.3), and replace $y_j(x)$ by $z_k(x_2)$ by using (5.4). Then we obtain

$$\begin{aligned} z_5(x_2) &= \frac{\Gamma(a_2-b_2+1)\Gamma(1-c_2)}{\Gamma(1-b_2)\Gamma(a_2-c_2+1)} z_1(x_2) + \frac{\Gamma(a_2-b_2+1)\Gamma(c_2-1)}{\Gamma(a_2)\Gamma(c_2-b_2)} z_2(x_2), \\ z_6(x_2) &= \frac{\Gamma(b_2-a_2+1)\Gamma(1-c_2)}{\Gamma(1-a_2)\Gamma(b_2-c_2+1)} z_1(x_2) + \frac{\Gamma(b_2-a_2+1)\Gamma(c_2-1)}{\Gamma(b_2)\Gamma(c_2-a_2)} z_2(x_2). \end{aligned}$$

In order to describe the connection relations for logarithmic case, we define logarithmic solutions by using z_j. From now on, we use (a, b, c, x) instead of (a_2, b_2, c_2, x_2). We repeat the definition of $z_j(x)$ $(j = 1, 2, 5, 6)$:

$$\begin{aligned} z_1(x) &= F(a, b, c; x), \\ z_2(x) &= (-x)^{1-c} F(a-c+1, b-c+1, 2-c; x), \\ z_5(x) &= (-x)^{-a} F\Big(a, a-c+1, a-b+1; \frac{1}{x}\Big), \\ z_6(x) &= (-x)^{-b} F\Big(b-c+1, b, b-a+1; \frac{1}{x}\Big). \end{aligned} \tag{5.5}$$

When $c = 1$, we define

$$\begin{aligned} \hat{z}_2(x) &= \lim_{c\to 1} \frac{1}{1-c}(z_2(x) - z_1(x)) \\ &= \log(-x) \sum_{k=0}^{\infty} \frac{(a,k)(b,k)}{(k!)^2} x^k \\ &\quad + \sum_{k=1}^{\infty} \frac{(a,k)(b,k)}{(k!)^2} \sum_{j=1}^{k} \left(\frac{1}{a+j-1} + \frac{1}{b+j-1} - \frac{2}{j} \right) x^k. \end{aligned} \tag{5.6}$$

When

$$c = m \in \mathbb{Z}_{\geq 2},\ a-c, b-c \notin \{-1, -2, \ldots, -m+1\},$$

we define

$$\hat{z}_2(x) = \lim_{c\to m}\left(\frac{(2-c,m-2)(m-1)!}{(a-c+1,m-1)(b-c+1,m-1)}z_2(x) - \frac{(-1)^{m-1}}{m-c}z_1(x)\right)$$
$$= (-1)^{m-1}\log(-x)\sum_{k=0}^{\infty}\frac{(a,k)(b,k)}{(m,k)k!}x^k$$
$$+ (-x)^{1-m}\frac{(2-m,m-2)(m-1)!}{(a-m+1,m-1)(b-m+1,m-1)}$$
$$\times\sum_{k=0}^{m-2}\frac{(a-m+1,k)(b-m+1,k)}{(2-m,k)k!}x^k$$
$$+ (-1)^{m-1}\sum_{k=1}^{\infty}\frac{(a,k)(b,k)}{(m,k)k!}\sum_{j=1}^{k}\left(\frac{1}{a+j-1}+\frac{1}{b+j-1}-\frac{1}{j}-\frac{1}{m+j-1}\right)x^k. \tag{5.7}$$

When

$$c = -m \in \mathbb{Z}_{\le 0},\ a,b \notin \{0,-1,\ldots,-m\},$$

we define

$$\hat{z}_1(x) = \lim_{c\to -m}\left(\frac{(c,m)(m+1)!}{(a,m+1)(b,m+1)}z_1(x) - \frac{(-1)^{m+1}}{c+m}z_2(x)\right)$$
$$= x^{m+1}\log(-x)\sum_{k=0}^{\infty}\frac{(a+m+1,k)(b+m+1,k)}{(m+2,k)k!}x^k$$
$$+ \frac{(-m,m)(m+1)!}{(a,m+1)(b,m+1)}\sum_{k=0}^{m}\frac{(a,k)(b,k)}{(-m,k)k!}x^k \tag{5.8}$$
$$+ x^{m+1}\sum_{k=1}^{\infty}\frac{(a+m+1,k)(b+m+1,k)}{(m+2,k)k!}$$
$$\times\sum_{j=1}^{k}\left(\frac{1}{a+m+j}+\frac{1}{b+m+j}-\frac{1}{j}-\frac{1}{m+1+j}\right)x^k.$$

When $a=b$, we define

$$\hat{z}_6(x) = \lim_{b\to a}\frac{1}{b-a}(z_6(x)-z_5(x))$$
$$= -(-x)^{-a}\log(-x)\sum_{k=0}^{\infty}\frac{(a,k)(a-c+1,k)}{(k!)^2}x^{-k}$$
$$+ (-x)^{-a}\sum_{k=0}^{\infty}\frac{(a,k)(a-c+1,k)}{(k!)^2}\sum_{j=1}^{k}\left(\frac{1}{a+j-1}+\frac{1}{a-c+j}-\frac{2}{j}\right)x^{-k}. \tag{5.9}$$

When

$$a-b = n \in \mathbb{Z}_{\ge 1},\ b, b-c+1 \notin \{0,-1,\ldots,-n+1\},$$

we define

$$\begin{aligned}
\hat{z}_6(x) &= \lim_{b\to a-n}\left(\frac{(b-a+1,n-1)n!}{(b,n)(b-c+1,n)}z_5(x)-\frac{(-1)^n}{b-a+n}z_5(x)\right)\\
&= -(-1)^n(-x)^{-a}\log(-x)\sum_{k=0}^{\infty}\frac{(a,k)(a-c+1,k)}{(n+1,k)k!}x^{-k}\\
&\quad+(-x)^{-a+n}\frac{(1-n,n-1)n!}{(a-n,n)(a-c-n+1,n)}\sum_{k=0}^{n-1}\frac{(a-n,k)(a-c-n+1,k)}{(1-n,k)k!}x^{-k}\\
&\quad+(-1)^n(-x)^{-a}\sum_{k=1}^{\infty}\frac{(a,k)(a-c+1,k)}{(n+1,k)k!}\\
&\qquad\times\sum_{j=1}^{k}\left(\frac{1}{a+j-1}+\frac{1}{a-c+j}-\frac{1}{j}-\frac{1}{n+j}\right)x^{-k}.
\end{aligned} \tag{5.10}$$

When

$$b-a=n\in\mathbb{Z}_{\geq 1},\ \ a,a-c+1\notin\{0,-1,\ldots,-n+1\},$$

we define

$$\begin{aligned}
\hat{z}_5(x) &= \lim_{a\to b-n}\left(\frac{(a-b+1,n-1)n!}{(a,n)(a-c+1,n)}z_5(x)-\frac{(-1)^n}{a-b+n}z_6(x)\right)\\
&= -(-1)^n(-x)^{-b}\log(-x)\sum_{k=0}^{\infty}\frac{(b,k)(b-c+1,k)}{(n+1,k)k!}x^{-k}\\
&\quad+(-x)^{-b+n}\frac{(1-n,n-1)n!}{(b-n,n)(b-c-n+1,n)}\sum_{k=0}^{n-1}\frac{(b-n,k)(b-c-n+1,k)}{(1-n,k)k!}x^{-k}\\
&\quad+(-1)^n(-x)^{-b}\sum_{k=1}^{\infty}\frac{(b,k)(b-c+1,k)}{(n+1,k)k!}\\
&\qquad\times\sum_{j=1}^{k}\left(\frac{1}{b+j-1}+\frac{1}{b-c+j}-\frac{1}{j}-\frac{1}{n+j}\right)x^{-k}.
\end{aligned} \tag{5.11}$$

Here we note that $z_1(x), \hat{z}_1(x), z_2(x)$ and $\hat{z}_2(x)$ are symmetric in (a,b). Also, when $a\neq b$, $z_6(x)$ (resp. $\hat{z}_6(x)$) is obtained from $z_5(x)$ (resp. $\hat{z}_5(x)$) by exchanging a and b.

By using $z_j(x), \hat{z}_j(x)$ $(j=1,2,5,6)$, we can describe the connection relations between $x=\infty$ and $x=0$. The connection relations for non-logarithmic cases can be directly obtained from Theorem 4.1 by applying the dictionary (5.3), (5.4). For the logarithmic cases, we derive the connection relations by taking the limits as in the proof of Theorem 4.1. In these ways, we obtain the following results.

Theorem 5.1. *Let $z_1(x), z_2(x), z_5(x), z_6(x)$ be defined by (5.5), $\hat{z}_2(x)$ by (5.6) and (5.7), $\hat{z}_1(x)$ by (5.8), $\hat{z}_6(x)$ by (5.9) and (5.10), and $\hat{z}_5(x)$ by (5.11). The following relations hold on the domain $D_{\infty 0}=\mathbb{C}\setminus[0,+\infty)$.*

(i) (generic:generic) *If*

$$a-b\notin\mathbb{Z}, c\notin\mathbb{Z},$$

we have

$$\begin{aligned}z_5(x)&=\frac{\Gamma(a-b+1)\Gamma(1-c)}{\Gamma(1-b)\Gamma(a-c+1)}z_1(x)+\frac{\Gamma(a-b+1)\Gamma(c-1)}{\Gamma(a)\Gamma(c-b)}z_2(x),\\ z_6(x)&=\frac{\Gamma(b-a+1)\Gamma(1-c)}{\Gamma(1-a)\Gamma(b-c+1)}z_1(x)+\frac{\Gamma(b-a+1)\Gamma(c-1)}{\Gamma(b)\Gamma(c-a)}z_2(x).\end{aligned}\tag{5.12}$$

(ii) (logarithmic:generic)
(ii-1) *If*

$$a-b=n\in\mathbb{Z}_{\geq 0},\ b,b-c+1\notin\{0,-1,\dots,-n+1\},\ c\notin\mathbb{Z},$$

we have

$$\begin{aligned}z_5(x)&=\frac{n!\Gamma(1-c)}{\Gamma(1-a+n)\Gamma(a-c+1)}z_1(x)+\frac{n!\Gamma(c-1)}{\Gamma(a)\Gamma(c-a+n)}z_2(x),\\ \hat{z}_6(x)&=\frac{(-n,n)\Gamma(1-c)}{\Gamma(1-a+n)\Gamma(a-c+1)}(\psi(1)+\psi(n+1)-\psi(a-c+1)-\psi(1-a))z_1(x)\\ &\quad+\frac{(-n,n)\Gamma(c-1)}{\Gamma(a)\Gamma(c-a+n)}(\psi(1)+\psi(n+1)-\psi(a)-\psi(c-a))z_2(x).\end{aligned}\tag{5.13}$$

(ii-2) *If*

$$b-a=n\in\mathbb{Z}_{\geq 1},\ a,a-c+1\notin\{0,-1,\dots,-n+1\},\ c\notin\mathbb{Z},$$

we have

$$\begin{aligned}\hat{z}_5(x)&=\frac{(-n,n)\Gamma(1-c)}{\Gamma(1-b+n)\Gamma(b-c+1)}(\psi(1)+\psi(n+1)-\psi(b-c+1)-\psi(1-b))z_1(x)\\ &\quad+\frac{(-n,n)\Gamma(c-1)}{\Gamma(b)\Gamma(c-b+n)}(\psi(1)+\psi(n+1)-\psi(b)-\psi(c-b))z_2(x),\\ z_6(x)&=\frac{n!\Gamma(1-c)}{\Gamma(1-b+n)\Gamma(b-c+1)}z_1(x)+\frac{n!\Gamma(c-1)}{\Gamma(b)\Gamma(c-b+n)}z_2(x).\end{aligned}\tag{5.14}$$

(iii) (apparent:generic)
(iii-1) *If*

$$a-b=n\in\mathbb{Z}_{\geq 1},\ b=-l\ (l\in\{0,1,\dots,n-1\}),\ c\notin\mathbb{Z},$$

we have

$$\begin{aligned}z_5(x)&=\frac{(l+1,n-l)}{(1-c,n-l)}z_1(x)+\frac{(n-l,l+1)}{(c-1,l+1)}z_2(x),\\ z_6(x)&=\begin{cases}\dfrac{(c,l)}{(n-l,l)}z_1(x) & (l<n-1),\\ z_2(x) & (l=n-1).\end{cases}\end{aligned}\tag{5.15}$$

(iii-2) *If*

$$b-a=n\in\mathbb{Z}_{\geq 1},\ a=-l\ (l\in\{0,1,\dots,n-1\}),\ c\notin\mathbb{Z},$$

we have

$$\begin{aligned} z_5(x)&=\begin{cases}\dfrac{(c,l)}{(n-l,l)}z_1(x) & (l<n-1),\\ z_2(x) & (l=n-1),\end{cases}\\ z_6(x)&=\frac{(l+1,n-l)}{(1-c,n-l)}z_1(x)+\frac{(n-l,l+1)}{(c-1,l+1)}z_2(x).\end{aligned} \tag{5.16}$$

(iv) (generic:logarithmic)
(iv-1) *If*

$$a-b\notin\mathbb{Z},\ c=m\in\mathbb{Z}_{\geq 1},\ a-c,b-c\notin\{-1,-2,\dots,-m+1\},$$

we have

$$\begin{aligned} z_5(x)&=\frac{\Gamma(a-b+1)}{(1-m,m-1)\Gamma(a-m+1)\Gamma(1-b)}(\psi(1)+\psi(m)-\psi(a)-\psi(1-b))z_1(x)\\ &\quad-\frac{\Gamma(a-b+1)}{(m-1)!\Gamma(a-m+1)\Gamma(1-b)}\hat{z}_2(x),\\ z_6(x)&=\frac{\Gamma(b-a+1)}{(1-m,m-1)\Gamma(b-m+1)\Gamma(1-a)}(\psi(1)+\psi(m)-\psi(b)-\psi(1-a))z_1(x)\\ &\quad-\frac{\Gamma(b-a+1)}{(m-1)!\Gamma(b-m+1)\Gamma(1-a)}\hat{z}_2(x).\end{aligned} \tag{5.17}$$

(iv-2) *If*

$$a-b\notin\mathbb{Z},\ c=-m\in\mathbb{Z}_{\leq 0},\ a,b\notin\{0,-1,\dots,-m\},$$

we have

$$\begin{aligned} z_5(x)&=-\frac{\Gamma(a-b+1)}{(m+1)!\Gamma(a)\Gamma(-b-m)}\hat{z}_1(x)\\ &\quad+\frac{\Gamma(a-b+1)}{(m+1)!\Gamma(a)\Gamma(-b-m)}\\ &\quad\times(\psi(1)+\psi(m+2)-\psi(a+m+1)-\psi(-b-m))z_2(x),\\ z_6(x)&=-\frac{\Gamma(b-a+1)}{(m+1)!\Gamma(b)\Gamma(-a-m)}\hat{z}_1(x)\\ &\quad+\frac{\Gamma(b-a+1)}{(m+1)!\Gamma(b)\Gamma(-a-m)}\\ &\quad\times(\psi(1)+\psi(m+2)-\psi(b+m+1)-\psi(-a-m))z_2(x).\end{aligned} \tag{5.18}$$

(v) (logarithmic:logarithmic)
(v-1) *If*

$$\begin{aligned} &a-b=n\in\mathbb{Z}_{\geq 0},\ b,b-c+1\notin\{0,-1,\dots,-n+1\},\\ &c=m\in\mathbb{Z}_{\geq 1},\ a-c,b-c\notin\{-1,-2,\dots,-m+1\},\end{aligned}$$

we have

$$
\begin{aligned}
z_5(x) = {}& \frac{n!}{(1-m,m-1)\Gamma(a-m+1)\Gamma(1-a+n)} \\
& \times (\psi(1)+\psi(m)-\psi(a)-\psi(1-a+n))z_1(x) \\
& - \frac{n!}{(m-1)!\Gamma(a-m+1)\Gamma(1-a+n)}\hat{z}_2(x), \\
\hat{z}_6(x) = {}& \frac{(-n,n)}{(1-m,m-1)\Gamma(a-m+1)\Gamma(1-a+n)}\Big((\psi(1)+\psi(m)-\psi(a)-\psi(1-a+n)) \\
& \times (\psi(1)+\psi(n+1)-\psi(a-m+1)-\psi(1-a)) - \frac{\pi^2}{\sin^2 \pi a}\Big)z_1(x) \\
& - \frac{(-n,n)}{(m-1)!\Gamma(a-m+1)\Gamma(1-a+n)} \\
& \times (\psi(1)+\psi(n+1)-\psi(a-m+1)-\psi(1-a))\hat{z}_2(x).
\end{aligned}
\tag{5.19}
$$

(v-2) *If*

$$
\begin{aligned}
& a-b=n\in\mathbb{Z}_{\geq 0},\ b, b-c+1\notin\{0,-1,\dots,-n+1\}, \\
& c=-m\in\mathbb{Z}_{\leq 0},\ a,b\notin\{0,-1,\dots,-m\},
\end{aligned}
$$

we have

$$
\begin{aligned}
z_5(x) = {}& -\frac{n!}{(m+1)!\Gamma(a)\Gamma(n-a-m)}\hat{z}_1(x) \\
& + \frac{n!}{(-m-1,m+1)\Gamma(a)\Gamma(n-a-m)} \\
& \times (\psi(1)+\psi(m+2)-\psi(a+m+1)-\psi(n-a-m))z_2(x), \\
\hat{z}_6(x) = {}& -\frac{(-n,n)}{(m+1)!\Gamma(a)\Gamma(n-a-m)} \\
& \times (\psi(1)+\psi(n+1)-\psi(a)-\psi(-a-m))\hat{z}_1(x) \\
& + \frac{(-n,n)}{(-m-1,m+1)\Gamma(a)\Gamma(n-a-m)}\Big((\psi(1)+\psi(n+1)-\psi(a)-\psi(-a-m)) \\
& \times (\psi(1)+\psi(m+2)-\psi(a+m+1)-\psi(n-a-m)) - \frac{\pi^2}{\sin^2 \pi a}\Big)z_2(x).
\end{aligned}
\tag{5.20}
$$

(v-3) *If*

$$
\begin{aligned}
& b-a=n\in\mathbb{Z}_{\geq 1},\ a, a-c+1\notin\{0,-1,\dots,-n+1\}, \\
& c=m\in\mathbb{Z}_{\geq 1},\ a-c, b-c\notin\{-1,-2,\dots,-m+1\},
\end{aligned}
$$

we have

$$\begin{aligned}
\hat{z}_5(x) &= \frac{(-n,n)}{(1-m,m-1)\Gamma(b-m+1)\Gamma(1-b+n)}\Big((\psi(1)+\psi(m)-\psi(b)-\psi(1-b+n))\\
&\qquad \times(\psi(1)+\psi(n+1)-\psi(b-m+1)-\psi(1-b)) - \frac{\pi^2}{\sin^2\pi b}\Big)z_1(x)\\
&\quad - \frac{(-n,n)}{(m-1)!\Gamma(b-m+1)\Gamma(1-b+n)}\\
&\qquad \times(\psi(1)+\psi(n+1)-\psi(b-m+1)-\psi(1-b))\hat{z}_2(x),\\
z_6(x) &= \frac{n!}{(1-m,m-1)\Gamma(b-m+1)\Gamma(1-b+n)}\\
&\qquad \times(\psi(1)+\psi(m)-\psi(b)-\psi(1-b+n))z_1(x)\\
&\quad - \frac{n!}{(m-1)!\Gamma(b-m+1)\Gamma(1-b+n)}\hat{z}_2(x).
\end{aligned} \tag{5.21}$$

(v-4) *If*

$$\begin{aligned}
&b-a=n\in\mathbb{Z}_{\ge 0},\ \ a, a-c+1\notin\{0,-1,\dots,-n+1\},\\
&c=-m\in\mathbb{Z}_{\le 0},\ \ a,b\notin\{0,-1,\dots,-m\},
\end{aligned}$$

we have

$$\begin{aligned}
\hat{z}_5(x) &= -\frac{(-n,n)}{(m+1)!\Gamma(b)\Gamma(n-b-m)}\\
&\qquad \times(\psi(1)+\psi(n+1)-\psi(b)-\psi(-b-m))\hat{z}_1(x)\\
&\quad + \frac{(-n,n)}{(-m-1,m+1)\Gamma(b)\Gamma(n-b-m)}\Big((\psi(1)+\psi(n+1)-\psi(b)-\psi(-b-m))\\
&\qquad \times(\psi(1)+\psi(m+2)-\psi(b+m+1)-\psi(n-b-m)) - \frac{\pi^2}{\sin^2\pi b}\Big)z_2(x),\\
z_6(x) &= -\frac{n!}{(m+1)!\Gamma(b)\Gamma(n-b-m)}\hat{z}_1(x)\\
&\quad + \frac{n!}{(-m-1,m+1)\Gamma(b)\Gamma(n-b-m)}\\
&\qquad \times(\psi(1)+\psi(m+2)-\psi(b+m+1)-\psi(n-b-m))z_2(x).
\end{aligned} \tag{5.22}$$

(vi) (apparent:logarithmic)

(vi-1) *If*

$$\begin{aligned}
&a-b=n\in\mathbb{Z}_{\ge 1},\ \ b=-l\ (0\le l\le n-1),\\
&c=m\in\mathbb{Z}_{\ge 1},\ \ a-c, b-c\notin\{-1,-2,\dots,-m+1\},
\end{aligned}$$

we have

$$z_5(x) = \frac{(n-l-m+1, m+l)}{(1-m, m-1)l!}(\psi(1)+\psi(m)-\psi(n-l)-\psi(l+1))z_1(x) - \frac{(n-l-m+1, m+l)}{(m-1)!l!}\hat{z}_2(x), \tag{5.23}$$
$$z_6(x) = \frac{(m,l)}{(n-l,l)}z_1(x).$$

(vi-2) *If*

$$a-b=n\in\mathbb{Z}_{\geq 1},\ b=-l\ (0\leq l\leq n-1),$$
$$c=-m\in\mathbb{Z}_{\leq 0},\ a,b\notin\{0,-1,\dots,-m\},$$

we have

$$z_5(x) = -\frac{(l-m, n-l+m+1)}{(m+1)!(n-l-1)!}\hat{z}_1(x) + \frac{(l-m, n-l+m+1)}{(-m-1, m+1)(n-l-1)!} \times (\psi(1)+\psi(m+2)-\psi(l-m)-\psi(n-l+m+1))z_2(x), \tag{5.24}$$
$$z_6(x) = \frac{l!}{(m+1)!(n-l+m+1, l-m-1)}z_2(x).$$

(vi-3) *If*

$$b-a=n\in\mathbb{Z}_{\geq 1},\ a=-l\ (0\leq l\leq n-1),$$
$$c=m\in\mathbb{Z}_{\geq 1},\ a-c, b-c\notin\{-1,-2,\dots,-m+1\},$$

we have

$$z_5(x) = \frac{(m,l)}{(n-l,l)}z_1(x),$$
$$z_6(x) = \frac{(n-l-m+1, m+l)}{(1-m, m-1)l!}(\psi(1)+\psi(m)-\psi(n-l)-\psi(l+1))z_1(x) - \frac{(n-l-m+1, m+l)}{(m-1)!l!}\hat{z}_2(x). \tag{5.25}$$

(vi-4) *If*

$$b-a=n\in\mathbb{Z}_{\geq 1},\ a=-l\ (0\leq l\leq n-1),$$
$$c=-m\in\mathbb{Z}_{\leq 0},\ a,b\notin\{0,-1,\dots,-m\},$$

we have

$$z_5(x) = \frac{l!}{(m+1)!(n-l+m+1, l-m-1)}z_2(x),$$
$$z_6(x) = -\frac{(l-m, n-l+m+1)}{(m+1)!(n-l-1)!}\hat{z}_1(x) + \frac{(l-m, n-l+m+1)}{(-m-1, m+1)(n-l-1)!} \times (\psi(1)+\psi(m+2)-\psi(l-m)-\psi(n-l+m+1))z_2(x). \tag{5.26}$$

(vii) (generic:apparent)
(vii-1) *If*

$$a-b\notin\mathbb{Z},\ c=m\in\mathbb{Z}_{\geq 2},\ b=l\in\{1,2,\dots,m-1\},$$

we have

$$\begin{aligned}
z_5(x)&=\begin{cases}\dfrac{(2-m,l-1)}{(1-a,l-1)}z_2(x) & (l<m-1),\\ \dfrac{m-a-1}{m-1}z_1(x)+\dfrac{(1,m-2)}{(a-m+2,m-2)}z_2(x) & (l=m-1),\end{cases}\\
z_6(x)&=\begin{cases}\dfrac{(a-l,l)}{(m-l,l)}z_1(x)+\dfrac{(l,m-l-1)}{(l-a+1,m-l-1)}z_2(x) & (l<m-1),\\ z_2(x) & (l=m-1).\end{cases}
\end{aligned}\tag{5.27}$$

(vii-2) *If*

$$a-b\notin\mathbb{Z},\ c=-m\in\mathbb{Z}_{\leq 0},\ a=-l\ (l\in\{0,1,\dots,m\}),$$

we have

$$\begin{aligned}
z_5(x)&=\begin{cases}\dfrac{(-m,l)}{(b,l)}z_1(x) & (l<m),\\ \dfrac{(1,m)}{(1-b-m,m)}z_1(x)+\dfrac{b+m}{m+1}z_2(x) & (l=m),\end{cases}\\
z_6(x)&=\begin{cases}\dfrac{(l+1,m-l)}{(b+l+1,m-l)}z_1(x)+\dfrac{(b,l+1)}{(-m-1,l+1)}z_2(x) & (l<m),\\ z_1(x) & (l=m).\end{cases}
\end{aligned}\tag{5.28}$$

(viii) (logarithmic:apparent)
(viii-1) *If*

$$\begin{aligned}&a-b=n\in\mathbb{Z}_{\geq 0},\ b,b-c+1\notin\{0,-1,\dots,-n+1\},\\ &c=m\in\mathbb{Z}_{\geq 2},\ b=l\in\{1,2,\dots,m-1\},\end{aligned}$$

we have

$$\begin{aligned}
z_5(x)&=\frac{(m-l,l-1)}{(n+1,l-1)}z_2(x),\\
\hat{z}_6(x)&=-\frac{(l-1)!n!}{(m-n-l,n+l)}z_1(x)\\
&\quad+\frac{(-1)^n(m-l,l-1)}{(n+1,l-1)}(\psi(1)+\psi(n+1)-\psi(m-l-n)-\psi(n+l))z_2(x).
\end{aligned}\tag{5.29}$$

(viii-2) *If*

$$\begin{aligned}&a-b=n\in\mathbb{Z}_{\geq 0},\ b,b-c+1\notin\{0,-1,\dots,-n+1\},\\ &c=-m\in\mathbb{Z}_{\leq 0},\ a=-l\ (l\in\{0,1,\dots,m\}),\end{aligned}$$

we have

$$\begin{aligned}
z_5(x) &= \frac{(m-l+1,l)}{(n+1,l)} z_1(x), \\
\hat{z}_6(x) &= \frac{(-1)^n (m-l+1,l)}{(n+1,l)} (\psi(1)+\psi(n+1)-\psi(m-l+1)-\psi(l+1)) z_1(x) \\
&\quad - \frac{n!l!}{(m-n-l+1,n+l+1)} z_2(x).
\end{aligned} \tag{5.30}$$

(viii-3) *If*

$$\begin{aligned}
&b-a=n\in\mathbb{Z}_{\geq 0},\ a, a-c+1 \notin \{0,-1,\dots,-n+1\}, \\
&c=m\in\mathbb{Z}_{\geq 2},\ a=l\in\{1,2,\dots,m-1\},
\end{aligned}$$

we have

$$\begin{aligned}
\hat{z}_5(x) &= -\frac{(l-1)!n!}{(m-n-l,n+l)} z_1(x) \\
&\quad + \frac{(-1)^n (m-l,l-1)}{(n+1,l-1)} (\psi(1)+\psi(n+1)-\psi(m-l-n)-\psi(n+l)) z_2(x), \\
z_6(x) &= \frac{(m-l,l-1)}{(n+1,l-1)} z_2(x).
\end{aligned} \tag{5.31}$$

(viii-4) *If*

$$\begin{aligned}
&b-a=n\in\mathbb{Z}_{\geq 0},\ a, a-c+1 \notin \{0,-1,\dots,-n+1\}, \\
&c=-m\in\mathbb{Z}_{\leq 0},\ b=-l\ (l\in\{0,1,\dots,m\}),
\end{aligned}$$

we have

$$\begin{aligned}
\hat{z}_5(x) &= \frac{(-1)^n (m-l+1,l)}{(n+1,l)} (\psi(1)+\psi(n+1)-\psi(m-l+1)-\psi(l+1)) z_1(x) \\
&\quad - \frac{n!l!}{(m-n-l+1,n+l+1)} z_2(x), \\
z_6(x) &= \frac{(m-l+1,l)}{(n+1,l)} z_1(x).
\end{aligned} \tag{5.32}$$

(ix) (apparent:apparent)
(ix-1) *If*

$$\begin{aligned}
&a-b=n\in\mathbb{Z}_{\geq 1},\ b=-l\ (l\in\{0,1,\dots,n-1\}), \\
&c=m\in\mathbb{Z}_{\geq 2},\ a-c\in\{-1,-2,\dots,-m+1\},
\end{aligned}$$

we have

$$\begin{aligned}
z_5(x) &= \begin{cases} \dfrac{(n-l+1,m-n+l-1)}{(n+1,m-n+l-1)} z_2(x) & (l<n-1), \\ -\dfrac{n}{m-1} z_1(x) + \dfrac{(1,m-2)}{(n+1,m-2)} z_2(x) & (l=n-1), \end{cases} \\
z_6(x) &= \begin{cases} \dfrac{(m,l)}{(n-l,l)} z_1(x) & (l<n-1), \\ z_2(x) & (l=n-1). \end{cases}
\end{aligned} \tag{5.33}$$

(ix-2) *If*

$$a-b=n\in\mathbb{Z}_{\geq 1},\ b=-l\ (l\in\{0,1,\dots,n-1\}),$$
$$c=-m\in\mathbb{Z}_{\leq 0},\ c-b\in\{0,-1,\dots,-m\},$$

we have

$$z_5(x)=\begin{cases}\dfrac{(1,m)}{(n+1,m)}z_1(x)-\dfrac{n}{m+1}z_2(x) & (l=0),\\ \dfrac{(l+1,m-l)}{(n+1,m-l)}z_1(x) & (l>0),\end{cases}$$
$$z_6(x)=\begin{cases}z_1(x) & (l=0),\\ \dfrac{(m-l+1,l)}{(1-n,l)}z_1(x)+\dfrac{(m+2,n-l-1)}{(l+1,n-l-1)}z_2(x) & (0<l<n-1),\\ z_2(x) & (0<l=n-1).\end{cases} \tag{5.34}$$

(ix-3) *If*

$$b-a=n\in\mathbb{Z}_{\geq 1},\ a=-l\ (l\in\{0,1,\dots,n-1\}),$$
$$c=m\in\mathbb{Z}_{\geq 2},\ b\in\{1,2,\dots,m+1\},$$

we have

$$z_5(x)=\begin{cases}nz_1(x)+z_2(x) & (1=n-l=m-1),\\ z_2(x) & (1=n-l<m-1),\\ \dfrac{(m,l)}{(n-l,l)}z_1(x) & (1<n-l\leq m-1),\end{cases}$$
$$z_6(x)=\begin{cases}\dfrac{(l+1,n-l)}{(1-m,n-l)}z_1(x)+\dfrac{(n-l,m-n+l-1)}{(n+1,m-n+l-1)}z_2(x) & (n-l<m-1),\\ z_2(x) & (n-l=m-1).\end{cases} \tag{5.35}$$

(ix-4) *If*

$$b-a=n\in\mathbb{Z}_{\geq 1},\ a=-l\ (l\in\{0,1,\dots,n-1\}),$$
$$c=-m\in\mathbb{Z}_{\leq 0},\ a\in\{0,-1,\dots,-m\},$$

we have

$$z_5(x)=\begin{cases}\dfrac{(m-l+1,l)}{(1-n,l)}z_1(x) & (l<n-1,l<m),\\ \dfrac{(1,m)}{(1-n,m)}z_1(x)+\dfrac{n}{m+1}z_2(x) & (l<n-1,l=m),\\ z_2(x) & (l=n-1),\end{cases} \tag{5.36}$$
$$z_6(x)=\begin{cases}\dfrac{(l+1,m-l)}{(n+1,m-l)}z_1(x)+\dfrac{(n-l,l+1)}{(-m-1,l+1)}z_2(x) & (l<m),\\ z_1(x) & (l=m).\end{cases}$$

Remark 5.1. Thanks to the symmetric nature with respect to (a, b), we can directly derive the relation (5.14) from (5.13) by replacing

$$(z_5(x), \hat{z}_6(x)) \to (z_6(x), \hat{z}_5(x)),\ (a, b) \to (b, a).$$

This way of derivation works also for obtaining

$$\begin{aligned} &(5.19) \to (5.21),\ (5.20) \to (5.22), \\ &(5.23) \to (5.25),\ (5.24) \to (5.26), \\ &(5.29) \to (5.31),\ (5.30) \to (5.32). \end{aligned}$$

6 Direct applications

6.1 Monodromy representation

Let $\mathcal{Y}(x)$ be a fundamental system of solutions of the hypergeometric differential equation (1.1) at $x = 1/2$. We denote the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, 1/2)$ by G. The monodromy representation

$$\rho : G \to \mathrm{GL}(2, \mathbb{C})$$

for (1.1) with respect to $\mathcal{Y}(x)$ is defined by

$$\gamma_* \mathcal{Y}(x) = \mathcal{Y}(x)\rho(\gamma)$$

for $\gamma \in G$, where γ_* denotes the analytic continuation along γ. Let γ_0 (resp. γ_1) be a loop with base point $1/2$ encircling $x = 0$ (resp. $x = 1$) once in the positive direction. For example, we may take

$$\gamma_0(t) = \frac{1}{2}e^{it},\ \gamma_1(t) = 1 - \frac{1}{2}e^{it} \quad (t \in [0, 2\pi]).$$

We regard γ_0, γ_1 as elements in the fundamental group G. Then G is generated by γ_0 and γ_1. Therefore the monodromy representation ρ is determined by two matrices M_0, M_1 which are defined by

$$(\gamma_0)_* \mathcal{Y}(x) = \mathcal{Y}(x)M_0,\ (\gamma_1)_* \mathcal{Y}(x) = \mathcal{Y}(x)M_1.$$

We shall explain that the connection relation given in Theorem 4.1 determines the monodromy representation.

According to the conditions for the parameters a, b, c, we take one of

$$(y_1(x), y_2(x)),\ (y_1(x), \hat{y}_2(x)),\ (\hat{y}_1(x), y_2(x))$$

as the fundamental system of solutions $\mathcal{Y}(x)$ that determines the monodromy representation ρ. If $c, c - a - b \notin \mathbb{Z}$, $(y_1(x), y_2(x))$ and $(y_3(x), y_4(x))$ are fundamental systems of solutions at $1/2$. By the definition, we readily see that

$$\begin{aligned} (\gamma_0)_*(y_1(x), y_2(x)) &= (y_1(x), y_2(x)) \begin{pmatrix} 1 & \\ & e^{2\pi i(-c)} \end{pmatrix}, \\ (\gamma_1)_*(y_3(x), y_4(x)) &= (y_3(x), y_4(x)) \begin{pmatrix} 1 & \\ & e^{2\pi i(c-a-b)} \end{pmatrix} \end{aligned}$$

hold. Then, if we take

$$\mathcal{Y}(x) = (y_1(x), y_2(x)),$$

we have

$$M_0 = \begin{pmatrix} 1 & \\ & e^{2\pi i(-c)} \end{pmatrix}.$$

On the other hand, Theorem 4.1 (i) explicitly gives the matrix $C \in \mathrm{GL}(2, \mathbb{C})$ satisfying

$$(y_1(x), y_2(x)) = (y_3(x), y_4(x))C.$$

Then the matrix M_1 is also explicitly given by

$$M_1 = C^{-1} \begin{pmatrix} 1 & \\ & e^{2\pi i(c-a-b)} \end{pmatrix} C.$$

Thus we obtain the monodromy representation for the case $c, c-a-b \notin \mathbb{Z}$. Note that the entries of the inverse C^{-1} are explicitly given in Corollary 4.4. However, we find that the entries of M_1 are not so simple.

In this way, we can obtain the monodromy representation from the connection relation. In order to get the monodromy representations for the other cases, we need to see the analytic continuations of fundamental systems of solutions at $x = 0$ (resp. $x = 1$) of logarithmic cases along γ_0 (resp. γ_1). By their definitions, we get

$$\begin{aligned}
(\gamma_0)_*(y_1(x), \hat{y}_2(x)) &= (y_1(x), \hat{y}_2(x)) \begin{pmatrix} 1 & 2\pi i \\ & 1 \end{pmatrix}, \\
(\gamma_0)_*(\hat{y}_1(x), y_2(x)) &= (\hat{y}_1(x), y_2(x)) \begin{pmatrix} 1 & \\ 2\pi i & 1 \end{pmatrix}, \\
(\gamma_1)_*(y_3(x), \hat{y}_4(x)) &= (y_3(x), \hat{y}_4(x)) \begin{pmatrix} 1 & 2\pi i \\ & 1 \end{pmatrix}, \\
(\gamma_1)_*(\hat{y}_3(x), y_4(x)) &= (\hat{y}_3(x), y_4(x)) \begin{pmatrix} 1 & \\ 2\pi i & 1 \end{pmatrix}.
\end{aligned}$$

We have only to notice $\hat{y}_2(x) = y_1(x) \log x + (\text{single-valued function})$ etc.. Thence the connection relations in Theorem 4.1 give the monodromy representations with respect to $\mathcal{Y}(x) = (y_1(x), y_2(x))$ or $(y_1(x), \hat{y}_2(x))$ or $(\hat{y}_1(x), y_2(x))$.

6.2 Connection relations for Legendre differential equation

The Legendre differential equation

$$(1-t^2)\frac{d^2u}{dt^2} - 2t\frac{du}{dt} + \nu(\nu+1)u = 0 \tag{6.1}$$

is also a fundamental differential equation in physics and mathematics. The Riemann scheme is given by

$$\begin{Bmatrix} t=1 & t=-1 & t=\infty \\ 0 & 0 & -\nu \\ 0 & 0 & \nu+1 \end{Bmatrix}.$$

This implies that the logarithmic cases occurs at $t = 1$ and $t = -1$. Since the Legendre equation (6.1) can be transformed to the Gauss equation (1.1), we can derive the connection relation for (6.1) from Theorem 4.1. Actually, if we change the independent variable t of (6.1) to

$$x = \frac{1-t}{2},$$

the transformed differential equation in x coincides with the hypergeometric differential equation (1.1) with parameter $(a, b, c) = (-\nu, \nu + 1, 1)$. In particular, we have

$$c = 1, \ c - a - b = 0,$$

which falls into the case studied in Theorem 4.1 (v-1).

We can get the local solutions of (6.1) at $t = 1$ and $t = -1$ by transforming the local solutions of (1.1) given in Section 3. For $x \in D_{01}$, we defined

$$\arg x = \arg(1 - x) = 0 \quad (x \in [0, 1]).$$

Then we consider the solutions of (6.1) on

$$D_{-1,1} = \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty)),$$

and determine the branches of the solutions by

$$\arg\left(\frac{1-t}{2}\right) = \arg\left(\frac{t+1}{2}\right) = 0 \quad (t \in [-1, 1]).$$

Now, from (3.2) and (3.9), we obtain the local solutions

$$\begin{aligned}
u_1(t) &= F\Big(-\nu, \nu+1, 1; \frac{1-t}{2}\Big), \\
\hat{u}_2(t) &= u_1(t) \log\left(\frac{1-t}{2}\right) \\
&\quad + \sum_{k=1}^{\infty} \frac{(-\nu, k)(\nu+1, k)}{(k!)^2} \sum_{j=1}^{k} \left(\frac{1}{-\nu + j - 1} + \frac{1}{\nu + j} - \frac{2}{j}\right) \left(\frac{1-t}{2}\right)^k
\end{aligned}$$

of (6.1) at $t = 1$. If $\nu \in \mathbb{Z}$, the infinite series in $u_1(t)$ and $\hat{u}_2(t)$ become finite sums. Similarly, from (3.4) and (3.14) we obtain the local solutions

$$\begin{aligned}
u_3(t) &= F\Big(-\nu, \nu+1, 1; \frac{t+1}{2}\Big), \\
\hat{u}_4(t) &= u_3(t) \log\left(\frac{t+1}{2}\right) \\
&\quad + \sum_{k=1}^{\infty} \frac{(-\nu, k)(\nu+1, k)}{(k!)^2} \sum_{j=1}^{k} \left(\frac{1}{-\nu + j - 1} + \frac{1}{\nu + j} - \frac{2}{j}\right) \left(\frac{t+1}{2}\right)^k
\end{aligned}$$

of (6.1) at $t=-1$. When $\nu\in\mathbb{Z}$, the infinite series in $u_3(t)$ and $\hat{u}_4(t)$ become finite sums. Since

$$u_1(t)=y_1(x),\ \hat{u}_2(t)=\hat{y}_2(x),\ u_3(t)=y_3(x),\ \hat{u}_4(t)=\hat{y}_4(x),$$

we obtain the connection relation between $(u_1(t),\hat{u}_2(t))$ and $(u_3(t),\hat{u}_4(t))$ from the connection relation (4.8) in Theorem 4.1 (v-1).

Proposition 6.1. *On the domain $D_{-1,1}$, the following relations hold:*

$$\begin{aligned} u_1(t)&=-\frac{\sin\pi\nu}{\pi}(2\psi(1)-\psi(-\nu)-\psi(\nu+1))u_3(x)+\frac{\sin\pi\nu}{\pi}\hat{u}_4(t),\\ \hat{u}_2(t)&=-\frac{\sin\pi\nu}{\pi}\left((2\psi(1)-\psi(-\nu)-\psi(\nu+1))^2-\frac{\pi^2}{\sin^2\pi\nu}\right)u_3(x)\\ &\quad+\frac{\sin\pi\nu}{\pi}(2\psi(1)-\psi(-\nu)-\psi(\nu+1))\hat{u}_4(t). \end{aligned}\tag{6.2}$$

Remark that the connection coefficients have removable singular points at $\nu\in\mathbb{Z}$. Therefore the connection relation (6.2) holds for all $\nu\in\mathbb{C}$. We note that the connection relation is also derived by using an integral representation [1, Theorem 9.5].

Appendix

For each connection relation given in Theorem 4.1 and Theorem 5.1, let A be the matrix of coefficients. Then the inverse relation is given by the inverse matrix A^{-1}. Here we give a table of A^{-1} or $|A|$ for all relations, which will be useful to derive their inverse relations.

Table 1: For relations in Theorem 4.1.

(i) $|A|=\dfrac{1-c}{a+b-c}$

(ii-1) $|A|=\dfrac{(-1)^m\Gamma(m)^2\Gamma(1-a)\Gamma(1-b)}{(m-a-b)\Gamma(m-a)\Gamma(m-b)}$

(ii-2) $|A|=\dfrac{(-1)^m\Gamma(m+2)^2\Gamma(-a-m)\Gamma(-b-m)}{(a+b+m)\Gamma(1-a)\Gamma(1-b)}$

(iii-1) For the case $l<m-1$,

$$A=\begin{pmatrix}\frac{(l,m-l)}{(l-b,m-l)} & \frac{(m-l,l)}{(b-l,l)}\\ \frac{(b-l+1,l-1)}{(m-l,l-1)} & 0\end{pmatrix},\ A^{-1}=\begin{pmatrix}0 & \frac{(m-l,l-1)}{(b-l+1,l-1)}\\ \frac{(b-l,l)}{(m-l,l)} & \frac{(l,m-l-1)}{(l-b+1,m-l-1)}\end{pmatrix},$$

and for the case $l=m-1$,

$$A=\begin{pmatrix}\frac{m-1}{m-b-1} & \frac{(1,m-1)}{(b-m+1,m-1)}\\ 0 & 1\end{pmatrix},\ A^{-1}=\begin{pmatrix}\frac{m-b-1}{m-1} & \frac{(1,m-2)}{(b-m+2,m-2)}\\ 0 & 1\end{pmatrix}$$

(iii-2) For the case $l < m$,

$$A = \begin{pmatrix} \frac{(m+b-l+1,l)}{(m-l+1,l)} & 0 \\ \frac{(l+1,m-l+1)}{(l-m-b,m-l+1)} & \frac{(m-l+1,l+1)}{(m-l+b,l+1)} \end{pmatrix}, \ A^{-1} = \begin{pmatrix} \frac{(m-l+1,l)}{(m+b-l+1,l)} & 0 \\ \frac{(l+1,m-l)}{(l-m-b+1,m-l)} & \frac{(m-l+b,l+1)}{(m-l+1,l+1)} \end{pmatrix},$$

and for the case $l = m$,

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{m+1}{b} & \frac{(1,m+1)}{(b,m+1)} \end{pmatrix}, \ A^{-1} = \begin{pmatrix} \frac{(1,m)}{(b+1,m)} & -\frac{b}{m+1} \\ 1 & 0 \end{pmatrix}$$

(iv-1) $|A| = \dfrac{(-1)^n(1-a-b+n)\Gamma(a)\Gamma(b)}{n!^2\Gamma(a-n)\Gamma(b-n)}$

(iv-2) $|A| = -\dfrac{(-1)^n(1-a-b-n)\Gamma(a+n)\Gamma(b+n)}{n!^2\Gamma(a)\Gamma(b)}$

(v-1) $|A| = \dfrac{(-1)^{m+n+1}\Gamma(m)^2\Gamma(a-m+1)\Gamma(1-a+n)}{n!^2\Gamma(m-a)\Gamma(a-n)}$

(v-2) $|A| = \dfrac{(-1)^{m+n+1}\Gamma(m)^2\Gamma(1-a)\Gamma(a-m+n+1)}{n!^2\Gamma(a)\Gamma(m-n-a)}$

(v-3) $|A| = \dfrac{(-1)^{m+n+1}\Gamma(m+2)^2\Gamma(a)\Gamma(n-a-m)}{n!^2\Gamma(1-a)\Gamma(a+m-n+1)}$

(v-4) $|A| = \dfrac{(-1)^{m+n+1}\Gamma(m+2)^2\Gamma(-a-m)\Gamma(a+n)}{n!^2\Gamma(a+m+1)\Gamma(1-a-n)}$

(vi-1) $|A| = \dfrac{\Gamma(m)(n+1,l-1)}{(-n,n)\Gamma(l)\Gamma(m-l-n)(m-l,l-1)}$

(vi-2) $|A| = -\dfrac{\Gamma(m)(n+1,l-n-1)}{(-n,n)\Gamma(m-l)\Gamma(l-n)(m-l+n,l-n-1)}$

(vi-3) $|A| = -\dfrac{\Gamma(m+2)(n+1,l)}{(-n,n)\Gamma(l+1)\Gamma(m-l-n+1)(m-l+1,l)}$

(vi-4) $|A| = \dfrac{\Gamma(m+2)(n+1,l-n)}{(-n,n)\Gamma(m-l+1)\Gamma(l-n+1)(m-l+n+1,l-n)}$

(vii-1) For the case $l < n-1$,

$$A = \begin{pmatrix} 0 & \frac{(n-l,l)}{(c,l)} \\ \frac{(c-1,l+1)}{(n-l,l+1)} & \frac{(l+1,n-l-1)}{(2-c,n-l-1)} \end{pmatrix}, \ A^{-1} = \begin{pmatrix} \frac{(l+1,n-l)}{(1-c,n-l)} & \frac{(n-l,l+1)}{(c-1,l+1)} \\ \frac{(c,l)}{(n-l,l)} & 0 \end{pmatrix},$$

and for the case $l = n-1$,

$$A = \begin{pmatrix} \frac{1-c}{n} & \frac{(1,n-1)}{(c,n-1)} \\ 0 & 1 \end{pmatrix}, \ A^{-1} = \begin{pmatrix} \frac{n}{1-c} & \frac{(1,n)}{(c-1,n)} \\ 0 & 1 \end{pmatrix}$$

(vii-2) For the case $l < n-1$,

$$A = \begin{pmatrix} \frac{(n-l,l)}{(c,l)} & 0 \\ \frac{(l+1,n-l-1)}{(2-c,n-l-1)} & \frac{(c-1,l+1)}{(n-l,l+1)} \end{pmatrix}, \ A^{-1} = \begin{pmatrix} \frac{(c,l)}{(n-l,l)} & 0 \\ \frac{(l+1,n-l)}{(1-c,n-l)} & \frac{(n-l,l+1)}{(c-1,l+1)} \end{pmatrix},$$

and for the case $l = n-1$,

$$A = \begin{pmatrix} \frac{(1,n-1)}{(c,n-1)} & \frac{1-c}{n} \\ 1 & 0 \end{pmatrix},\ A^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{n}{1-c} & \frac{(1,n)}{(c-1,n)} \end{pmatrix}$$

(viii-1) $|A| = \dfrac{(n-l,l)(1,l)(1-m,m-1)}{(m,l)(n-m-l+1,m+l)}$

(viii-2) $|A| = -\dfrac{(n-l,l)(1,l)(1-m,m-1)}{(m,l)(n-m-l+1,m+l)}$

(viii-3) $|A| = -\dfrac{(l+1,n-l-1)(1,l-m-1)(-m-1,m+1)}{(m+2,n-l-1)(n-l,l+1)}$

(viii-4) $|A| = \dfrac{(l+1,n-l-1)(1,l-m-1)(-m-1,m+1)}{(m+2,n-l-1)(n-l,l+1)}$

(ix-1) For the case $l < m-1$,

$$A = \begin{pmatrix} 0 & \frac{(m-l,n+l-m)}{(m,n+l-m)} \\ \frac{(n+1,l-1)}{(m-l,l-1)} & 0 \end{pmatrix},\ A^{-1} = \begin{pmatrix} 0 & \frac{(m-l,l-1)}{(n+1,l-1)} \\ \frac{(m,n+l-m)}{(m-l,n+l-m)} & 0 \end{pmatrix},$$

and for the case $l = m-1$,

$$A = \begin{pmatrix} -\frac{m-1}{n} & \frac{(1,n-1)}{(m,n-1)} \\ 0 & 1 \end{pmatrix},\ A^{-1} = \begin{pmatrix} -\frac{n}{m-1} & \frac{(1,n)}{(m-1,n)} \\ 0 & 1 \end{pmatrix}$$

(ix-2) For the case $l = 1$,

$$A = \begin{pmatrix} \frac{(1,n-1)}{(m,n-1)} & -\frac{m-1}{n} \\ 1 & 0 \end{pmatrix},\ A^{-1} = \begin{pmatrix} 0 & 1 \\ -\frac{n}{m-1} & \frac{(1,n)}{(m-1,n)} \end{pmatrix},$$

for the case $1 < l < m-1$,

$$A = \begin{pmatrix} \frac{(l,n-l)}{(m,n-l)} & 0 \\ \frac{(n-l+1,l-1)}{(2-m,l-1)} & \frac{(n+1,m-l-1)}{(l,m-l-1)} \end{pmatrix},\ A^{-1} = \begin{pmatrix} \frac{(n,m-l)}{(l,m-l)} & 0 \\ \frac{(n-l+1,l)}{(1-m,l)} & \frac{(l,n-l+1)}{(m-1,n-l+1)} \end{pmatrix},$$

and for the case $l = m-1$,

$$A = \begin{pmatrix} \frac{m-1}{n} & 0 \\ 0 & 1 \end{pmatrix},\ A^{-1} = \begin{pmatrix} \frac{n}{m-1} & 0 \\ 0 & 1 \end{pmatrix}$$

(ix-3) For the case $m-l = n-1 = 0$,

$$A = \begin{pmatrix} m+1 & 1 \\ 0 & 1 \end{pmatrix},\ A^{-1} = \begin{pmatrix} \frac{1}{m+1} & -\frac{1}{m+1} \\ 0 & 1 \end{pmatrix},$$

for the case $0 = m-l < n-1$,

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{m+1}{n} & \frac{(1,n-1)}{(m+2,n-1)} \end{pmatrix},\ A^{-1} = \begin{pmatrix} \frac{(1,n)}{(m+1,n)} & -\frac{n}{m+1} \\ 1 & 0 \end{pmatrix},$$

for the case $0 < m - l = n - 1$,

$$A = \begin{pmatrix} \frac{n+l}{n} & 0 \\ 0 & 1 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} \frac{n}{n+l} & 0 \\ 0 & 1 \end{pmatrix},$$

and for the case $0 < m - l < n - 1$,

$$A = \begin{pmatrix} \frac{(n+1,l)}{(m-l+1,l)} & 0 \\ \frac{(l+1,m-l+1)}{(-n,m-l+1)} & \frac{(m-l+1,n-m+l-1)}{(m+2,n-m+l-1)} \end{pmatrix}, \; A^{-1} = \begin{pmatrix} \frac{(m-l+1,l)}{(n+1,l)} & 0 \\ \frac{(l+1,m-l)}{(1-n,m-l)} & \frac{(n,l+1)}{(m-l+1,l+1)} \end{pmatrix}$$

(ix-4) For the case $l < m, l < n - 1$,

$$A = \begin{pmatrix} \frac{(n-l,l)}{(-m,l)} & 0 \\ \frac{(l+1,n-l-1)}{(m+2,n-l-1)} & \frac{(m-l+1,l+1)}{(-n,l+1)} \end{pmatrix}, \; A^{-1} = \begin{pmatrix} \frac{(-m,l)}{(n-l,l)} & 0 \\ \frac{(l+1,n-l)}{(m+1,n-l)} & \frac{(n-l,l+1)}{(-m-1,l+1)} \end{pmatrix},$$

for the case $l < m, l = n - 1$,

$$A = \begin{pmatrix} \frac{(1,n-1)}{(-m,n-1)} & \frac{m+1}{n} \\ 1 & 0 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{n}{m+1} & \frac{(1,n)}{(-m-1,n)} \end{pmatrix},$$

for the case $l = m < n - 1$,

$$A = \begin{pmatrix} 0 & 1 \\ \frac{m+1}{n} & \frac{(1,m+1)}{(-n,m+1)} \end{pmatrix}, \; A^{-1} = \begin{pmatrix} \frac{(1,m)}{(1-n,m)} & \frac{n}{m+1} \\ 1 & 0 \end{pmatrix},$$

and for the case $l = m = n - 1$,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Table 2: For relations in Theorem 5.1.

(i) $|A| = \dfrac{b-a}{c-1}$

(ii-1) $|A| = \dfrac{n!(-n,n)\Gamma(1-a)\Gamma(c-a)}{(c-1)\Gamma(1-a+n)\Gamma(c-a+n)}$

(ii-2) $|A| = -\dfrac{n!(-n,n)\Gamma(1-b)\Gamma(c-b)}{(c-1)\Gamma(1-b+n)\Gamma(c-b+n)}$

(iii-1) For the case $l < n - 1$,

$$A = \begin{pmatrix} \frac{(l+1,n-l)}{(1-c,n-l)} & \frac{(n-l,l+1)}{(c-1,l+1)} \\ \frac{(c,l)}{(n-l,l)} & 0 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} 0 & \frac{(n-l,l)}{(c,l)} \\ \frac{(c-1,l+1)}{(n-l,l+1)} & \frac{(l+1,n-l-1)}{(2-c,n-l-1)} \end{pmatrix},$$

and for the case $l = n - 1$,

$$A = \begin{pmatrix} \frac{n}{1-c} & \frac{(1,n)}{(c-1,n)} \\ 0 & 1 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} \frac{1-c}{n} & \frac{(1,n-1)}{(c,n-1)} \\ 0 & 1 \end{pmatrix}$$

(iii-2) For the case $l < n-1$,

$$A = \begin{pmatrix} \frac{(c,l)}{(n-l,l)} & 0 \\ \frac{(l+1,n-l)}{(1-c,n-l)} & \frac{(n-l,l+1)}{(c-1,l+1)} \end{pmatrix}, \ A^{-1} = \begin{pmatrix} \frac{(n-l,l)}{(c,l)} & 0 \\ \frac{(l+1,n-l-1)}{(2-c,n-l-1)} & \frac{(c-1,l+1)}{(n-l,l+1)} \end{pmatrix},$$

and for the case $l = n-1$,

$$A = \begin{pmatrix} 0 & 1 \\ \frac{n}{1-c} & \frac{(1,n)}{(c-1,n)} \end{pmatrix}, \ A^{-1} = \begin{pmatrix} \frac{(1,n-1)}{(c,n-1)} & \frac{1-c}{n} \\ 1 & 0 \end{pmatrix}$$

(iv-1) $|A| = \dfrac{(a-b)\Gamma(a)\Gamma(b)}{(1-m,m-1)(m-1)!\Gamma(a-m+1)\Gamma(b-m+1)}$

(iv-2) $|A| = \dfrac{(b-a)\Gamma(a+m+1)\Gamma(b+m+1)}{(m+1)!^2\Gamma(a)\Gamma(b)}$

(v-1) $|A| = -\dfrac{n!^2\Gamma(a-n)\Gamma(m-a)}{(m-1)!^2\Gamma(a-m+1)\Gamma(1-a+n)}$

(v-2) $|A| = \dfrac{n!^2\Gamma(1-a)\Gamma(a+m-n+1)}{(m+1)!^2\Gamma(a)\Gamma(n-a-m)}$

(v-3) $|A| = \dfrac{n!^2\Gamma(b-n)\Gamma(m-b)}{(m-1)!^2\Gamma(b-m+1)\Gamma(1-b+n)}$

(v-4) $|A| = -\dfrac{n!^2\Gamma(1-b)\Gamma(b+m-n+1)}{(m+1)!^2\Gamma(b)\Gamma(n-b-m)}$

(vi-1) $|A| = \dfrac{(n-l-m+1,m+l)(m,l)}{(m-1)!l!(n-l,l)}$

(vi-2) $|A| = -\dfrac{l!(l-m,n-l+m+1)}{(m+1)!^2(n-l-1)!(n-l+m+1,l-m-1)}$

(vi-3) $|A| = -\dfrac{(n-l-m+1,m+l)(m,l)}{(m-1)!l!(n-l,l)}$

(vi-4) $|A| = \dfrac{l!(l-m,n-l+m+1)}{(m+1)!^2(n-l-1)!(n-l+m+1,l-m-1)}$

(vii-1) For the case $l < m-1$,

$$A = \begin{pmatrix} 0 & \frac{(m-l,l-1)}{(a-l+1,l-1)} \\ \frac{(a-l,l)}{(m-l,l)} & \frac{(l,m-l-1)}{(l-a+1,m-l-1)} \end{pmatrix}, \ A^{-1} = \begin{pmatrix} \frac{(l,m-l)}{(l-a,m-l)} & \frac{(m-l,l)}{(a-l,l)} \\ \frac{(a-l+1,l-1)}{(m-l,l-1)} & 0 \end{pmatrix},$$

and for the case $l = m-1$,

$$A = \begin{pmatrix} \frac{m-a-1}{m-1} & \frac{(1,m-2)}{(a-m+2,m-2)} \\ 0 & 1 \end{pmatrix}, \ A^{-1} = \begin{pmatrix} \frac{m-1}{m-a-1} & \frac{(1,m-1)}{(a-m+1,m-1)} \\ 0 & 1 \end{pmatrix}$$

(vii-2) For the case $l < m$,

$$A = \begin{pmatrix} \frac{(-m,l)}{(b,l)} & 0 \\ \frac{(l+1,m-l)}{(b+l+1,m-l)} & \frac{(b,l+1)}{(-m-1,l+1)} \end{pmatrix}, \ A^{-1} = \begin{pmatrix} \frac{(b,l)}{(-m,l)} & 0 \\ \frac{(l+1,m-l+1)}{(b+l,m-l+1)} & \frac{(-m-1,l+1)}{(b,l+1)} \end{pmatrix},$$

and for the case $l = m$,

$$A = \begin{pmatrix} \frac{(1,m)}{(1-b-m,m)} & \frac{b+m}{m+1} \\ 1 & 0 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{m+1}{b+m} & \frac{(1,m+1)}{(-b-m,m+1)} \end{pmatrix}$$

(viii-1) $|A| = \dfrac{(l-1)!n!(m-l,l-1)}{(m-n-l,n+l)(n+1,l-1)}$

(viii-2) $|A| = -\dfrac{n!l!(m-l+1,l)}{(m-n-l+1,n+l+1)(n+1,l)}$

(viii-3) $|A| = -\dfrac{(l-1)!n!(m-l,l-1)}{(m-n-l,n+l)(n+1,l-1)}$

(viii-4) $|A| = \dfrac{n!l!(m-l+1,l)}{(m-n-l+1,n+l+1)(n+1,l)}$

(ix-1) For the case $l < n-1$,

$$A = \begin{pmatrix} 0 & \frac{(n-l+1,m-n+l-1)}{(n+1,m-n+l-1)} \\ \frac{(m,l)}{(n-l,l)} & 0 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} 0 & \frac{(n-l,l)}{(m,l)} \\ \frac{(n+1,m-n+l-1)}{(n-l+1,m-n+l-1)} & 0 \end{pmatrix},$$

and for the case $l = n-1$,

$$A = \begin{pmatrix} -\frac{n}{m-1} & \frac{(1,m-2)}{(n+1,m-2)} \\ 0 & 1 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} -\frac{m-1}{n} & \frac{(1,m-1)}{(n,m-1)} \\ 0 & 1 \end{pmatrix}$$

(ix-2) For the case $l = 0$,

$$A = \begin{pmatrix} \frac{(1,m)}{(n+1,m)} & -\frac{n}{m+1} \\ 1 & 0 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} 0 & 1 \\ -\frac{m+1}{n} & \frac{(1,m+1)}{(n,m+1)} \end{pmatrix},$$

for the case $0 < l < n-1$,

$$A = \begin{pmatrix} \frac{(l+1,m-l)}{(n+1,m-l)} & 0 \\ \frac{(m-l+1,l)}{(1-n,l)} & \frac{(m+2,n-l-1)}{(l+1,n-l-1)} \end{pmatrix}, \; A^{-1} = \begin{pmatrix} \frac{(m+1,n-l)}{(l+1,n-l)} & 0 \\ \frac{(m-l+1,l+1)}{(-n,l+1)} & \frac{(l+1,m-l+1)}{(n,m-l+1)} \end{pmatrix},$$

and for the case $0 < l = n-1$,

$$A = \begin{pmatrix} \frac{n}{m+1} & 0 \\ 0 & 1 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} \frac{m+1}{n} & 0 \\ 0 & 1 \end{pmatrix}$$

(ix-3) For the case $1 = n-l = m-1$,

$$A = \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix}, \; A^{-1} = \begin{pmatrix} \frac{1}{n} & -\frac{1}{n} \\ 0 & 1 \end{pmatrix},$$

for the case $1 = n-l < m-1$,

$$A = \begin{pmatrix} 0 & 1 \\ \frac{n}{1-m} & \frac{(1,m-2)}{(n+1,m-2)} \end{pmatrix}, \; A^{-1} = \begin{pmatrix} \frac{(1,m-1)}{(n,m-1)} & \frac{1-m}{n} \\ 1 & 0 \end{pmatrix},$$

for the case $1 < n - l < m - 1$,

$$A = \begin{pmatrix} \frac{(m,l)}{(n-l,l)} & 0 \\ \frac{(l+1,n-l)}{(1-m,n-l)} & \frac{(n-l,m-n+l-1)}{(n+1,m-n+l-1)} \end{pmatrix},\ A^{-1} = \begin{pmatrix} \frac{(n-l,l)}{(m,l)} & 0 \\ \frac{(l+1,n-l-1)}{(2-m,n-l-1)} & \frac{(m-1,l+1)}{(n-l,l+1)} \end{pmatrix},$$

and for the case $1 < n - l = m - 1$,

$$A = \begin{pmatrix} \frac{n}{m-1} & 0 \\ 0 & 1 \end{pmatrix},\ A^{-1} = \begin{pmatrix} \frac{m-1}{n} & 0 \\ 0 & 1 \end{pmatrix}$$

(ix-4) For the case $l < n - 1, l < m$,

$$A = \begin{pmatrix} \frac{(m-l+1,l)}{(1-n,l)} & 0 \\ \frac{(l+1,m-l)}{(n+1,m-l)} & \frac{(n-l,l+1)}{(-m-1,l+1)} \end{pmatrix},\ A^{-1} = \begin{pmatrix} \frac{(n-l,l)}{(-m,l)} & 0 \\ \frac{(l+1,m-l+1)}{(n,m-l+1)} & \frac{(m-l+1,l+1)}{(-n,l+1)} \end{pmatrix},$$

for the case $l < n - 1, l = m$,

$$A = \begin{pmatrix} \frac{(1,m)}{(1-n,m)} & \frac{n}{m+1} \\ 1 & 0 \end{pmatrix},\ A^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{m+1}{n} & \frac{(1,m+1)}{(-n,m+1)} \end{pmatrix},$$

for the case $l = n - 1 < m$,

$$A = \begin{pmatrix} 0 & 1 \\ \frac{n}{m+1} & \frac{(1,n)}{(-m-1,n)} \end{pmatrix},\ A^{-1} = \begin{pmatrix} \frac{(1,n-1)}{(-m,n-1)} & \frac{m+1}{n} \\ 1 & 0 \end{pmatrix},$$

and for the case $l = n - 1 = m$,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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