Kumamoto J. Math.
Vol.**36** (2023), 1–16

The best constant of the Sobolev inequality corresponding to $2M$-th order differential equation with periodic boundary conditions

Hiroyuki Yamagishi and Yoshinori Kametaka

(Received March 21, 2022)
(Accepted November 1, 2022)

Abstract. The Sobolev inequality shows that the supremum of a function is estimated from above by a constant multiple of the potential energy. We have found the best constant and function, which attain the equality. In the background, there is a $2M$-th order linear ordinary differential equation $(-D^2+a_0^2)\cdots(-D^2+a_{M-1}^2)u = f(x)$ on an interval $(0, L)$ with periodic boundary conditions, where $D = d/dx$, $M = 1, 2, 3, \cdots$ and $0 < a_0 < \cdots < a_{M-1}$. The solution u is expressed by using the Green function $G(x-y)$. The best constant of the Sobolev inequality is the diagonal value of the Green function $G(0)$. The equality of the Sobolev inequality holds for the Green function $G(x-y_0)$ where any fixed $y_0 \in [0, L]$.

1 Introduction

We prepare some notations which are used in this paper. Let $M = 1, 2, 3, \cdots$ and $L > 0$. We define hyperbolic functions $\mathrm{ch}(x) = \cosh(x)$, $\mathrm{sh}(x) = \sinh(x)$ and $\mathrm{th}(x) = \tanh(x)$ for short. The eigenvalue problem

$$\text{EVP} \quad \begin{cases} (-1)^M \varphi_j^{(2M)} = \lambda_j \varphi_j & (0 < x < L) \\ \varphi_j^{(i)}(L) - \varphi_j^{(i)}(0) = 0 & (0 \le i \le 2M-1) \end{cases} \qquad (j \in \mathbf{Z})$$

has countably many eigenvalues

$$\lambda_j = \left(\frac{\omega_j}{L}\right)^{2M}, \qquad \omega_j = 2\pi j \qquad (j \in \mathbf{Z})$$

and corresponds to eigenfunctions

$$\varphi_j(x) = \frac{1}{\sqrt{L}} e^{\sqrt{-1}\frac{\omega_j}{L}x} \qquad (j \in \mathbf{Z},\ x \in \mathbf{R}).$$

Mathematical Subject Classification (2020): Primary 46E35; Secondary 34B27
Key words: Sobolev inequality, Green function, reproducing kernel

As is well-known, the system of eigenfunctions $\{\varphi_j(x)\,|\,j\in\mathbf{Z}\}$ is a C.O.N.S. in $L^2(0,L)$. $\varphi_j(x)$ satisfies $\varphi_j(x+L)=\varphi_j(x)$, $\varphi_j(-x)=\varphi_{-j}(x)$ and the additional formula

$$\varphi_j(x)\varphi_j(y)=\frac{1}{\sqrt{L}}\varphi_j(x+y)\qquad (j\in\mathbf{Z},\ 0<x,y<L).$$

For any $u\in\{u\in L^2(0,L)\,|\,u(L+x)=u(x)\}$, we introduce the Fourier transform $\widehat{\ }$ defined by

$$u(x)=\sum_{k\in\mathbf{Z}}\widehat{u}(k)\,\varphi_k(x)\qquad \stackrel{\widehat{\ }}{\longrightarrow}\qquad \widehat{u}(k)=\int_0^L u(x)\,\overline{\varphi}_k(x)\,dx. \tag{1.1}$$

We define the convolution:

$$(u*v)(x)=\int_0^L u(x-y)\,v(y)\,dy.$$

We introduce the characteristic polynomial:

$$P(z)=\prod_{j=0}^{M-1}(z+a_j^2)=\sum_{j=0}^{M}p_{M-j}\,z^j, \tag{1.2}$$
$$0<a_0<a_1<\cdots<a_{M-1},$$
$$p_0=1,\qquad p_1=a_0^2+a_1^2+\cdots+a_{M-1}^2,\qquad \cdots,\qquad p_M=a_0^2a_1^2\cdots a_{M-1}^2.$$

We put $D=d/dx$. We consider the periodic boundary value problem for a $2M$-th order linear ordinary differential operator $P(-D^2)$:

$$\text{BVP}(M)\quad \begin{cases} P(-D^2)u=f(x) & (0<x<L),\\ u^{(i)}(L)-u^{(i)}(0)=0 & (0\le i\le 2M-1).\end{cases}$$

Concerning the uniqueness and existence of the solution to BVP(M), we have the following theorem.

Theorem 1.1. For any bounded continuous function $f(x)$ $(0<x<L)$, BVP(M) has a unique solution $u(x)$ expressed as

$$u(x)=\int_0^L G(x-y)\,f(y)\,dy\qquad (0<x<L), \tag{1.3}$$

where $G(x-y)$ $(0<x,y<L)$ is the Green function. Using the function

$$G_j(x)=\frac{\operatorname{ch}(\,a_j(|x|-L/2)\,)}{2a_j\operatorname{sh}(a_jL/2)}\qquad (0\le j\le M-1,\ -L<x<L),$$

$G(x)$ $(-L < x < L)$ has three equivalent expressions:

$$G(x) = \begin{cases} G_0(x) & (M=1), \\ (-1)^{M-1} \left| \begin{array}{c} a_j^{2i} \\ \hline G_j(x) \end{array} \right| \Big/ \left| \begin{array}{c} a_j^{2i} \end{array} \right| & (M=2,3,4,\cdots), \end{cases} \tag{1.4}$$

$$G(x) = \sum_{j=0}^{M-1} \frac{1}{P'(-a_j^2)} G_j(x), \tag{1.5}$$

$$G(x) = (G_0 * \cdots * G_{M-1})(x). \tag{1.6}$$

In the numerator of (1.4), i and j satisfy $0 \le i \le M-2$ and $0 \le j \le M-1$, respectively. In the denominator of (1.4), i and j satisfy $0 \le i, j \le M-1$.

The concrete forms of $G(x) = G(M;x)$ $(M=2,3,4)$ in (1.4) are as follows:

$$G(2;x) = -\left| \begin{array}{cc} 1 & 1 \\ \hline G_0(x) & G_1(x) \end{array} \right| \Big/ \left| \begin{array}{cc} 1 & 1 \\ a_0^2 & a_1^2 \end{array} \right|,$$

$$G(3;x) = \left| \begin{array}{ccc} 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 \\ \hline G_0(x) & G_1(x) & G_2(x) \end{array} \right| \Big/ \left| \begin{array}{ccc} 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 \\ a_0^4 & a_1^4 & a_2^4 \end{array} \right|,$$

$$G(4;x) = -\left| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^4 & a_1^4 & a_2^4 & a_3^4 \\ \hline G_0(x) & G_1(x) & G_2(x) & G_3(x) \end{array} \right| \Big/ \left| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^4 & a_1^4 & a_2^4 & a_3^4 \\ a_0^6 & a_1^6 & a_2^6 & a_3^6 \end{array} \right|.$$

The graphs of Green functions $G(x-y) = G(M;x-y)$ $(M=1,2,3,4,\ 0<x,y<1)$ are shown as Figure 1.

We introduce the Sobolev space

$$H = \left\{ u \,\middle|\, u,\ u^{(M)} \in L^2(0,L), \quad u^{(i)}(L) - u^{(i)}(0) = 0 \quad (0 \le i \le M-1) \right\},$$

the Sobolev inner product

$$(u,v)_H = \int_0^L \sum_{j=0}^{M} p_{M-j} u^{(j)}(x)\, \overline{v}^{(j)}(x)\, dx$$

and the Sobolev energy

$$\| u \|_H^2 = (u,u)_H = \int_0^L \sum_{j=0}^{M} p_{M-j} \left| u^{(j)}(x) \right|^2 dx.$$

H is a Hilbert space with inner product $(\cdot,\cdot)_H$.

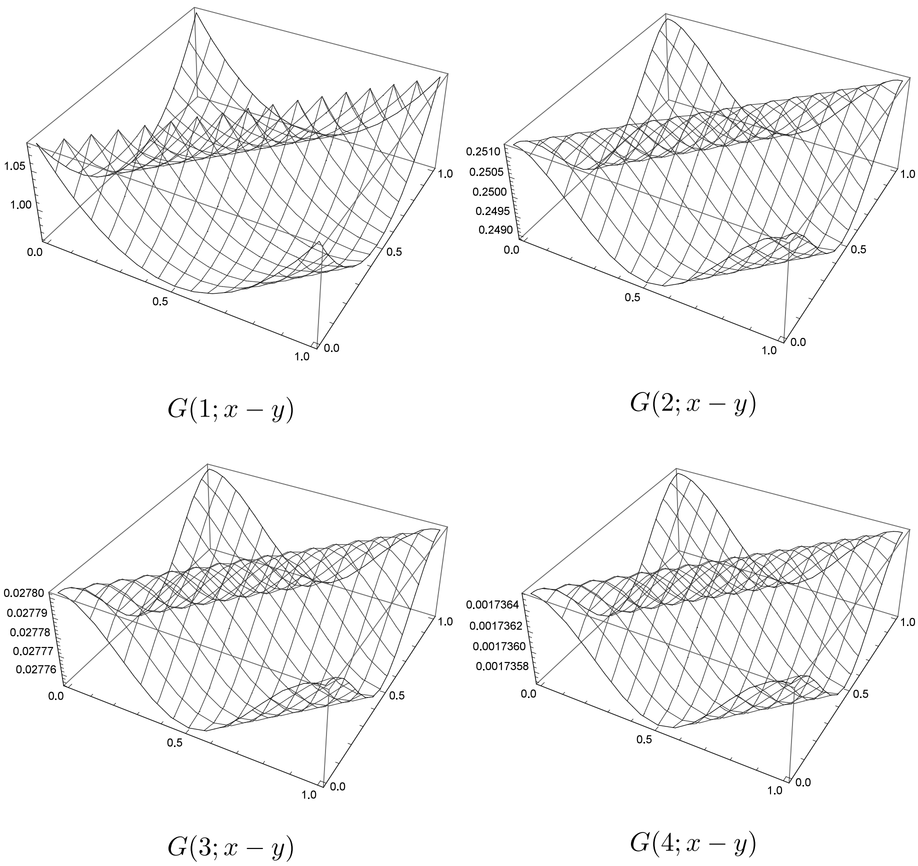

Figure 1: Green functions $G(M; x-y)$ ($a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4,\ 0 < x, y < 1$).

Theorem 1.2. *There exists a positive constant C such that for any $u \in H$ the Sobolev inequality*

$$\left(\sup_{0 \le y \le L} |\, u(y)\,| \right)^2 \le C \|\, u \,\|_H^2 \tag{1.7}$$

holds. Among such C the best constant C_0 is

$$C_0 = G(0) =$$
$$\begin{cases} \dfrac{1}{2a_0 \operatorname{th}(a_0 L/2)} & (M=1), \\[2ex] \dfrac{(-1)^{M-1}}{2a_0 \cdots a_{M-1}} \left| \dfrac{a_j^{2i+1}}{(\operatorname{th}(a_j L/2))^{-1}} \right| \Big/ \left| \; a_j^{2i} \; \right| & (M = 2,3,4,\cdots). \end{cases} \tag{1.8}$$

In the numerator of (1.8) ($M = 2,3,4,\cdots$), i and j satisfy $0 \le i \le M-2$ and $0 \le j \le M-1$, respectively. In the denominator of (1.8) ($M = 2,3,4,\cdots$), i and j satisfy $0 \le i,j \le M-1$. If we replace C for C_0 in the above inequality (1.7), the equality holds if and only if the constant multiple of $u(x) = G(x-y_0)$ $(0 < x < L)$, where $y_0 \in [0,L]$ is an arbitrary real number.

The concrete forms of $C_0 = G(M;0)$ $(M = 2,3,4)$ in (1.8) are as follows:

$$G(2;0) = -\frac{1}{2a_0a_1} \begin{vmatrix} a_0 & a_1 \\ \hline (\operatorname{th}(a_0L/2))^{-1} & (\operatorname{th}(a_1L/2))^{-1} \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 \\ a_0^2 & a_1^2 \end{vmatrix},$$

$$G(3;0) = \frac{1}{2a_0a_1a_2} \begin{vmatrix} a_0 & a_1 & a_2 \\ a_0^3 & a_1^3 & a_2^3 \\ \hline (\operatorname{th}(a_0L/2))^{-1} & (\operatorname{th}(a_1L/2))^{-1} & (\operatorname{th}(a_2L/2))^{-1} \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 \\ a_0^4 & a_1^4 & a_2^4 \end{vmatrix},$$

$$G(4;0) = -\frac{1}{2a_0a_1a_2a_3} \times \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 \\ a_0^5 & a_1^5 & a_2^5 & a_3^5 \\ \hline (\operatorname{th}(a_0L/2))^{-1} & (\operatorname{th}(a_1L/2))^{-1} & (\operatorname{th}(a_2L/2))^{-1} & (\operatorname{th}(a_3L/2))^{-1} \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^4 & a_1^4 & a_2^4 & a_3^4 \\ a_0^6 & a_1^6 & a_2^6 & a_3^6 \end{vmatrix}.$$

The engineering meaning of the Sobolev inequality (1.7) is that the square of the maximum bending of a string ($M = 1$) or a beam ($M = 2$) is estimated from above by a constant multiple of the potential energy. The best constant is the maximum of the diagonal value of the Green function associated with bending problem of a string or a beam.

In our previous study, we have the best constants of Sobolev inequalities corresponding to some differential equations with boundary conditions. The bending of a string on an interval $(-D^2+a_0^2)u=f(x)$ $(x\in(0,L))$ with Dirichlet [13], Neumann [13] and periodic [5, 13] boundary conditions, and its discrete version [16, 17] are given. The bending of a beam on a half line $(-D^2+a_0^2)(-D^2+a_1^2)u=f(x)$ $(x\in(0,\infty))$ with clamped, Dirichlet, Neumann and free boundary conditions [6, 8, 11] are given. The bending of a beam on an interval $(-D^2+a_0^2)(-D^2+a_1^2)u=f(x)$ $(x\in(0,L))$ with clamped [15], Dirichlet [13, 15], Neumann [13, 15], free [15] and periodic [13] boundary conditions are given. $2M$-th order differential equation on a whole line $P(-D^2)u=f(x)$ $(x\in\mathbf{R})$ [1] and on an N dimensional Euclidean space $P(-\Delta)u=f(x)$ $(x\in\mathbf{R}^N)$ [3, 4] are given. $2M$-th order simple type $(a_0=\cdots=a_{M-1}=0)$ differential equation on an interval $(-1)^M D^{2M}u=f(x)$ $(x\in(0,L))$ with clamped [12], Dirichlet [14], Neumann [14], free [10] and periodic [14] (the discrete version [7]) boundary conditions are given. Moreover, we have the best constants of Sobolev-type inequalities corresponding to M-th order differential equations $P(D)u=f(x)$ $(x\in\mathbf{R})$ [2] and $P(D)u=f(x)$ $(x\in(0,1))$ with periodic boundary conditions [9]. In this paper, we have the best constant of the Sobolev inequality corresponding to a $2M$-th order differential equation on an interval $P(-D^2)u=f(x)$ $(x\in(0,L))$ with periodic boundary conditions.

This paper is organized as follows. In section 2, we explain the Fourier transform. The section 3 is devoted to the proof of Theorem 1.1. The section 4 is devoted to the proof of the main Theorem 1.2.

2 Fourier transform

We explain the properties of the Fourier transform defined by (1.1). We introduce the trigger function

$$\{x\}=L\left(\frac{x}{L}-\left\lfloor\frac{x}{L}\right\rfloor\right)\qquad(-\infty<x<\infty),$$

where $\lfloor x\rfloor$ is an integer part of a real number x defined by

$$\lfloor x\rfloor=\sup\{n\in\mathbf{Z}\,|\,n\leq x\,\}. \tag{2.1}$$

Hence, $\{x\}$ is a periodic function of x with period L as Figure 2.

Lemma 2.1. Several important properties of the Fourier transform (1.1) are listed as follows.

$$D^i u(x) \qquad \overset{\frown}{\longrightarrow} \qquad \left(\sqrt{-1}\,\frac{\omega_k}{L}\right)^i \widehat{u}(k), \tag{2.2}$$

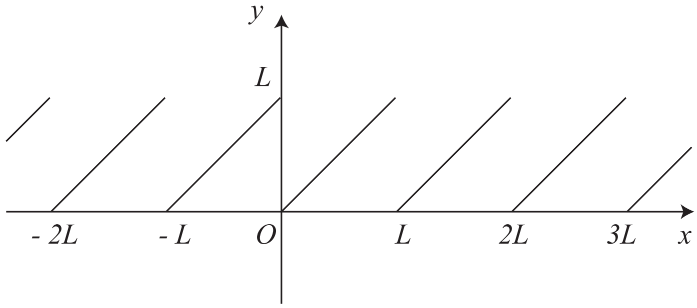

Figure 2: Trigger function $\{x\}$ $(-\infty < x < \infty)$.

where $u(x)$ satisfies $u^{(i-1)}(L) - u^{(i-1)}(0) = 0 \quad (i = 1, 2, 3, \cdots)$.

$$u(x-y) \quad \stackrel{\frown}{\longrightarrow} \quad e^{-\sqrt{-1}\frac{\omega_k}{L}y}\widehat{u}(k). \tag{2.3}$$

$$u(-x) \quad \stackrel{\frown}{\longrightarrow} \quad \widehat{u}(-k). \tag{2.4}$$

$$\sqrt{L}\frac{1}{1-e^{-a_j L}}\, e^{-a_j\{x\}} \quad \stackrel{\frown}{\longrightarrow} \quad \frac{1}{\sqrt{-1}\frac{\omega_k}{L} + a_j}. \tag{2.5}$$

$$G_j(x) = \frac{\mathrm{ch}(\, a_j(|x| - L/2)\,)}{2a_j\, \mathrm{sh}(a_j L/2)} \quad \stackrel{\frown}{\longrightarrow} \quad \widehat{G}_j(k) = \frac{1}{\sqrt{L}}\frac{1}{\left(\frac{\omega_k}{L}\right)^2 + a_j^2}. \tag{2.6}$$

$$(u*v)(x) = \int_0^L u(x-y)\, v(y)\, dy \quad \stackrel{\frown}{\longrightarrow} \quad (u*v)\widehat{\ }(k) = \sqrt{L}\,\widehat{u}(k)\,\widehat{v}(k). \tag{2.7}$$

$$\int_0^L u(x)\,\overline{v}(x)\, dx = \sum_{k\in\mathbf{Z}} \widehat{u}(k)\,\overline{\widehat{v}}(k). \tag{2.8}$$

$$\int_0^L |u(x)|^2\, dx = \sum_{k\in\mathbf{Z}} |\widehat{u}(k)|^2\,. \tag{2.9}$$

(2.9) is the Parseval equality.

Proof of Lemma 2.1 (2.2) is shown as

$$\begin{aligned}
&D^i u(x) \quad \stackrel{\frown}{\longrightarrow} \quad \int_0^L D^i u(x)\overline{\varphi}_k(x)dx = \int_0^L u^{(i)}(x)\overline{\varphi}_k(x)dx \\
&= u^{(i-1)}(L)\overline{\varphi}_k(L) - u^{(i-1)}(0)\overline{\varphi}_k(0) - \int_0^L u^{(i-1)}(x)\overline{\left(\sqrt{-1}\frac{\omega_k}{L}\right)}\overline{\varphi}_k(x)dx \\
&= \left(\sqrt{-1}\frac{\omega_k}{L}\right)\int_0^L u^{(i-1)}(x)\overline{\varphi}_k(x)dx = \cdots = \left(\sqrt{-1}\frac{\omega_k}{L}\right)^i\int_0^L u(x)\overline{\varphi}_k(x)dx \\
&= \left(\sqrt{-1}\frac{\omega_k}{L}\right)^i \widehat{u}(k),
\end{aligned}$$

where we use $u^{(i-1)}(L)-u^{(i-1)}(0)=0\ \ (i=1,2,3,\cdots)$. (2.3) is shown as

$$
\begin{aligned}
&u(x-y) \quad \stackrel{\frown}{\longrightarrow} \quad \int_0^L u(x-y)\overline{\varphi}_k(x)dx=\int_{-y}^{L-y} u(z)\overline{\varphi}_k(y+z)dz\\
&=\sqrt{L}\left[\int_{-y}^{0} u(z)\overline{\varphi}_k(z)dz+\int_0^{L-y} u(z)\overline{\varphi}_k(z)dz\right]\overline{\varphi}_k(y)\\
&=\sqrt{L}\int_0^L u(z)\overline{\varphi}_k(z)dz\overline{\varphi}_k(y)=\sqrt{L}\,\overline{\varphi}_k(y)\widehat{u}(k)=e^{-\sqrt{-1}\frac{\omega_k}{L}y}\widehat{u}(k),
\end{aligned}
$$

where we use $u(z)=u(z+L),\ \varphi_k(z)=\varphi_k(z+L)$. (2.4) is shown as

$$
\begin{aligned}
&u(-x) \quad \stackrel{\frown}{\longrightarrow} \quad \int_0^L u(-x)\overline{\varphi}_k(x)dx=\int_0^L u(L-x)\overline{\varphi}_k(x)dx\\
&=\int_0^L u(y)\overline{\varphi}_k(L-y)dy=\int_0^L u(y)\overline{\varphi}_k(-y)dy=\int_0^L u(y)\overline{\varphi}_{-k}(y)dy=\widehat{u}(-k).
\end{aligned}
$$

(2.5) is shown as

$$
\begin{aligned}
&\sqrt{L}\,\frac{1}{1-e^{-a_jL}}e^{-a_j\{x\}} \quad \stackrel{\frown}{\longrightarrow} \quad \int_0^L \sqrt{L}\,\frac{1}{1-e^{-a_jL}}e^{-a_j\{x\}}\overline{\varphi}_k(x)dx\\
&=\frac{1}{1-e^{-a_jL}}\int_0^L e^{-(\sqrt{-1}\frac{\omega_k}{L}+a_j)x}dx=\frac{1}{\sqrt{-1}\frac{\omega_k}{L}+a_j}.
\end{aligned}
$$

We consider (2.6). Noting $\{-x\}=L-\{x\}$ and

$$
\begin{aligned}
&\mathrm{ch}(\,a_j(\{-x\}-L/2)\,)=\mathrm{ch}(\,a_j(L-\{x\}-L/2)\,)\\
&=\mathrm{ch}(\,a_j(L/2-\{x\})\,)=\mathrm{ch}(\,a_j(\{x\}-L/2)\,),
\end{aligned}
$$

we have

$$
\mathrm{ch}(\,a_j(\{x\}-L/2)\,)=\mathrm{ch}(\,a_j(|x|-L/2)\,) \qquad (-L<x<L).
$$

(2.6) is shown as

$$
\begin{aligned}
G_j(x)&=\frac{\mathrm{ch}(\,a_j(|x|-L/2)\,)}{2a_j\,\mathrm{sh}(a_jL/2)}=\frac{\mathrm{ch}(\,a_j(\{x\}-L/2)\,)}{2a_j\,\mathrm{sh}(a_jL/2)}\\
&=\frac{1}{2a_j}\frac{1}{1-e^{-a_jL}}\left[e^{-a_j\{x\}}+e^{-a_j(L-\{x\})}\right]\\
&=\frac{1}{\sqrt{L}}\frac{1}{2a_j}\sqrt{L}\frac{1}{1-e^{-a_jL}}\left[e^{-a_j\{x\}}+e^{-a_j\{-x\}}\right]\\
&\stackrel{\frown}{\longrightarrow}\ \frac{1}{\sqrt{L}}\frac{1}{2a_j}\left[\frac{1}{\sqrt{-1}\frac{\omega_k}{L}+a_j}+\frac{1}{\sqrt{-1}\frac{\omega_{-k}}{L}+a_j}\right]\\
&=\frac{1}{\sqrt{L}}\frac{1}{\left(\frac{\omega_k}{L}\right)^2+a_j^2}=\widehat{G}_j(k),
\end{aligned}
$$

where we used (2.4). (2.7) is shown as

$$\begin{aligned}(u*v)(x) &= \int_0^L u(x-y)\,v(y)\,dy = \int_0^L \sum_{k\in\mathbf{Z}} \widehat{u}(k)\varphi_k(x-y)\sum_{\ell\in\mathbf{Z}}\widehat{v}(\ell)\varphi_\ell(y)dy \\ &= \sum_{k\in\mathbf{Z}}\sum_{\ell\in\mathbf{Z}}\sqrt{L}\widehat{u}(k)\widehat{v}(\ell)\varphi_k(x)\int_0^L \overline{\varphi_k}(y)\varphi_\ell(y)dy \\ &= \sum_{k\in\mathbf{Z}}\sqrt{L}\widehat{u}(k)\widehat{v}(k)\varphi_k(x) = \sum_{k\in\mathbf{Z}}(u*v)\widehat{\ }(k)\varphi_k(x).\end{aligned}$$

(2.8) is shown as

$$\begin{aligned}\int_0^L u(x)\,\overline{v}(x)\,dx &= \int_0^L \sum_{k\in\mathbf{Z}}\widehat{u}(k)\varphi_k(x)\overline{\sum_{\ell\in\mathbf{Z}}\widehat{v}(\ell)\varphi_\ell(x)}\,dx \\ &= \sum_{k\in\mathbf{Z}}\sum_{\ell\in\mathbf{Z}}\widehat{u}(k)\overline{\widehat{v}}(\ell)\int_0^L \varphi_k(x)\,\overline{\varphi}_\ell(x)\,dx = \sum_{k\in\mathbf{Z}}\widehat{u}(k)\overline{\widehat{v}}(k).\end{aligned}$$

Applying (2.8) to $v(x)=u(x)$, we have (2.9). Hence we have Lemma 2.1. ■

3 Green function of BVP(M)

In this section, we prove Theorem 1.1.

Proof of Theorem 1.1 ($M=1$) Applying the Fourier transform to BVP(1) and using (2.2), we have

$$\left(\left(\frac{\omega_k}{L}\right)^2 + a_0^2\right)\widehat{u}(k) = \widehat{f}(k) \qquad (k\in\mathbf{Z})$$

$$\Updownarrow$$

$$\widehat{u}(k) = \sqrt{L}\,\widehat{G}_0(k)\widehat{f}(k), \qquad \widehat{G}_0(k) = \frac{1}{\sqrt{L}}\frac{1}{\left(\frac{\omega_k}{L}\right)^2 + a_0^2} \qquad (k\in\mathbf{Z}).$$

Using the inverse Fourier transform for $\widehat{u}(k)$ and using (2.7), we have (1.3). Using the inverse Fourier transform for $\widehat{G}_0(k)$, we have the Green function (1.4) ($M=1$). Since the right-hand side of (1.3) includes only a data function $f(x)$, the solution to BVP(1) is unique.

Using the properties (3.1), (3.2) and (3.3) of Lemma 3.1, we can show that $u(x)$ defined by (1.3) satisfies BVP(1), which guarantees the existence of the solution. ■

Lemma 3.1. $G_j(x-y)$ $(0\le j\le M-1,\ 0<x,y<L)$ *satisfies the following*

properties:

$$(-\partial_x^2 + a_j^2)\, G_j(x-y) = 0 \qquad (0 < x, y < L,\ x \neq y), \tag{3.1}$$

$$\partial_x^i G_j(x-y)\Big|_{x=L} - \partial_x^i G_j(x-y)\Big|_{x=0} = 0 \qquad (i = 0, 1,\ 0 < y < L), \tag{3.2}$$

$$\partial_x^i G_j(x-y)\Big|_{y=x-0} - \partial_x^i G_j(x-y)\Big|_{y=x+0} = \begin{cases} 0 & (i=0) \\ -1 & (i=1) \end{cases} \qquad (0 < x < L), \tag{3.3}$$

$$G_j(x-y) > 0 \qquad (0 < x, y < L). \tag{3.4}$$

Proof of Lemma 3.1 We introduce the signum function $\mathrm{sgn}(x) = 1$ $(0 \le x < \infty), -1$ $(-\infty < x < 0)$. Differentiating $G_j(x-y)$ with respect to x, we have

$$G_j(x-y) = \frac{\mathrm{ch}(\, a_j(|x-y| - L/2)\,)}{2a_j\,\mathrm{sh}(a_jL/2)} > 0 \qquad (0 < x, y < L),$$

$$\partial_x G_j(x-y) = \mathrm{sgn}(x-y)\frac{\mathrm{sh}(\, a_j(|x-y| - L/2)\,)}{2\,\mathrm{sh}(a_jL/2)} \qquad (0 < x, y < L,\ x \neq y),$$

$$\partial_x^2 G_j(x-y) = \frac{a_j\,\mathrm{ch}(\, a_j(|x-y| - L/2)\,)}{2\,\mathrm{sh}(a_jL/2)} = a_j^2 G_j(x-y) \qquad (0 < x, y < L,\ x \neq y).$$

Using these relations, we have (3.1), (3.2), (3.3) and (3.4). ■

Lemma 3.2. *The partial fraction expansion of* $1/P(z)$ *is*

$$\frac{1}{P(z)} = \prod_{j=0}^{M-1} \frac{1}{z + a_j^2} = \sum_{j=0}^{M-1} b_j \frac{1}{z + a_j^2}$$

$$= (-1)^{M-1} \left| \begin{matrix} a_j^{2i} \\ \hline (z + a_j^2)^{-1} \end{matrix} \right|_{0 \le i \le M-2,\ 0 \le j \le M-1} \Bigg/ \left| \quad a_j^{2i} \quad \right|_{0 \le i, j \le M-1}, \tag{3.5}$$

$$b_j = \frac{1}{P'(-a_j^2)} = \prod_{k=0,\ k \neq j}^{M-1} \frac{1}{-a_j^2 + a_k^2}. \tag{3.6}$$

Proof of Lemma 3.2 Multiplying

$$\frac{1}{P(z)} = \sum_{k=0}^{M-1} b_k \frac{1}{z + a_k^2}$$

into $(z + a_j^2)$ and taking the limit as $z \to -a_j^2$, we have (3.6). Using

$$\begin{pmatrix} \\ b_i \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \dfrac{1}{P'(-a_i^2)} \\ \\ \end{pmatrix} = \begin{pmatrix} & & \\ & (-a_j^2)^i & \\ & & \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and

$$
{}^t\boldsymbol{a}\boldsymbol{A}^{-1}\boldsymbol{b} = -\left|\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{b} \\ \hline {}^t\boldsymbol{a} & 0 \end{array}\right| \Big/ \left|\begin{array}{c} \boldsymbol{A} \end{array}\right|,
$$

where an $M \times M$ regular matrix $\boldsymbol{A}$ and $M \times 1$ matrices $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$
\begin{aligned}
\sum_{j=0}^{M-1} b_j (z + a_j^2)^{-1} &= \left(\ (z + a_j^2)^{-1}\ \right) \begin{pmatrix} b_i \end{pmatrix} \\
&= \left(\ (z + a_j^2)^{-1}\ \right) \begin{pmatrix} (-a_j^2)^i \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\
&= -\left|\begin{array}{c|c} (-a_j^2)^i & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \\ \hline (z + a_j^2)^{-1} & 0 \end{array}\right| \Big/ \left|\begin{array}{c} (-a_j^2)^i \end{array}\right| \\
&= \left|\begin{array}{c} (-a_j^2)^i \\ \hline (z + a_j^2)^{-1} \end{array}\right| \Big/ \left|\begin{array}{c} (-a_j^2)^i \end{array}\right| \\
&= \frac{(-1)^{0+1+\cdots+(M-2)}}{(-1)^{0+1+\cdots+(M-2)+(M-1)}} \left|\begin{array}{c} a_j^{2i} \\ \hline (z + a_j^2)^{-1} \end{array}\right| \Big/ \left|\begin{array}{c} a_j^{2i} \end{array}\right|,
\end{aligned}
$$

where i, j satisfy $0 \le i \le M-2,\ 0 \le j \le M-1$ in the numerator and i, j satisfy $0 \le i, j \le M-1$ in the denominator. Thus, we have (3.5). ■

Proof of Theorem 1.1 ($M = 2, 3, 4, \cdots$) Applying the Fourier transform to BVP(M) and using (2.2), we have

$$
\begin{aligned}
& P\left(\left(\frac{\omega_k}{L}\right)^2\right) \widehat{u}(k) = \widehat{f}(k) \qquad (k \in \mathbf{Z}) \\
& \Updownarrow \\
& \widehat{u}(k) = \sqrt{L}\, \widehat{G}(k) \widehat{f}(k), \qquad \widehat{G}(k) = \frac{1}{\sqrt{L}} \frac{1}{P\left(\left(\frac{\omega_k}{L}\right)^2\right)} \qquad (k \in \mathbf{Z}).
\end{aligned} \tag{3.7}
$$

Using the inverse Fourier transform for $\widehat{u}(k)$ and (2.7), we have (1.3). Applying

$\widehat{G}(k)$ to Lemma 3.2, we have

$$\widehat{G}(k) = \frac{1}{\sqrt{L}} \frac{1}{P\Big(\big(\frac{\omega_k}{L}\big)^2\Big)} = \Big(\sqrt{L}\Big)^{M-1} \prod_{j=0}^{M-1} \widehat{G}_j(k) = \sum_{j=0}^{M-1} b_j \widehat{G}_j(k)$$

$$= (-1)^{M-1} \left| \begin{array}{c} a_j^{2i} \\ \hline \widehat{G}_j(k) \end{array} \right|_{0 \le i \le M-2,\ 0 \le j \le M-1} \Bigg/ \left| \begin{array}{c} a_j^{2i} \end{array} \right|_{0 \le i,j \le M-1} .$$

Using the inverse Fourier transform, we have the Green function (1.4), (1.5) and (1.6). Since the right-hand side of (1.3) includes only a data function $f(x)$, the solution to BVP(M) is unique.

Using the properties (3.8), (3.9) and (3.10) of Lemma 3.3, we can show that $u(x)$ defined by (1.3) satisfies BVP(M), which guarantees the existence of the solution. ■

Lemma 3.3. *The Green function $G(x-y)$ satisfies the following properties:*

$$P(-\partial_x^2)\, G(x-y) = 0 \qquad (0 < x, y < L,\ x \ne y), \tag{3.8}$$

$$\partial_x^i G(x-y)\Big|_{x=L} - \partial_x^i G(x-y)\Big|_{x=0} = 0 \qquad (0 \le i \le 2M-1,\ 0 < y < L), \tag{3.9}$$

$$\partial_x^i G(x-y)\Big|_{y=x-0} - \partial_x^i G(x-y)\Big|_{y=x+0} = \begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \end{cases} \quad (0 < x < L), \tag{3.10}$$

$$G(x-y) > 0 \qquad (0 < x, y < L). \tag{3.11}$$

Proof of Lemma 3.3 Using (3.1), we have (3.8) as

$$P(-\partial_x^2)\, G(x-y) = \prod_{k=0}^{M-1} \left(-\partial_x^2 + a_k^2\right) \sum_{j=0}^{M-1} b_j\, G_j(x-y)$$
$$= \sum_{j=0}^{M-1} b_j \prod_{k=0}^{M-1} \left(-a_j^2 + a_k^2\right) G_j(x-y) = 0.$$

If we set $i = 2k + \varepsilon$ $(0 \le k \le M-1,\ \varepsilon = 0, 1)$, then we have

$$\partial_x^{2k+\varepsilon} G(x-y) = \sum_{j=0}^{M-1} b_j\, \partial_x^{2k+\varepsilon} G_j(x-y) = \sum_{j=0}^{M-1} b_j\, a_j^{2k} \partial_x^{\varepsilon} G_j(x-y).$$

Using (3.2), we have (3.9) as

$$\partial_x^{2k+\varepsilon} G(x-y)\Big|_{x=L} - \partial_x^{2k+\varepsilon} G(x-y)\Big|_{x=0}$$
$$= \sum_{j=0}^{M-1} b_j\, a_j^{2k} \left[\ \partial_x^{\varepsilon} G_j(x-y)\Big|_{x=L} - \partial_x^{\varepsilon} G_j(x-y)\Big|_{x=0}\ \right] = 0.$$

Using (3.3), we have (3.10) as

$$\begin{aligned}
&\partial_x^{2k+\varepsilon} G(x-y)\Big|_{y=x-0} - \partial_x^{2k+\varepsilon} G(x-y)\Big|_{y=x+0} \\
&= \sum_{j=0}^{M-1} b_j a_j^{2k} \left[\partial_x^{\varepsilon} G_j(x-y)\Big|_{y=x-0} - \partial_x^{\varepsilon} G_j(x-y)\Big|_{y=x+0} \right] \\
&= \begin{cases} 0 & (\varepsilon = 0) \\ -\displaystyle\sum_{j=0}^{M-1} b_j a_j^{2k} & (\varepsilon = 1) \end{cases} \qquad (0 \le k \le M-1),
\end{aligned}$$

where

$$\begin{aligned}
-\sum_{j=0}^{M-1} b_j a_j^{2k} &= (-1)^M \left| \begin{array}{c} a_j^{2i} \\ \hline a_j^{2k} \end{array} \right| \Big/ \left| \quad a_j^{2i} \quad \right| \\
&= \begin{cases} 0 & (0 \le k \le M-2), \\ (-1)^M & (k = M-1). \end{cases}
\end{aligned}$$

Using (1.6) and (3.4), we have (3.11). This completes the proof of Lemma 3.3. ■

Concerning the uniqueness of the Green function, we show the following lemma.

Lemma 3.4. The smooth function $G(x-y)$ $(0 < x, y < L)$ satisfying Lemma 3.3 (3.8), (3.9) and (3.10) is unique.

Proof of Lemma 3.4 Suppose that we have another function $\widetilde{G}(x-y)$ satisfying Lemma 3.3 (3.8), (3.9) and (3.10). For any function $f(x)$,

$$u(x) = \int_0^L \widetilde{G}(x-y)\, f(y)\, dy \qquad (0 < x < L)$$

satisfies BVP(M). From Theorem 1.1, we have

$$\int_0^L \widetilde{G}(x-y)\, f(y)\, dy = \int_0^L G(x-y)\, f(y)\, dy \qquad (0 < x < L).$$

This shows $\widetilde{G}(x-y) = G(x-y)$ $(0 < x, y < L)$. ■

4 Sobolev inequality

In this section, we show that the Green function $G(x-y)$ is a reproducing kernel for a set of Hilbert space H and its inner product $(\cdot,\cdot)_H$ introduced in section 1.

Lemma 4.1. *For any $u \in H$ and any fixed $y \in [0, L]$, we have the reproducing relation*

$$u(y) = (\, u(\cdot), G(\cdot - y)\,)_H \qquad (0 \le y \le L) \tag{4.1}$$

holds. Applying $u(x)=G(x-y)\in H$ *to* (4.1), *we have*

$$G(0)=\|\,G(\cdot-y)\,\|_H^2 \qquad (0\le y\le L). \tag{4.2}$$

Proof of Lemma 4.1 Applying $(\cdot,\cdot)_H$ to the Parseval equality (2.8), we have

$$(u,v)_H=\sum_{k\in\mathbf{Z}}P\Big(\Big(\frac{\omega_k}{L}\Big)^2\Big)\,\widehat{u}(k)\,\overline{\widehat{v}}(k). \tag{4.3}$$

Inserting

$$v(x)=G(x-y) \qquad \overset{\frown}{\longrightarrow} \qquad \widehat{v}(k)=e^{-\sqrt{-1}\frac{\omega_k}{L}y}\widehat{G}(k)$$

into (4.3), we have

$$(\,u(\cdot),G(\cdot-y)\,)_H=\sum_{k\in\mathbf{Z}}P\Big(\Big(\frac{\omega_k}{L}\Big)^2\Big)\,\widehat{u}(k)\,\overline{e^{-\sqrt{-1}\frac{\omega_k}{L}y}\widehat{G}(k)}=\sum_{k\in\mathbf{Z}}\widehat{u}(k)\,\varphi_k(y)=u(y),$$

where we use $\sqrt{L}\,P\Big(\big(\frac{\omega_k}{L}\big)^2\Big)\,\widehat{G}(k)=1$ ($k\in\mathbf{Z}$) in (3.7). We have Lemma 4.1. ■

Proof of Theorem 1.2 Applying the Schwarz inequality to (4.1) and using (4.2), we have

$$|u(y)|^2\le\|u\|_H^2\|\,G(\cdot-y)\,\|_H^2=G(0)\|u\|_H^2.$$

Noting $C_0=G(0)$ and taking the supremum with respect to $y\in[0,L]$, we have the Sobolev inequality

$$\Big(\sup_{0\le y\le L}|\,u(y)\,|\Big)^2\le C_0\,\|u\|_H^2. \tag{4.4}$$

This inequality shows that $(\cdot,\cdot)_H$ is positive definite. In fact, $\|u\|_H=0$ yields $u\equiv 0$ ($x\in[0,L]$). Applying (4.4) to $u(x)=G(x-y_0)\in H$ where $y_0\in[0,L]$ is an arbitrarily fixed number, we have

$$\Big(\sup_{0\le y\le L}|\,G(y-y_0)\,|\Big)^2\le C_0\,\|\,G(\cdot-y_0)\,\|_H^2=C_0^2.$$

Combining this with the trivial inequality

$$C_0^2=(G(y_0-y_0))^2\le\Big(\sup_{0\le y\le L}|\,G(y-y_0)\,|\Big)^2,$$

we have

$$\Big(\sup_{0\le y\le L}|\,G(y-y_0)\,|\Big)^2=C_0\,\|\,G(\cdot-y_0)\,\|_H^2\,.$$

The concrete form of (1.8) is followed by (1.4). This completes the proof of Theorem 1.2. ■

Acknowledgement

This research is supported by J. S. P. S. Grant-in-Aid for Scientific Research (C) No.17K05374 and 18K03340.

References

[1] Y. Kametaka, A. Nagai, K. Watanabe, K. Takemura and H. Yamagishi, *Giambelli's formula and the best constant of Sobolev inequality in one dimensional Euclidean space*, Sci. Math. Jpn., **e-2009** (2009), 621–635.

[2] Y. Kametaka, K. Takemura, H. Yamagishi, A. Nagai and K. Watanabe, *Heaviside cable, Thomson cable and the best constant of a Sobolev-type inequality*, Sci. Math. Jpn., **e-2007** (2007), 739–755.

[3] Y. Kametaka, K. Watanabe and A. Nagai, *The best constant of Sobolev inequality in an n dimensional Euclidean space*, Proc. Japan Acad., **81**, Ser.A, No.3 (2005), 57–60.

[4] Y. Kametaka, K. Watanabe, A. Nagai and S. Pyatkov, *The best constant of Sobolev inequality in an n dimensional Euclidean space*, Sci. Math. Jpn., **e-2004** (2004), 295–303.

[5] Y. Kametaka, K. Watanabe, A. Nagai, H. Yamagishi and K. Takemura, *The best constant of Sobolev inequality which correspond to a bending problem of a string with periodic boundary condition*, Sci. Math. Jpn., **e-2007** (2007), 283–300.

[6] Y. Kametaka, K. Watanabe, A. Nagai, K. Takemura and H. Yamagishi, *Positivity and Hierarchical Structure of four Green Functions Corresponding to a Bending Problem of a Beam on a half line*, Math. J. Okayama Univ., **65** (2023) 145–173.

[7] A. Nagai, Y. Kametaka, H. Yamagishi, K. Takemura and K. Watanabe, *Discrete Bernoulli polynomials and the best constant of discrete Sobolev inequality*, Funkcial. Ekvac., **51** (2008), 307–327.

[8] K. Takemura, Y. Kametaka, K. Watanabe, A. Nagai and H. Yamagishi, *The best constant of Sobolev inequality corresponding to a bending problem of a beam on a half line*, Far East J. Appl. Math., **51** No.1 (2011), 45–71.

[9] K. Takemura, Y. Kametaka, K. Watanabe, A. Nagai and H. Yamagishi, *Sobolev type inequalities of time-periodic boundary value problems for Heaviside and Thomson Cables*, Bound. Value Probl., (2012), 2012:95.

[10] K. Takemura, A. Nagai, Y. Kametaka, K. Watanabe and H. Yamagishi, *Sobolev inequality of free boundary value problem for* $(-1)^M(d/dx)^{2M}$, J. Inequal. Appl., (2015) 2015:54, DOI 10.1186/s13660-015-0568-9.

[11] K. Watanabe, Y. Kametaka, A. Nagai, K. Takemura and H. Yamagishi, *The best constants of Sobolev and Kolmogorov type inequalities on a half line*, Far East J. Appl. Math., **52** No.2 (2011), 101–129.

[12] K. Watanabe, Y. Kametaka, H. Yamagishi, A. Nagai and K. Takemura, *The best constant of Sobolev inequality corresponding to clamped boundary value problem*, Bound. Value Probl., (2011), Article ID 875057, 17 pages.

[13] H. Yamagishi, *The best constant of Sobolev inequality corresponding to a bending problem of a beam under tension on an elastic foundation II*, Trans. Jpn. Soc. Ind. Appl. Math., **29** (2019), 294–324 [in Japanese].

[14] H. Yamagishi, Y. Kametaka, A. Nagai, K. Watanabe and K. Takemura, *Riemann zeta function and the best constants of five series of Sobolev inequalities*, RIMS Kokyuroku Bessatsu, **B13** (2009), 125–139.

[15] H. Yamagishi, Y. Kametaka, K. Takemura, K. Watanabe and A. Nagai, *The best constant of Sobolev inequality corresponding to a bending problem of a beam under tension on an elastic foundation*, Trans. Jpn. Soc. Ind. Appl. Math., **19** (2009), 489–518 [in Japanese].

[16] H. Yamagishi and A. Nagai, *The best constant of discrete Sobolev inequality corresponding to a discrete bending problem of a string*, Saitama Math. J., **34** (2022), 19–46.

[17] H. Yamagishi, A. Nagai, K. Watanabe, K. Takemura and Y. Kametaka, *The best constant of discrete Sobolev inequality corresponding to a bending problem of a string*, Kumamoto J. Math., **25** (2012), 1–15.

Hiroyuki Yamagishi
Tokyo Metropolitan College of Industrial Technology
1-10-40 Higashi-oi, Shinagawa Tokyo 140-0011, Japan
e-mail: yamagisi@metro-cit.ac.jp

Yoshinori Kametaka
Faculty of Engineering Science, Osaka University
1-3 Machikaneyama-cho, Toyonaka 560-8531, Japan
e-mail: kametaka@sigmath.es.osaka-u.ac.jp