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On curves approaching asymptotic critical value set of a polynomial map

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Abstract. In this note, we show that the asymptotic critical value set of a polynomial map contains the critical values of a polynomial associated to so called "bad face" of the Newton polyhedron. For this purpose, we present an effective method to construct rational curves that make the polynomial to approach asymptotic critical values. In the case when the polynomial map is Newton non-degenerate at infinity, we give also a superset of the asymptotic critical value set including the bifurcation locus. Our main technical tool is the toric geometry that has been introduced into the study of this question by A.Némethi and A.Zaharia.

1 Introduction

The bifurcation locus of a polynomial map $f\colon \mathbb{C}^n \to \mathbb{C}$ is the smallest subset $\mathcal{B}(f) \subset \mathbb{C}$ such that f is a locally trivial fibration over $\mathbb{C} \setminus \mathcal{B}(f)$, [17, A1], [21, Cor 5.1]. It is known that $\mathcal{B}(f)$ is the union of the set of critical values $f(\mathrm{Sing} f)$ and the set of bifurcation values at infinity $\mathcal{B}_\infty(f)$ which may be non-empty and disjoint from $f(\mathrm{Sing} f)$ even in very simple examples [1]. Finding the bifurcation locus in the cases $n > 2$ is a difficult task and it still remains to be an unreachable ideal. Nevertheless, one can obtain approximations by supersets of $\mathcal{B}(f)$ by exploiting asymptotical regularity conditions at infinity.

Jelonek and Kurdyka [11, 12] introduced the set of asymptotic critical values $\mathcal{K}_\infty(f)$ and established an algorithm for finding them. Parusiński [16] has shown that $\mathcal{K}_\infty(f)$ is finite and includes $\mathcal{B}_\infty(f)$. Under the condition that the projective closure of the generic fibre of f in $\mathbb{P}^n$ has only isolated singularities, it is shown in [15] that the equality $\mathcal{B}(f) = \mathcal{K}_\infty(f) \cup f(\mathrm{Sing} f)$ holds.

A precedent work [5] established a method to detect the bifurcation set in an efficient way. It gave an answer to a question raised in [12] and [6] about the detection of the bifurcation locus by rational curve with parametric representation. More concretely, for a real polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree $\leq d$, authors of [5] consider a real rational curve $X(t)$, $lim_{t\to 0}\|X(t)\| \to \infty$, with parametric

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representation of length $(d+1)d^{n-1}+1$ to attain the asymptotic critical value $lim_{t\to 0} f(X(t)) \in \mathcal{K}_\infty(f)$.

In our present note, we propose a method to construct a rational curve that corresponds to an asymptotic critical value of a polynomial (Theorem 3.1). This allows us to find out efficiently a subset of $\mathcal{K}_\infty(f)$ for a polynomial f whose Newton polyhedron has full dimension n. The rational curve in question has drastically reduced number of terms present in its parametric representation. Further in this article, we shall use the terminology "parametric length of a curve" to denote this number. Thus in our Example 5.1, the parametric length of a real rational curve has been reduced to 4 in comparison with 3601 proposed in [5].

Starting from Lemma 3.1, we take into account the condition (μ) on the vector q (2.17) that is always satisfied for a proper choice of the toric data W (2.4) (Lemma 3.2). We follow [13, 22] as for the use of toric geometry in the investigation of the asymptotically non-regular values of f. Our main Theorem 3.1 states the inclusion into $\mathcal{K}_\infty(f)$ of critical values of a certain polynomial f^W_γ, with possibly non-isolated singularities, constructed on a "bad face" γ of the Newton polyhedron of f. Thus Corollary 3.2 establishes an inclusion relation

$$\bigcup_{\gamma: bad\ face} f_\gamma(\text{Sing } f_\gamma \cap (\mathbb{C}^\times)^{dim\ \gamma}) \subset \mathcal{K}_\infty(f) \tag{1.1}$$

that is valid even in the case of f^W_γ with non-isolated singularities. The inclusion (1.1) holds for every f that effectively depends on n variables.

For the case where f is Newton non-degenerate at infinity this gives an approximation of $\mathcal{K}_\infty(f)$ formulated in Corollary 3.3 that determines a superset of $\mathcal{K}_\infty(f)$

$$\mathcal{K}_\infty(f) \subset \bigcup_{\gamma: bad\ face} f_\gamma(\text{Sing } f_\gamma \cap (\mathbb{C}^\times)^{dim\ \gamma}) \cup \{0\}. \tag{1.2}$$

In this case our Corollary 3.5 gives a refined upper bound estimate of the cardinality $\#\mathcal{K}_\infty(f)$ in terms of volumes of polyhedra explicitly obtained from bad faces. We remark that this estimation gives an approximation sharper than [11, Theorem 2.2, 2.3] under conditions imposed in Corollary 3.5.

In [19] it is shown that the left hand side set of the relation (1.1) is contained in the bifurcation set $\mathcal{B}(f)$ for γ relatively simple bad face (see Definition 4.1) and f^W_γ with isolated singularities on $(\mathbb{C}^\times)^{dim\ \gamma}$. As the inclusion $\mathcal{B}(f) \subset \mathcal{K}_\infty(f)$ is known from [11],[16], our Corollary 3.3 represents a new result only for γ non-relatively simple bad face, if the condition of the isolated singularities at infinity is assumed.

In Section 4 we examine an example of a polynomial in 5 variables with non-relatively simple bad face. Even in this situation, we can construct a curve approaching an asymptotic critical value of f. This gives an example to Corollary 3.3 that is not covered by [19]. As [19] imposes the condition of isolated singularity at infinity, it does not concern our Example 5.1 treating the non-isolated singularities.

It is worthy noticing that M.Ishikawa [10] established a precise description of $\mathcal{B}(f)$ analogous to [13, Proposition 6] for any polynomial map in two variables, i.e. possibly for Newton degenerate polynomials.

Our method heavily relies on various kinds of Newton polyhedra constructed in two different chart systems. The core technique is explained in Proposition 2.1 where the key data like the integer vector $q \in \mathbb{Z}^n$ and the integer $\rho > 0$ are introduced. The vector $q \in \mathbb{Z}^n$ is used to calculate the number L_0 (3.7) that determines the parametric length together with the integer ρ (2.18).

Finally, we remark that efficiency to detect asymptotically non-regular values can be applied to optimisation problems e.g. see [9]. We recall that [2] had recourse to effective use of Newton polyhedra in the investigation of the order of coerciveness of a polynomial f. After [18, Theorem 5], if a polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}$ with rational coefficients admits a bounded infimum, this infimum must belong to the set $\mathcal{K}_\infty(f) \cup f(\text{Sing}f)$. As for the real setting of the problem, see [20]. Thus we hope that our approach represents not only purely theoretical interests, but also certain utility in the optimisation problem.

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2 Approach with unimodular subdivision of the dual cone

To fix notations and fundamental notions, we follow [13, 22].

Let us consider a polynomial in $x = (x_1, \cdots, x_n)$,

$$f(x) = \sum_{\alpha} a_\alpha x^\alpha \tag{2.1}$$

with $f(0) = 0$ where the multi-index α runs within the set of integer points $supp(f) = \{\alpha \in (\mathbb{Z}_{\geq 0})^n; a_\alpha \neq 0\}$. We introduce a convex polyhedron of finite volume $\Delta(f)$ defined as the convex hull of $supp(f)$ in $\mathbb{R}^n$ that is assumed to be of the maximal dimension, i.e. $dim\ \Delta(f) = n$. We denote the convex hull of $supp(f) \cup \{0\}$ in $\mathbb{R}^n$ by $\widetilde{\Gamma}_-(f)$.

Definition 2.1. *For $a \in (\mathbb{R}^n)^*$ we denote by Δ^a a face of $\widetilde{\Gamma}_-(f)$ determined by the condition $\langle a, y\rangle \leq \langle a, x\rangle$ for every pair $x \in \widetilde{\Gamma}_-(f)$ and $y \in \Delta^a$. For a face $\gamma \subset \Delta(f)$ of the Newton polyhedron of f (2.1), we define $f_\gamma(x) = \sum_{\alpha\in\gamma} a_\alpha x^\alpha$.*

Definition 2.2. *For a set $\Lambda \subset (\mathbb{R}_{\geq 0})^n$ we denote by $C(\Lambda) = \{tv; t \in \mathbb{R}_{\geq 0},\ v \in \Lambda\}$ the cone with the base Λ.*

Definition 2.3. *Let K be a unimodular simplicial subdivision of $(\widetilde{\Gamma}_-(f))^*$ where $(\widetilde{\Gamma}_-(f))^*$ is the dual to $\widetilde{\Gamma}_-(f)$.*

$$\begin{aligned}(\widetilde{\Gamma}_-(f))^* &= \{a \in (\mathbb{R}^n)^*;\ \langle a, x\rangle \geq 0,\ \forall x \in \widetilde{\Gamma}_-(f))\} \\ &= \{a \in (\mathbb{R}^n)^*;\ \langle a, x\rangle \geq 0,\ \forall x \in C(\widetilde{\Gamma}_-(f)))\}.\end{aligned}$$

Definition 2.4. *([13]) We call a face $\gamma \subset \Delta(f)$ bad, if it satisfies the following two properties.*

(i) The affine subspace of dimension $=$ dim γ spanned by γ contains the origin.

(ii) ($\pm$ condition for the bad face) There exists a hyperplane $H \subset \mathbb{R}^n$ such that $\gamma = H \cap \Delta(f)$ defined by an equation $\sum_{j=1}^n p_j x_j = 0$ be provided with a pair of indices $i \neq j$ satisfying $p_i p_j < 0$.

Assume that γ is a bad face and a $k-$dimensional cone $\sigma \in K \subset (\widetilde{\Gamma}_-(f))^* \subset (\mathbb{R}^n)^*$ satisfies

$$\gamma \subset \sigma^* = \{x \in \mathbb{R}^n; \langle \alpha, x\rangle \geq 0, \forall \alpha \in \sigma\} \tag{2.2}$$

with a basis $a_1, \dots, a_k \in \mathbb{Z}^n$ such that

$$\gamma = \{v \in \Delta(f); \langle a_i, v\rangle = 0,\ i = 1, \dots, k\}. \tag{2.3}$$

Such a basis exists by virtue of Definition 2.4 (ii).

If $a_1, \cdots, a_k$ is a unimodular basis of a $k-$ dimensional cone $\sigma \in K$, i.e. $\sigma = \Sigma_{i=1}^k t_i a_i, t_i \geq 0$, we can choose $m_1, \cdots, m_n \in \mathbb{Z}^n$ a basis of the dual cone $\sigma^* = \{x \in \mathbb{R}^n; \langle x, a\rangle \geq 0, \forall a \in \sigma\}$ such that $\langle a_i, m_j\rangle = \delta_{ij}, i \in [1;k], j \in [1;n]$ where δ_{ij} denotes Kronecker Delta. From here on we shall use the notation $i \in [r_1; r_2] \Leftrightarrow i \in \{r_1, \cdots, r_2\}$ for two integers $r_1 \leq r_2$. We can further extend the basis $a_1, \cdots, a_k$ to an $n-$dimensional basis $a_1, \cdots, a_n$ with the aid of supplementary vectors $a_{k+1}, \cdots, a_n$ in such a way that $det(a_1, \cdots, a_n) = 1$. We pose $\sigma^* = \{\sum_{i=1}^n \lambda_i m_i; \lambda_j \geq 0, j \in [1;k]\}$.

Definition 2.5. *Let $\sigma \in K$ be a unimodular simplicial cone with $dim(\sigma) = k$. An algebraic torus of dimension $n-k$ associated to the cone σ can be defined as*

$$\Phi[\sigma] = (\mathbb{C}^\times)^n / \{(t^{b_1}, \dots, t^{b_n}); t \in (\mathbb{C}^\times)^k, (b_1, \dots, b_n) \in \sigma\}.$$

We also consider a disjoint union of tori given by $M_{\overline{\sigma}} = \cup_{\sigma' \subset \sigma} \Phi[\sigma'] \cong \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k} \ni (u_1, \dots, u_k, u_{k+1}, \dots, u_n)$ with $\overline{\sigma} = \cup_{\sigma' \subset \sigma} \sigma'$ where σ' runs over all subcones of σ.

Definition 2.6. *For an integer vector $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$ we denote by $\alpha' = (\alpha_1, \cdots, \alpha_k) \in \mathbb{Z}^k$ and $\alpha'' = (\alpha_{k+1}, \cdots, \alpha_n) \in \mathbb{Z}^{n-k}$ its respective components. In a parallel way, we introduce two sets of variables $u' \in \mathbb{C}^k$ (called affine) and $u'' \in (\mathbb{C}^\times)^{n-k}$ (called toric), $u = (u', u'') \in \mathbb{C}^n_k$ where*

$$\mathbb{C}^n_k = \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}.$$

We introduce the following unimodular matrices M and W with integer entries (i.e. complementary vectors $a_{k+1}, \cdots, a_n \in \sigma^\perp$) associated to the cone $\sigma \in K$.

$$W = ({a_1}^T, \dots, {a_n}^T) = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}, W^{-1} = M = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix} \tag{2.4}$$

where $(m_1, \ldots, m_n)$ is a basis of σ^* and $\sigma^* = \Sigma_{i=1}^k \mathbb{R}_{\geq 0} m_i + \Sigma_{j=k+1}^n \mathbb{R} m_j$. Further we use the following notation also (see Lemma 3.1)

$$M^T = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}. \tag{2.5}$$

Under the change of variables

$$(x_1, \cdots, x_n) = (u^{w_1}, \cdots, u^{w_n}) \tag{2.6}$$

we consider

$$f^W(u) = \sum_{\alpha \in supp(f)} a_\alpha u^{\alpha \cdot W} \tag{2.7}$$

where W as in (2.4). For $\alpha \in \mathbb{Z}^n$, we represent $\alpha.W \in \mathbb{Z}^n$ the integer vector with the aid of its components

$$\alpha.W = (\lambda_1(\alpha), \cdots, \lambda_n(\alpha)).$$

Due to Definition 2.3, the choice of the basis $(a_1, \cdots, a_k)$ and (2.2), we have the following.

Lemma 2.1. *For general $v \in \Delta(f)$ that is not necessarily located on the bad face γ, we have $\lambda_1(v), \ldots, \lambda_k(v) \geq 0$. In particular, $\forall j \in [1; k]$, $\exists v \in \Delta(f)$ such that $\lambda_j(v) > 0$.*

Lemma 2.2. *The Laurent polynomial $f^W(0, u'') = f_\gamma^W(u) = \sum_{\alpha \in \gamma} a_\alpha u^{\alpha.W}$ is a polynomial (with positive power terms) in u'' variables.*

Thus only $\lambda_{k+1}(v), \ldots, \lambda_n(v)$ may be negative for $v \in \Delta(f)$ in general. Because of (2.2), $f_\gamma^W(0, u'')$ is a polynomial in u'' variables. In other words, $\lambda_1(v) = \langle a_1, v \rangle = 0, \ldots, \lambda_k(v) = \langle a_k, v \rangle = 0, \lambda_{k+1}(v) = \langle a_{k+1}, v \rangle \geq 0, \ldots, \lambda_n(v) = \langle a_n, v \rangle \geq 0$, $\forall v \in \gamma$.

The expression $f^W(u)$ is a Laurent polynomial with possibly negative power exponents in toric variables u'', but being restricted to affine variables u', it gives a polynomial in $u' = (u_1, \ldots, u_k)$. We denote

$$\vartheta_u f^W(u) = (\vartheta_{u_1} f^W(u), \cdots, \vartheta_{u_n} f^W(u)),$$

with $\vartheta_{u_j} = u_j \frac{\partial}{\partial u_j}$, $j \in [1; n]$. For a critical point $u^* = (0, u''_*) \in \mathbb{C}_k^n$ such that $\vartheta_u f_\gamma^W(u^*) = 0$, we introduce the notation

$$u' = (u_1, \cdots, u_k), U'' = (U_{k+1}, \cdots, U_n) = (u_{k+1} - u^*_{k+1}, \cdots, u_n - u^*_n)$$

and consider a local expansion of the Laurent polynomial $f^W(u)$ at $u = u^* = (0, u''_*) \in \mathbb{C}_k^n$ as follows

$$f^W(u) = \sum_{\beta \in supp_{u^*}(f^W)} a^*_\beta (u-u^*)^\beta = \sum_{\beta \in supp_{u^*}(f^W)} a^*_\beta u'^{\beta'} U''^{\beta''}, \tag{2.8}$$

for $supp_{u^*}(f^W) := \{\beta \in (\mathbb{Z}_{\geq 0})^n; a^*_\beta \neq 0\}$. Here the expression corresponding to the term $\alpha.W \in (\mathbb{Z}^k_{\geq 0} \setminus \{0\}) \times \mathbb{Z}^{n-k}_{<0}$ in (2.7) shall produce a series in (2.8) with $(\beta', \beta'') \in (\mathbb{Z}^k_{\geq 0} \setminus \{0\}) \times (\mathbb{Z}_{\geq 0})^{n-k}$ according to the rule

$$\frac{1}{u_j} = \frac{1}{u^*_j} \sum_{\ell \geq 0} (-\frac{U_j}{u^*_j})^\ell \tag{2.9}$$

for $j \in [k+1; n]$.

We calculate logarithmic derivatives of $f^W(u)$ (2.8) as follows

$$\vartheta_{u_j} f^W(u) = \sum_{\beta \in supp_{u^*}(f^W)} \beta_j a^*_\beta u'^{\beta'} U''^{\beta''}, \ \ j \in [1,k], \tag{2.10}$$

$$\vartheta_{u_\ell} f^W(u) = \sum_{\beta \in supp_{u^*}(f^W)} \beta_\ell a^*_\beta u'^{\beta'} \frac{U''^{\beta''}}{U_\ell} (U_\ell + u^*_\ell), \ \ \ell \in [k+1, n]. \tag{2.11}$$

In (2.10), (2.11) the condition $|\beta''| \geq 2$ for $\beta' = 0$ follows from Lemma 2.2 and the fact that $\vartheta_u f^W(u^*) = 0$ at $u^* = (0, u''_*) \in \mathbb{C}^n_k$.

Definition 2.7. *We denote by $\Delta_{u^*}(f^W)$ a polyhedron obtained as convex hull of $supp_{u^*}(f^W - f^W(u^*))$. Similarly we define $\Delta_{u^*}(\vartheta_{u_j} f^W)$ as convex hull of*

$$supp_{u^*}(\vartheta_{u_j} f^W)$$

for $j \in [1; n]$. With the aid of polyhedra $\Delta_{u^*}(\vartheta_{u_j} f^W), j \in [1; n]$, we define

$$\Delta^* = \text{convex hull of } \cup_{j=1}^n \Delta_{u^*}(\vartheta_{u_j} f^W). \tag{2.12}$$

From (2.10), (2.11) we see that $\Delta_{u^*}(f^W) \subset \Delta^*$.

As the Laurent polynomial $f^W(u)$ effectively depends on the variable $u \in \mathbb{C}^n_k$ and $|\beta''| \geq 2$ for $\beta' = 0$ for (2.10), (2.11), the statement below holds.

Lemma 2.3. *For every face $\delta \subset \Delta^*$ with dim $\delta \leq n-2$ there exists a vector $q_\delta = (q'_\delta, q''_\delta) \in (\mathbb{Z}^k \setminus \{0\}) \times \mathbb{Z}^{n-k}$ orthogonal to δ.*

For the cone σ mentioned in Definition 2.5 we consider the decomposition

$$f^W(u) = \tilde{f}^W(u) + R(u) \tag{2.13}$$

with

$$\tilde{f}^W(u) = \sum_{\alpha \in supp(f) \cap (\mathbb{Z}_{\geq 0})^n M} a_\alpha u^{\alpha.W}.$$

The condition $\alpha \in (\mathbb{Z}_{\geq 0})^n M$ is equivalent to $\alpha.W \in (\mathbb{Z}_{\geq 0})^n$. Here $R(u)$ corresponds to terms with exponent vectors $\alpha.W \notin (\mathbb{Z}_{\geq 0})^n$ such that negative powers appear, i.e. some of $\lambda_{k+1}(\alpha), \dots, \lambda_n(\alpha)$ are strictly negative and $\lambda_1(\alpha), \dots, \lambda_k(\alpha) \geq 0$. We shall note that if some of $\lambda_{k+j}(\alpha)$ is strictly negative for $\alpha \in \Delta(f)$, then $\lambda_i(\alpha)$ for some $i \in [1; k]$ must be strictly positive. See Lemmata 2.1, 2.2.

Lemma 2.4. *We have the following relations for the intersection of Δ^* with the $(n-k)-$dimensional positive octant $(\mathbb{R}_{\geq 0})^{n-k}$,*

$$dim \ \big(\Delta^* \cap \{(0, \alpha''); \alpha'' \in (\mathbb{R}_{\geq 0})^{n-k}\}\big) \geq n - k - 1, \tag{2.14}$$

$$\Delta^* \cap \{(0, \alpha''); \alpha'' \in (\mathbb{R}_{\geq 0})^{n-k}\} \subset \{(0, \alpha''); |\alpha''| \geq 1\}. \tag{2.15}$$

Consequently, there exists a face Γ of Δ^* containing the LHS of (2.15) *as its face. Thereby vector q orthogonal to Γ is of the form*

$$q = (q', q'') \in (\mathbb{Z}^k \setminus \{0\}) \times (\mathbb{Z}^{n-k}_{\geq 0} \setminus \{0\}). \tag{2.16}$$

Proof. Because of $\vartheta_u f^W(u''_*) = 0$, we see that the exponent β'' from (2.10), (2.11) satisfies $|\beta''| \geq 2$.

As $f^W(u)$ effectively depends on $(U_{k+1}, \cdots, U_n) \in (\mathbb{C}^\times)^{n-k}$ there exist subsets $(\mathbb{R}_{>0})^{n_i} \subset (\mathbb{R}_{\geq 0})^{n-k}, i \in [1; L]$, satisfying $\sum_{i=1}^{L} n_i = n - k$ under the condition that in $(\mathbb{R}_{>0})^{n_i}$ we find a point $(0, \tilde{\beta}''(i)) \in (\mathbb{R}_{>0})^{n_i} \cap \Gamma_i$ for a face Γ_i of $\Delta_{u^*}(f^W)$. On each of $(\mathbb{R}_{>0})^{n_i}$ we find a set of generator unit vectors $\{\mathbf{e}_{\ell(i)}\}_{\ell(i) \in L(i)}$ for an index set $L(i)$ such that $(\mathbb{R}_{>0})^{n_i} = \sum_{\ell(i) \in L(i)} \mathbb{R}_{>0} \mathbf{e}_{\ell(i)}, \#L(i) = n_i$.

It follows that $(0, \tilde{\beta}''(i) - \mathbf{e}_{\ell(i)}) \in \Delta^* \setminus \Delta_{u^*}(f^W), \ell(i) \in L(i)$ thanks to (2.11). Thus on $(\mathbb{R}_{>0})^{n_i}$ we find a set of points from Δ^* with cardinality n_i as follows

$$\{(0, \tilde{\beta}''(i) - \mathbf{e}_{\ell(i)}); \ell(i) \in L(i)\}.$$

The inequality (2.14) follows from the equality

$$dim\big(\text{convex hull of } \cup_{i=1}^{L} \cup_{\ell(i) \in L(i)} (0, \tilde{\beta}''(i) - \mathbf{e}_{\ell(i)})\big) = n - k - 1.$$

The equalities (2.11) and $\vartheta_u f^W_\gamma(u^*) = 0$ for $u^* = (0, u''_*) \in \mathbb{C}^n_k$ show (2.15).

Lemma 2.3 and (2.15) imply the existence of the vector q and the face Γ as in (2.16).

In summary, we establish the following.

Proposition 2.1. *Assume that $\vartheta_u f^W_\gamma(u^*) = 0$ for $u^* = (0, u''_*) \in \mathbb{C}^n_k$. The face Γ of the polyhedron Δ^* found in Lemma 2.4 with its orthogonal vector $q = (q', q'') \in (\mathbb{Z}^k \setminus \{0\}) \times (\mathbb{Z}^{n-k}_{\geq 0} \setminus \{0\})$* (2.16) *can be characterised as follows*

$$\Gamma = \{\beta \in \Delta^*; \langle \beta, q \rangle \leq \left\langle \tilde{\beta}, q \right\rangle \text{ for every } \tilde{\beta} \in \Delta^*\}. \tag{2.17}$$

Thus for any $\beta \in \Delta^*$ *the inequality* $\langle \beta, q\rangle \leq \left\langle \tilde{\beta}, q \right\rangle$ *holds with every*

$$\tilde{\beta} \in \Delta_{u^*}(\langle \mu_i, \vartheta_u f^W(u)\rangle),$$

$i \in [1;n]$. *We shall further denote by* ρ *the following integer*

$$\rho = min_{\tilde{\alpha}\in\Delta^*} \langle \tilde{\alpha}, q\rangle \tag{2.18}$$

that is equal to $\langle \alpha, q\rangle$ *for* $\alpha \in \Gamma$.

This proposition follows from Lemma 2.4 and the inclusion $\Delta_{u^*}(\langle \mu_i, \vartheta_u f^W(u)\rangle) \subset \Delta^*$, $i \in [1;n]$. In fact, by a proper choice of vectors $a_k, \cdots, a_n$ that form a part of a unimodular basis of $\mathbb{R}^n$ (2.4) (Lemma 3.2), we can assume that $\Delta_{u^*}(\langle \mu_i, \vartheta_u f^W(u)\rangle) = \Delta^*$, $\forall i \in [1;n]$. See Corollary 3.1 below.

See Figure 5.1 where the face Γ is illustrated for the Example 5.1.

3 Curve construction by means of Newton polyhedron

In this section we construct a curve $X(t)$ that goes to the infinity on which the value of f tends to an asymptotic critical value $lim_{t\to 0} f(X(t)) \in \mathcal{K}_\infty(f)$.

Let us introduce a curve with parametric representation

$$Q(t) = (u'(t), u''(t)) = (c't^{q'} + h.o.t., u^{''}_{*} + c''t^{q''} + h.o.t.) \tag{3.1}$$

where $q = (q', q'') \in (\mathbb{Z}^k \setminus \{0\}) \times (\mathbb{Z}^{n-k}_{\geq 0} \setminus \{0\})$ found in Proposition 2.1 and $u^{''}_{*} \neq 0$, as $u^{''}_{*} \in (\mathbb{C}^\times)^{n-k}$. Here $c't^{q'} = (c'_1 t^{q'_1}, \cdots, c'_k t^{q'_k})$ etc.

Definition 3.1. *([11, 12]) Consider a curve* $X(t) = (x_1(t), \cdots, x_n(t))$ *that satisfies the following two conditions*

$$lim_{t\to 0}||X(t)|| = \infty, \tag{3.2}$$

$$lim_{t\to 0} x_i(t)\frac{\partial f(X(t))}{\partial x_j} \to 0 \tag{3.3}$$

for every pair $(i,j) \in [1;n]^2$. *We call the value* $lim_{t\to 0} f(X(t))$ *an asymptotic critical value of* f. *We denote by* $\mathcal{K}_\infty(f)$ *the set of asymptotic critical values of* f.

After [4], the image value of f that is not asymptotic critical is called a $t-$*regular value of* f. If the limit $lim_{t\to 0} f(X(t)) = p_0$ exists for the curve (3.2), the negation of the condition (3.3) is known as Malgrange condition for the fibre $f^{-1}(p_0)$, i.e. $\exists \epsilon > 0$ such that

$$lim_{t\to 0}||X(t)||||grad\ f(X(t))|| > \epsilon.$$

To construct a curve $\|X(t)\| \to \infty$ as above, it is enough to consider only one torus chart $\Phi[\sigma]$ from Definition 2.5.

Lemma 3.1. *For* $q = (q', q'') \in (\mathbb{Z}^k \setminus \{0\}) \times (\mathbb{Z}^{n-k}_{\geq 0} \setminus \{0\})$ *found in Proposition 2.1, the following equivalence holds.* (i) $\exists w_i$ (2.4) *such that* $\langle (q', 0), w_i \rangle < 0 \Leftrightarrow$ (ii) $(q', 0) \notin \sum_{j=1}^{n} \mathbb{R}_{\geq 0}\mu_j$. *We call this condition* (μ).

Proof. $(i) \Rightarrow (ii)$. We show the contraposition. For the vector $r = \sum_{j=1}^{n} t_j \mu_j$, $t_j \geq 0$ for every $j \in [1;n]$, $\langle r, w_i \rangle = t_i \geq 0$ for every i.

$(ii) \Rightarrow (i)$. Also by contraposition. Take $r = \sum_{j=1}^{n} s_j \mu_j \in \mathbb{R}^n$ such that $\langle r, w_i \rangle = s_i \geq 0$ for every i. As $r = (q', 0) \neq (0,0)$ not every s_i equals to zero, thus $s_j > 0$ for some j. Compare with [7, 2.3] Claim 1, Claim 2, Exercise.

Lemma 3.2. *The condition* (μ) *of Lemma 3.1 is satisfied for properly chosen vectors* $a_k, \cdots, a_n$ *that form a part of a unimodular basis of* $\mathbb{R}^n$ (2.4)*. Especially, we can choose vectors* $m_j, j \in [1;n]$ *from* $(\mathbb{Z}_{>0})^n$.

Proof. The condition (ii) of Lemma 3.1 is satisfied if $m_i \in (\mathbb{Z}_{>0})^n, \forall i \in [1;n]$.

From Definition 2.4 (ii) we can choose $a_1, \cdots, a_k$ in such a way that $\forall \ell \in [1;k], \exists j \in [1;n]$ satisfying $a_\ell^j < 0$ for $a_\ell = (a_\ell^1, \cdots, a_\ell^n)$. In other words, $\{x \in \mathbb{R}^n; \langle a_\ell, x \rangle = 0\} \cap (\mathbb{R}_{>0})^n \neq \emptyset, \forall \ell \in [1;k]$. Let $a_1, \cdots, a_n$ be a unimodular basis of $\mathbb{Z}^n$ obtained as an extension of $a_1, \cdots, a_k$ chosen as above. We see that it is possible to construct a unimodular basis $a_i \notin (\mathbb{R}_{\geq 0})^n, \forall i \in [1;n]$ as it is allowed to replace $a_j, j \in [k+1;n]$ by $a_j + C_{j,\ell} a_\ell$ with large enough $C_{j,\ell} \in \mathbb{Z}_{>0}, \ell \in [1;k]$. Thus we can assume that $a_j, \forall j \in [1;n]$ has at least one strictly negative component.

For the $n-$dimensional cone $\tau = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} a_i$ consider its dual cone $\tau^\vee$ that admits the expression $\tau^\vee = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} \bar{m}_i$ with generators $\bar{m}_i \in \mathbb{Z}^n$ satisfying $\langle \bar{m}_i, a_j \rangle = B_i \delta_{i,j}$ for certain $B_i \in \mathbb{Z}_{>0}, i \in [1;n]$. See [7, §1.2, (8)].

In view of the unimodular property of $(a_1, \cdots, a_n)$ we see that the basis $(\bar{m}_1, \cdots, \bar{m}_n)$ of the dual cone $\tau^\vee$ must be also unimodular. This means that $(\bar{m}_1, \cdots, \bar{m}_n)$ can be taken as $M = W^{-1}$ in (2.4).

By the above construction, for the cone $\tau = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} a_i$ the set $\tau \setminus \{0\}$ contains $(\mathbb{R}_{>0})^n$ as its strict subset. The dual cone $\tau^\vee$ satisfies $\tau^\vee \setminus \{0\} \subsetneq (\mathbb{R}^*_{>0})^n$. This implies that the generators $\tau^\vee$ of $m_i, \forall i \in [1;n]$ are from $(\mathbb{Z}_{>0})^n$.

From Lemma 3.2 it follows that the vectors $\mu_i, i \in [1;n]$ from (2.5) lie in $(\mathbb{Z}_{>0})^n$. This yields the following.

Corollary 3.1. *Under the choice of the unimodular basis* $(a_1, \cdots, a_n)$ *as in Lemma 3.2 we can assume that the Newton polyhedron* $\Delta_{u^*}(\langle \mu_i, \vartheta_u f^W(u) \rangle)$ *coincides with* Δ^*, $\forall i \in [1;n]$.

We remark also the following.

Lemma 3.3. *The integer* ρ *(2.18) is strictly positive for* q *determined for the face* Γ *constructed in Proposition 2.1.*

Proof. By Definition 2.7, for every $\beta \in \Delta_{u^*}(\langle \mu_j, f^W \rangle)$ there exists $\tilde{\alpha} \in \{\langle q, \cdot \rangle = \rho\}$ such that $\beta = t\tilde{\alpha}$ for $t \geq 1$. The number ρ was defined as the minimal value of the linear function $\langle q, \cdot \rangle$ on Δ^* and $\langle q, \beta \rangle = t\rho \geq \rho$ thus ρ must be strictly positive.

Let us denote by $X(t)$ the image of the curve $Q(t)$ defined in (3.1) by the map (2.6).

Lemma 3.4. *The condition (μ) of Lemma 3.1 is sufficient so that there exists a curve $\|X(t)\| \rightarrow \infty$ with finite limit $lim_{t\rightarrow 0} f(X(t)) = lim_{t\rightarrow 0} f^W(Q(t))$. The equality $lim_{t\rightarrow 0}\ \vartheta_u f^W(Q(t)) = 0$ holds and the limit $lim_{t\rightarrow 0} f^W(Q(t))$ corresponds to a critical value of the polynomial $f^W_\gamma(u)$.*

Proof. By (3.1) and $x_i = u^{w_i}$, we have

$$x_i(t) = c_i t^{\langle (q',0), w_i \rangle}(1 + h.o.t.).$$

The existence of the value $lim_{t\rightarrow 0} f(X(t)) = lim_{t\rightarrow 0} f^W(Q(t))$ is clear from the definition of the curve (3.1).

By means of the vectors introduced in Lemma 3.1 (μ), we deduce the following relation

$$\begin{pmatrix} \vartheta_{x_1} f(x) \\ \vartheta_{x_2} f(x) \\ \vdots \\ \vartheta_{x_n} f(x) \end{pmatrix} = M^T \begin{pmatrix} \vartheta_{u_1} f^W(u) \\ \vartheta_{u_2} f^W(u) \\ \vdots \\ \vartheta_{u_n} f^W(u) \end{pmatrix}. \tag{3.4}$$

Let $\vec{\ell} = (\ell_1, \cdots, \ell_n) \in (\mathbb{R}^*)^n$ be a vector in general position with non-zero components and denote $\left\langle \vec{\ell}, u^W \right\rangle = \sum_{j=1}^n \ell_j u^{w_j}$. Then we have

$$\left\langle \vec{\ell}, x \right\rangle \begin{pmatrix} \partial_{x_1} f(x) \\ \partial_{x_2} f(x) \\ \vdots \\ \partial_{x_n} f(x) \end{pmatrix} = \left\langle \vec{\ell}, u^W \right\rangle \begin{pmatrix} \frac{\mu_1}{u^{w_1}} \\ \frac{\mu_2}{u^{w_2}} \\ \vdots \\ \frac{\mu_n}{u^{w_n}} \end{pmatrix} \begin{pmatrix} \vartheta_{u_1} f^W(u) \\ \vartheta_{u_2} f^W(u) \\ \vdots \\ \vartheta_{u_n} f^W(u) \end{pmatrix}. \tag{3.5}$$

From this equality we see that it is enough to look for a curve $Q(t)$ given by (3.1) such that

$$\min_{i \neq j} \langle (q',0), w_i - w_j \rangle + ord\left(\left\langle \mu_j, \vartheta_u f^W \right\rangle (Q(t))\right) > 0 \tag{3.6}$$

for every $j \in [1;n]$ so that to ensure the condition (3.3). In fact, a linear combination of LHS of (3.5) for various vectors $\vec{\ell}$ will produce all $n \times n$ functions present in (3.3).

We define also

$$L_0 = max_{i \neq j} \langle (q',0), w_i - w_j \rangle. \tag{3.7}$$

Definition 3.2. *We shall use the set of indices $\mathbb{J} \subset [1;n]$ defined by*

$$\mathbb{J} = \{j \in [1;n]; min_{i \neq j} \langle (q',0), w_i - w_j \rangle < 0\}.$$

The cardinality of $\mathbb{J}$ is at most $n-1$.

To formulate the main theorem of this section, we introduce a coordinate system on the (arc) space of rationally parametrised curves of the form (3.1),

$$Q(t) = (u'(t), u''(t)) = (c'(0)t^{q'} + c'(1)t^{q'+1} + h.o.t., u''_* + c''(0)t^{q''} + c''(1)t^{q''+1} + h.o.t.) \tag{3.8}$$

where $q = (q', q'') \in \mathbb{Z}^n$ with coprime elements characterised in Proposition 2.1 and Lemma 3.3.

Here we take into account finite number of coefficients

$$c'(j) = (c_1(j), \cdots, c_k(j)) \in \mathbb{C}^k, \quad c''(j) = (c_{k+1}(j), \cdots, c_n(j)) \in \mathbb{C}^{n-k},$$

$j \in \mathbb{Z}_{\geq 0}$. We denote the space of coefficients $\mathcal{C}$ in such a way that $\mathbf{c} = (c', c'') \in \mathcal{C}$, $c' = (c'(0), c'(1), c'(2), \cdots)$, $c'' = (c''(0), c''(1), c''(2), \cdots)$.

The following theorem tells us that every critical value of the polynomial

$$f_\gamma^W(u) = \sum_{\alpha \in \gamma \cap supp(f)} a_\alpha u^{\alpha . W}$$

with γ bad face is an asymptotic critical value. It is worthy noticing that the singular points of $f_\gamma(x)$ can be non-isolated and no restriction is assumed on the dimension of the bad face γ in question.

Theorem 3.1. *Let $f \in \mathbb{C}[x]$ be a polynomial whose Newton polyhedron $\Delta(f)$ has full dimension n. Assume that γ is one of its bad faces like in Definition 2.4. (i) We can find a curve $X(t)$ satisfying (3.2), (3.3) of Definition 3.1 such that $lim_{t \to 0} f(X(t))$ equals to a critical value of the polynomial $f_\gamma^W(u)$. (ii) This curve is obtained as the image by the map (2.6) of a curve $Q(t)$ whose coefficients $\mathbf{c} \in \mathcal{C}$ satisfy $(L_0 - \rho + 1) \mid \mathbb{J} \mid -$tuple of algebraic equations for ρ* (2.18)*, L_0 (3.7). (iii) The curve $Q(t)$ mentioned in (ii) has a parametric representation (3.8) of parametric length $L_0 - \rho + 2$, i.e. we can assume its parametrisation coefficients $(c'(j), c''(j)) = 0$ for $j > L_0 - \rho + 1$.*

Proof. By Lemma 3.2 and Lemma 3.4 we have already shown that the curve under question satisfies (3.2) of Definition 3.1.

Now we need to show that there is a curve (3.8) satisfying (3.3). For this purpose we look for a curve that makes the inequality (3.6) valid. Proposition 2.1 and Lemma 3.3 tell us that it is enough to verify (3.6) for $j \in \mathbb{J}$ as $ord\left(\left\langle \mu_j, \vartheta_u f^W \right\rangle (Q(t))\right) \geq \rho > 0$.

The expansion of $\left\langle \mu_j, \vartheta_u f^W \right\rangle (Q(t)), j \in \mathbb{J}$ in t has the following form

$$g_\rho^j(\mathbf{c}) t^\rho + g_{\rho+1}^j(\mathbf{c}) t^{\rho+1} + h.o.t.$$

For each $j \in \mathbb{J}$, the vector with polynomial entries $g_\rho^j(\mathbf{c})$ depends on all n variables $(c'(0), c''(0)) \in \mathbb{C}^n \subset \mathcal{C}$ in view of the choice of $q \in \mathbb{Z}^n$ made in Proposition 2.1.

As $\mid \mathbb{J} \mid < n$, the system of algebraic equations $g_\rho^j(\mathbf{c}) = 0, \forall j \in \mathbb{J}$ has non-trivial solutions in $\mathbb{C}$ while $g_\rho^j(\mathbf{c})$ effectively depends on $(c'(0), c''(0))$.

The vector with polynomial entries $g_{\rho+1}^j(\mathbf{c})$ effectively depends on $(c'(0), c''(0), c'(1), c''(1)) \in \mathbb{C}^{2n} \subset \mathcal{C}$ thus the system of equations $g_{\rho+1}^j(\mathbf{c}) = 0, \forall j \in \mathbb{J}$ has also non-trivial solutions in $\mathbb{C}$.

In this way, we can find non-trivial solutions to $(L_0 + 1 - \rho) \mid \mathbb{J} \mid -$tuple of algebraic equations

$$g_\rho^j(\mathbf{c}) = g_{\rho+1}^j(\mathbf{c}) = \cdots = g_{L_0}^j(\mathbf{c}) = 0, \forall j \in \mathbb{J}$$

for L_0 (3.7).

To prove this, it is enough to show that $g^j_{\rho+\ell}(\mathbf{c})$ effectively depends on $(c'(\ell), c''(\ell))$ that are absent in $g^j_{\rho+\tilde{\ell}}(\mathbf{c})$ for $\tilde{\ell} \in [0;\ell-1]$.

First we remark that $g^j_{\rho+\ell}(\mathbf{c})$ is a sum of monomials of the form

$$const. \prod_{\nu=1}^{n} \prod_{i_\nu \in I_\nu} c_\nu(i_\nu)^{m_{i_\nu,\nu}} \tag{3.9}$$

satisfying the following homogeneity condition

$$\rho + \ell = \sum_{\nu=1}^{n} \sum_{i_\nu \in I_\nu} (q_\nu + i_\nu) m_{i_\nu,\nu} \tag{3.10}$$

with $I_\nu \subset [0;\ell]$, $m_{i_\nu,\nu} \geq 0$, $\forall i_\nu \in I_\nu, \forall \nu \in [1;n]$.

By a simple calculation, we see that non vanishing terms of the following form appear in $g^j_{\rho+\ell}(\mathbf{c}), \ell \geq 1$,

$$const. \left(\prod_{\nu=1,\nu\neq\kappa}^{k} \prod_{i_\nu \in I_\nu} c_\nu(i_\nu)^{m_{i_\nu,\nu}} \right) \left(\prod_{i_\kappa \in I_\kappa \setminus \{\ell\}} c_\kappa(i_\kappa)^{m_{i_\kappa,\kappa}} \right) c_\kappa(\ell) \tag{3.11}$$

for $\kappa \in [1;k]$ and

$$const. \left(\prod_{\nu=k+1,\nu\neq\kappa}^{n} \prod_{i_\nu \in I_\nu} c_\nu(i_\nu)^{m_{i_\nu,\nu}} \right) \left(\prod_{i_\kappa \in I_\kappa \setminus \{\ell\}} c_\kappa(i_\kappa)^{m_{i_\kappa,\kappa}} \right) c_\kappa(\ell) \tag{3.12}$$

for $\kappa \in [k+1;n]$. Non vanishing of (3.11) with $\kappa \in [1;k]$ is due to the presence of a term proportional to $u'^{\alpha'}(t)U''^{\beta''}(t)$ such that $\langle q, (\alpha', \beta'')\rangle = \rho$ in $\left\langle \mu_j, \vartheta_u f^W(u)\right\rangle$. That of (3.12) with $\kappa \in [k+1;n]$ is due to the presence of a term proportional to $U''^{\alpha''}(t)$ such that $\langle q, (0, \alpha'')\rangle = \rho$ in $\left\langle \mu_j, \vartheta_u f^W(u)\right\rangle$ and $u''_* \neq 0$. This can be seen from Proposition 2.1 and Corollary 3.1. These originating monomials $u'^{\alpha'}(t)U''^{\beta''}(t)$, $U''^{\alpha''}(t)$ are uniquely determined from power exponents $\{m_{i_\nu,\nu}\}_{i_\nu\in I_\nu}$ that can be seen from (3.9), (3.10):

$$\alpha_\kappa = 1 + \sum_{i_\kappa \in I_\kappa} m_{i_\kappa,\kappa} \text{ for } \kappa \in [1;k]; \quad (resp.\ \kappa \in [k+1;n]),$$

$$\alpha_\nu = \sum_{i_\nu \in I_\nu} m_{i_\nu,\nu} \text{ for } \nu \in [1;k] \setminus \{\kappa\} \quad (resp.\ \nu \in [k+1;n] \setminus \{\kappa\}).$$

Thus no cancellation of terms (3.11), (3.12) happens. As $\left\langle \mu_j, \vartheta_u f^W(u)\right\rangle$, $j \in \mathbb{J}$ contains monomials $u'^{\alpha'}(t)U''^{\beta''}(t)$, $U''^{\alpha''}(t)$ of the above type, the factor $c_\kappa(\ell)$, $\kappa \in [1;n]$ appears in $g^j_{\rho+\ell}(\mathbf{c})$ but it does not appear in $g^j_{\rho+\tilde{\ell}}(\mathbf{c})$, $\tilde{\ell} \in [0;\ell-1]$ because of (3.10).

Corollary 3.2. *For $f \in \mathbb{C}[x]$ with dim $\Delta(f) = n$, the following inclusion holds*

$$\bigcup_{\gamma} f_\gamma(\text{Sing } f_\gamma \cap (\mathbb{C}^\times)^{dim\ \gamma}) \subset \mathcal{K}_\infty(f) \tag{3.13}$$

where γ runs among bad faces of $\Delta(f)$.

Proof. Theorem 3.1 tells us

$$f_\gamma^W(\text{Sing } f_\gamma^W \cap (\mathbb{C}^\times)^{dim\ \gamma}) \subset \mathcal{K}_\infty(f).$$

It is enough to show that

$$f_\gamma^W(\text{Sing } f_\gamma^W \cap (\mathbb{C}^\times)^{dim\ \gamma}) = f_\gamma(\text{Sing } f_\gamma \cap (\mathbb{C}^\times)^{dim\ \gamma})$$

for $f_\gamma(x) = \sum_{\alpha \in \gamma \cap supp(f)} a_\alpha x^\alpha$.

From Lemma 2.2, $f_\gamma^W(u)$ is a polynomial depending effectively on toric variables u'' and independent of affine variables u' (the condition (i) of the Definition 2.4). This means that $\vartheta_{u_1} f_\gamma^W(u) = \cdots = \vartheta_{u_k} f_\gamma^W(u) = 0$. Thus for $u''_* \in \text{Sing } f_\gamma^W \cap (\mathbb{C}^\times)^{dim\ \gamma}$, the vanishing of the logarithmic gradient vector holds: $\vartheta_u f_\gamma^W(0, u''_*) = 0$. By using the map $u''(x) = (x^{m_{k+1}}, \cdots, x^{m_n})$ induced by the inverse to (2.6), we see $f_\gamma(x) = f_\gamma^W(0, u''(x))$. Taking the relation (3.4) into account, we see that this entails $\vartheta_x f_\gamma(x_*) = 0$ for $x_* \in (\mathbb{C}^\times)^n$ that satisfies $u''(x_*) = u''_*$.

Conversely, if $\vartheta_x f_\gamma(x_*) = 0$ for $x_* \in (\mathbb{C}^\times)^n$, by (3.4), we see $\vartheta_u f_\gamma^W(0, u''_*) = 0$ for $u''_* = u''(x_*)$ the image of the map (2.6).

Remark 3.1. After [18, Theorem 5], if a polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}$ with rational coefficients admits a bounded infimum, it must belong to the set $\mathcal{K}_\infty(f) \cup f(\text{Sing} f)$. Our Corollary 3.2 exhibits the candidate for infimum of a real polynomial map, if it is possible to construct a real curve satisfying (3.2), (3.3) after the method in Lemma 3.4. See [20]. Thus our approach represents a potential utility in the optimisation problem.

In [3] Theorem 1.1, for f Newton non-degenerate at infinity it is stated that

$$\mathcal{K}_\infty(f) \subset \{0\} \cup \bigcup_{\Delta} f_\Delta(\text{Sing } f_\Delta \cap (\mathbb{C}^\times)^{dim\ \Delta}), \tag{3.14}$$

where the union runs over all "atypical faces" of f (faces that satisfy our Definition 2.4 (ii)).

Here we remark the following:

Corollary 3.3. *For f Newton non-degenerate at infinity in the sense of [3], the following inclusion holds*

$$\mathcal{K}_\infty(f) \subset \bigcup_{\gamma} f_\gamma(\text{Sing } f_\gamma \cap (\mathbb{C}^\times)^{dim\ \gamma}) \cup \{0\}, \tag{3.15}$$

where γ runs among bad faces of $\Delta(f)$.

Proof. For Δ an atypical face satisfying (ii) of Definition 2.4, but not (i), we see that

$$f_\Delta(\mathrm{Sing}\ f_\Delta \cap (\mathbb{C}^\times)^{dim\ \Delta}) \subset \{0\}. \tag{3.16}$$

In this case, the face Δ is contained in a ($dim\ \Delta$) dimensional affine space that does not pass through the origin. As f_Δ is a weighted homogeneous polynomial such that $f_\Delta = \sum_{i=1}^n w_i \vartheta_i f_\Delta$ for a non-zero rational vector $(w_i)_{i=1}^n$, we have (3.16).

The relations (3.14) and (3.16) yield (3.15).

In combining Corollaries 3.2, 3.3, we determine $\mathcal{K}_\infty(f)$ up to $\{0\}$ under the assumption of Theorem 3.1 for f Newton non-degenerate at infinity.

For f depending effectively on two variables, M.Ishikawa [10, Theorem 6.5] established a precise description of $\mathcal{B}(f)$ where a set essentially larger than the LHS of (3.13) appears. This situation suggests that the superset of $\mathcal{K}_\infty(f)$ can be essentially larger than the RHS of (3.15), if f is not Newton non-degenerate at infinity.

Parusiński [15] established the equality

$$\mathcal{K}_\infty(f) \cup f(\mathrm{Sing}\ f) = \mathcal{B}(f)$$

for the case where the projective closure in $\mathbb{P}^n$ of the generic fibre of f has only isolated singularities on the hyperplane at infinity $H_\infty \subset \mathbb{P}^n$. In this setting, we see

Corollary 3.4. *Let f be a polynomial decomposed into homogeneous terms $f(x) = \sum_{j=0}^d f_j(x), deg\ f_j = j$, in such a way that the Newton polyhedron $\Delta(f_d)$ is of full dimension ($= n-1$). Furthermore, the projective closure in $\mathbb{P}^n$ of the generic fibre of f has only isolated singularities on the hyperplane at infinity $H_\infty \subset \mathbb{P}^n$. Then we have*

$$\Sigma_f \subset \mathcal{B}(f) \subset \Sigma_f \cup \{0\}. \tag{3.17}$$

for

$$\Sigma_f = f(\mathrm{Sing}\ f) \cup \bigcup_\gamma f_\gamma(\mathrm{Sing}\ f_\gamma \cap (\mathbb{C}^\times)^{dim\ \gamma}).$$

Thus to decide exactly $\mathcal{B}(f)$ in this case, it is enough to verify $\{0\} \in \mathcal{B}(f)$ or not.

Now we consider an affine lattice $\mathbb{L}_\gamma$ (i.e. a principal homogeneous space of a free Abelian group) with rank $dim\ \gamma$ generated by

$$\gamma_{\mathbb{Z}} \cdot W = \{\alpha.W; \alpha \in \gamma \cap \mathbb{Z}^n\} \tag{3.18}$$

for γ a bad face and W (2.4). We denote by $\mathbb{L}_\gamma \otimes \mathbb{R}$ the real affine space spanned by $\mathbb{L}_\gamma$. In $\mathbb{L}_\gamma \otimes \mathbb{R}$, we introduce the volume form Vol_γ by setting the volume of an elementary simplex with vertices in $\mathbb{L}_\gamma$ equal to 1 ([8, §2 C]). After [8, Theorem 3A.2] the principal $A-$determinant $D_A(f_\gamma^W + a_0)$ of the polynomial $\sum_{\alpha \in supp(f_\gamma)} a_\alpha u^{\alpha.W} + a_0$ is a homogeneous polynomial of degree $Vol_\gamma(\overline{\gamma_{\mathbb{Z}} \cdot W})$

in $(a_\alpha)_{\alpha\in suppf_\gamma}$ and a_0 for $\overline{\gamma_{\mathbb{Z}}\cdot W}$ the convex hull of $\gamma_{\mathbb{Z}}\cdot W$ and $\{0\}$ in $\mathbb{L}_\gamma\otimes\mathbb{R}$. Let us define the volume $Vol_\gamma(\overline{\gamma_{\mathbb{Z}}})$ in an analogous way to $Vol_\gamma(\overline{\gamma_{\mathbb{Z}}\cdot W})$ in replacing W by the identity matrix $Id_{dim\,\gamma}$.

We remark that $Vol_\gamma(\overline{\gamma_{\mathbb{Z}}\cdot W})$ equals to $Vol_\gamma(\overline{\gamma_{\mathbb{Z}}})$ due to the unimodularity of W.

Thus we establish the following evaluation on the cardinality of $\mathcal{K}_\infty(f)$ and $\mathcal{B}(f)$.

Corollary 3.5. *For f like in Corollary 3.3 (resp. like in 3.4), the following inequalities hold*

$$\#\mathcal{K}_\infty(f)\le 1+\sum_{\gamma:bad\ face} Vol_\gamma(\overline{\gamma_{\mathbb{Z}}}), \tag{3.19}$$

[resp.

$$\#\mathcal{B}(f)\le 1+\sum_{\gamma:bad\ face} Vol_\gamma(\overline{\gamma_{\mathbb{Z}}})+\#f(\mathrm{Sing}\ f).] \tag{3.20}$$

We remark that the estimation above (3.19) gives a sharper approximation than [11, Theorem 2.2, 2.3] under conditions imposed in Corollary 3.5.

4 Non relatively simple face

In [19], the notion of relatively simple face has been introduced.

Definition 4.1. *([19, Definition 1.4]) A face $\gamma\subset\tilde{\Gamma}_-(f)\cap\Delta(f)$ is called relatively simple, if $C(\gamma)^*\subset(\tilde{\Gamma}_-(f))^*$ is simplicial or $dim\ C(\gamma)^*\le 3$.*

The main result Theorem 1.6 of [19] relies heavily on the notion of relatively simple faces. It shows that the set $\bigcup_\gamma f_\gamma(\mathrm{Sing}\ f_\gamma\cap(\mathbb{C}^\times)^{dim\,\gamma})$ where γ runs relatively simple bad faces is contained in the bifurcation value set of a polynomial mapping f under the condition of Newton non-degeneracy and isolated singularities at infinity. We say that f has isolated singularities at infinity over $b\in\bigcup_\gamma f_\gamma(\mathrm{Sing}\ f_\gamma\cap(\mathbb{C}^\times)^{dim\,\gamma})$ if $f_\gamma^{-1}(b)\cap(\mathbb{C}^\times)^{dim\,\gamma}$ has only isolated singular points for every bad face γ.

In this section, we examine an example of a polynomial in 5 variables with non-relatively simple bad face (see (4.2)). Even in this situation, we can construct a curve $X(t)$ satisfying (3.2), (3.3) of Definition 3.1 approaching the value $f_\gamma(\mathrm{Sing}\ f_\gamma\cap(\mathbb{C}^\times)^{dim\,\gamma})$ for non-relatively simple bad face γ. This gives an example to Corollary 3.2 that is not covered by [19].

1) Let us begin with a simplicial cone in $\mathbb{R}^4$ generated by four 1 dimensional cones $C(\bar v_1),C(\bar v_2),C(\bar v_3)$, $C(\bar v_4)$ where $\bar v_1=(3,3,4,2)$, $\bar v_2=(1,3,5,2)$, $\bar v_3=(3,1,4,2)$, $\bar v_4=(1,1,1,1)$. Here we recall the Definition 2.2.

Each face of the simplicial cone

$$F_{i,j,k}:=C\left(\sum_{s_i+s_j+s_k=1} s_i\bar v_i+s_j\bar v_j+s_k\bar v_k\right) \tag{4.1}$$

where $\{i,j,k\} = \{1,2,3,4\} \setminus \{\ell\}$ for $\ell \neq i,j,k$, is defined as a subset of a plane $\{v \in \mathbb{R}^4; \langle A_{i,j,k}, v\rangle = 0\}$. The orthogonal vector to each of the faces is given by: $A_{1,2,3} = (-2,0,-4,11)$, $A_{1,3,4} = (-2,0,1,1)$, $A_{1,2,4} = (1,-5,2,2)$, $A_{2,3,4} = (1,1,0,-2)$. We choose the direction of the orthogonal vector in such a way that $\langle A_{i,j,k}, \bar{v}_\ell\rangle > 0$ for every quadruple indices $\{i,j,k,\ell\} = \{1,2,3,4\}$.

We shall construct a non-simplicial cone by means of an additional cone $C(\bar{v}_5)$ that will be built with the aid of the vector $A_{2,3,4}$. Namely we choose $\bar{v}_5 = \bar{v}_2 + \bar{v}_3 + \bar{v}_4 - A_{2,3,4} = (4,4,10,7)$.

We shall convince ourselves that the new non-simplicial cone geneated by five 1 dimensional cones $C(\bar{v}_1), C(\bar{v}_2), C(\bar{v}_3)$, $C(\bar{v}_4), C(\bar{v}_5)$ is a convex cone with six faces. In fact, we calculate the orthogonal vector to each of faces $F_{i,j,k}$ defined in a manner similar to (4.1) for $\{i,j,k\} = \{1,2,3,4,5\} \setminus \{\ell,p\}$ such that $\{i,j,k,\ell,p\} = \{1,2,3,4,5\}$. $A_{2,3,5} = (17,29,-24,8)$, $A_{2,4,5} = (2,-1,1,-2)$, $A_{3,4,5} = (-1,3,2,-4)$ in addition to $A_{1,2,3}, A_{1,3,4}, A_{1,2,4}$ already known ($F_{2,3,4}$ is not a face of the newly constructed cone any more). We have again $\langle A_{i,j,k}, \bar{v}_\ell\rangle > 0$, $\langle A_{i,j,k}, \bar{v}_p\rangle > 0$ for every quintuple indices $\{i,j,k,\ell,p\}$ as above and see thus the newly constructed cone is convex.

2) Now we consider a shift of the apex of the cone towards a vector $\bar{v}_0 \in (\mathbb{R}_{>0})^4$, say $\bar{v}_0 = (1,2,3,1)$. We denote the face of the shifted cone $B_{i,j,k} = \bar{v}_0 + F_{i,j,k}$, $(i,j,k) \in I_F := \{(1,2,3),(1,3,4),(1,2,4),(2,3,5),(2,4,5),(3,4,5)\}$. The face $B_{i,j,k}$ is a subset of a plane $\{v \in \mathbb{R}^4; \langle A_{i,j,k}, v\rangle = \langle A_{i,j,k}, \bar{v}_0\rangle\}$ for $(i,j,k) \in I_F$. In "homogenising" the defining equation of a plane containing $B_{i,j,k}$ we get a plane in $\mathbb{R}^5$: $H_{i,j,k} = \{(x,y,z,w,r) \in \mathbb{R}^5; \langle A_{i,j,k}, (x,y,z,w)\rangle = \langle A_{i,j,k}, \bar{v}_0\rangle\, r\}$ for $(i,j,k) \in I_F$. In this way we get six ($= \sharp I_F$) planes in $\mathbb{R}^5$ passing through the origin and the intersection

$$\bar{C} = \cap_{(i,j,k)\in I_F}\{(x,y,z,w,r) \in \mathbb{R}^5; \langle A_{i,j,k}, (x,y,z,w)\rangle - \langle A_{i,j,k}, \bar{v}_0\rangle\, r \geq 0\}$$

produces a convex cone. By construction, every plane $H_{i,j,k}$ contains a 1 dimensional cone $C(v_0)$. Consequently, the cone $\bar{C}$ contains $C(v_0) = \cap_{(i,j,k)\in I_F} H_{i,j,k}$ for $v_0 = (\bar{v}_0, 1)$.

If we use the choice done in 1) and $v_0 = (1,2,3,1,1)$, the vectors $L_{i,j,k}$ orthogonal to the planes $H_{i,j,k}$ are given by $L_{1,2,3} = (-2,0,-4,11,3)$, $L_{1,3,4} = (-2,0,1,1,-2)$, $L_{1,2,4} = (1,-5,2,2,1)$, $L_{2,3,5} = (17,29,-24,8,-11)$, $L_{2,4,5} = (2,-1,1,-2,-1)$, $L_{3,4,5} = (-1,3,2,-4,-7)$. We define vectors $v_i = (\bar{v}_i, 0), i \in [1;5]$ in $\mathbb{R}^5$. The vector $L_{i,j,k} \in (\mathbb{R}^5)^*$ is orthogonal to v_i, v_j, v_k in addition to v_0. We shall check that $\langle L_{i,j,k}, v_\ell\rangle \geq 0$ for every v_ℓ, $\ell \in [0;5]$. Except 6 triples $(i,j,k) \in I_F$, this positivity property is not satisfied for other triples from $\{1,2,3,4,5\}$. In particular, for $L_{2,3,4} = (1,1,0,-2,-1)$ we have $\langle L_{2,3,4}, v_5\rangle = -6$.

The following polynomial

$$f = -3x^{v_0} + x^{3v_0} + x^{v_1+v_0} + x^{v_2+v_0} + x^{v_3+v_0} + x^{v_4+v_0} + x^{v_5+v_0} \tag{4.2}$$

has a $1-$ dimensional bad face contained in $C(v_0)$ that is not relatively simple in the sense of Definition 4.1. In fact, the cone $C(v_0)^* \in (\mathbb{R}^5)^*$ in the dual fan $(\widetilde{\Gamma}_-(f))^*$ is a non-simplicial 4 dimensional cone with 6 generators $L_{i,j,k}, (i,j,k) \in I_F$ calculated above. In the sequel, we shall show the inclusion

$$\{\pm 2\} \subset \mathcal{K}_\infty(f) \subset \{0, \pm 2\}. \tag{4.3}$$

3) Now we shall construct a unimodular cone $\sigma \in K$ of the unimodular simplicial subdivision $(\widetilde{\Gamma}_-(f))^*$. For example, if we choose

$$a_1 = \frac{1}{5}(L_{1,3,4} + L_{1,2,4} + 2L_{1,2,3}), \quad a_2 = \frac{1}{15}(L_{1,2,4} + 3L_{1,2,3} + 10L_{2,4,5}),$$
$$a_3 = L_{1,2,3}, \quad a_4 = L_{2,4,5}, \quad a_5 = (1,1,-1,1,0),$$

as column vectors of

$$W = ({a_1}^T, {a_2}^T, {a_3}^T, a_4^T, a_5^T) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -2 & 2 & 1 \\ -1 & -1 & 0 & -1 & 1 \\ -1 & 0 & -4 & 1 & -1 \\ 5 & 1 & 11 & -2 & 1 \\ 1 & 0 & 3 & -1 & 0 \end{pmatrix},$$

they generate a unimodular cone σ. One shall also verify that $\langle a_5, \alpha \rangle > 0$ for all $\alpha \in supp(f)$. As for the method to obtain unimodular simplicial subdivision of a cone see [14].

In this situation, the polynomial (4.2) will have the following form

$$f^W(u) = {u_1}^{17}{u_2}^7{u_3}^{29}{u_5}^6 + {u_1}^2{u_2}^4{u_4}^5{u_5}^3 + {u_1}^2u_2{u_3}^5{u_5}^3 +$$
$$u_1{u_5}^2 + {u_2}^2{u_4}^3{u_5}^5 + {u_5}^3 - 3u_5.$$

Consider the expansion (2.8) around the singular point $u^* = (0,0,0,0,1)$ where $\vartheta_u f^W(u^*) = 0$. Then we see that $supp_{u^*}(\langle \mu_j, f^W \rangle) \subset \{v; \langle q, v \rangle \geq 5\}$ for $j \in [1;4]$ and vector $q = (5,-20,3,15,5)$. The face $\Gamma \subset supp_{u^*}(\langle \mu_j, f^W \rangle)$ treated in Proposition 2.1, i.e. $\Gamma \subset \{v; \langle q, v \rangle = 5\}$ can be found as a convex hull of points $(0,0,0,\ 0,\ 1)$, $(0,2,0,3,0)$, $(1,0,0,0,0)$, $(2,1,5,0,0)$, $(2,4,0,5,0)$.

The relation $(q',0).W = (-1,0,-2,8,-1)$ for $(q',0) = (5,-20,3,\ 15,0)$ shows that the condition (μ) of Lemma 3.1 is satisfied. From this relation, we see that $\mathbb{J} = \{1,2,4,5\}$ and $min_{i \neq j} \langle (q',0), w_i - w_j \rangle = \langle (q',0), w_3 - w_4 \rangle = -10 = -L_0$, $\rho = min_{\alpha \in \Delta_{u^*}(\langle \mu_j, f^W \rangle)} \langle q, \alpha \rangle = 5$.

We consider a curve $Q(t)$ with real coefficients of length $11 = L_0 + 1$ namely

$$u_1 = \sum_{j=0}^{10} c_1(j) t^{j+5}, u_2 = \sum_{j=0}^{10} c_2(j) t^{j-20},$$

$$u_3 = \sum_{j=0}^{10} c_3(j) t^{j+3}, u_4 = \sum_{j=0}^{10} c_4(j) t^{j+15}, u_5 = 1 + \sum_{j=0}^{10} c_5(j) t^{j+5}.$$

The system of equations (that corresponds to the coefficients of t^5 terms)

$$g_1^1(\mathbf{c}) = g_1^2(\mathbf{c}) = g_1^4(\mathbf{c}) = g_1^5(\mathbf{c}) = 0$$

where

$g_1^1(\mathbf{c}) = 4c_1(0)^2c_2(0)^4c_4(0)^5 + 2c_1(0)^2c_2(0)c_3(0)^5 + 2c_1(0) + 4c_2(0)^2c_4(0)^3 + 6c_5(0)$,

$g_1^2(\mathbf{c}) = 3c_1(0)^2c_2(0)^4c_4(0)^5 + 3c_1(0)^2c_2(0)c_3(0)^5 + 5c_1(0) + 5c_2(0)^2c_4(0)^3 + 12c_5(0)$,
$g_1^4(\mathbf{c}) = 3c_1(0)^2c_2(0)^4c_4(0)^5+2c_1(0)^2c_2(0)c_3(0)^5+3c_1(0)+3c_2(0)^2c_4(0)^3+6c_5(0)$
$g_1^5(\mathbf{c}) = c_1(0)^2c_2(0)^4c_4(0)^5 + c_1(0)^2c_2(0)c_3(0)^5 + c_1(0) + c_2(0)^2c_4(0)^3 + 6c_5(0)$
admits non-trivial solutions because each equation effectively depends on $c_j(0)_{j\in[1;5]}$. In a similar manner, the system of equations (that corresponds to the coefficients of t^{k+1} terms)

$$g_k^1(\mathbf{c}) = g_k^2(\mathbf{c}) = g_k^4(\mathbf{c}) = g_k^5(\mathbf{c}) = 0$$

for $k \in [2,6]$ also admits non-trivial solutions by virtue of Theorem 3.1.

In this way, we can find non-trivial solutions for a system of 24 algebraic equations $g_k^j(\mathbf{c}) = 0, \mathbf{c} \in \mathcal{C}, j = 1,2,4,5, k \in [1,6]$. This means that we can construct a curve $Q(t)$ of parametric length 7 sastisfying the condition (3.6) $-10 + ord\left\langle \mu_j, \vartheta_u f^W(Q(t))\right\rangle > 0$ for $j \in \mathbb{J} = \{1,2,4,5\}$. The image $X(t)$ of the curve $Q(t)$ by the map

$$x_1 = u^{(-1,1,-2,2,1)}, x_2 = u^{(-1,-1,0,-1,1)}, x_3 = u^{(-1,0,-4,1,-1)},$$

$$x_4 = u^{(5,1,11,-2,1)}, x_5 = u^{(1,0,3,-1,0)}$$

satisfies (3.2), (3.3) of Definition 3.1 and $lim_{t\to 0} f(X(t)) = -2 \in \mathcal{K}_\infty(f)$. A similar arguments shows $2 \in \mathcal{K}_\infty(f)$. We see that the polynomial (4.2) is Newton non-degenerate at infinity in the sense of [3, Theorem 1.1]. There is no contribution in the right hand side superset in (3.14) from "atypical faces" except that from the "strongly atypical face" [3, Definition 3.2] corresponding to the bad face of $\Delta(f)$ for (4.2). Thus we conclude the inclusion relation (4.3).

For the polynomial (4.2) the method of [5, Theorem 3.5.] proposes construction of a curve of parametric length $d^4(d+1)+1 = 3360001$ with $d = 20 = |v_0 + v_1|$ satisfying the required properties.

5 Examples

We shall give an example that illustrates our Theorem 3.1.

Example 5.1. (Non-isolated singularity on a two dimensional bad face)

Consider a polynomial $f(x) = x^{v_1} + (x^{v_2} - x^{v_3} + 1)^2 + (x^{v_2} - x^{v_3} + 1)^3 + x^{v_4} - 2$ with $v_1 = (2,1,1), v_2 = (2,2,1), v_3 = (1,2,1), v_4 = (3,1,1)$. This case with non-isolated singularities at infinity has not been treated in [19].

We remark that

$$M = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = ({\mu_1}^T, {\mu_2}^T, {\mu_3}^T) = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

is unimodular. Thus we can set

$$M^{-1} = W = ({a_1}^T, {a_2}^T, {a_3}^T) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -3 & 2 \end{pmatrix}.$$

The only bad face γ of $\Delta(f)$ is located on the plane spanned by v_2, v_3. For the above W, we have

$$f^W(u) = -2 + u_1 + (u_2 - u_3 + 1)^2 + (u_2 - u_3 + 1)^3 + \frac{u_1 u_2}{u_3}.$$

After (3.19), $\#\mathcal{K}_\infty(f) \leq 10$ while $Vol_\gamma(\overline{\gamma_{\mathbb{Z}} \cdot W}) = 9$ for $\overline{\gamma_{\mathbb{Z}} \cdot W}$: the convex hull of $\{(0,0),(3,0),(0,3)\}$.

The polynomial $f_\gamma^W(0, u_2, u_3) = (u_2 - u_3 + 1)^2 + (u_2 - u_3 + 1)^3$ has non-isolated singularities along a line $u_2 - u_3 + 1 = 0$. We can choose, for example $u^* = (0, -1/3, 2/3)$. In the neighbourhood of this point the rational function $f^W(u)$ has the expansion

$$f^W(u) = -2+u_1+(U_2-U_3)^2+(U_2-U_3)^3+\frac{3u_1}{2}(U_2-1/3)(1-\frac{3U_3}{2}+(\frac{3U_3}{2})^2+\cdots),$$

for $U_2 = u_2 + 1/3, U_3 = u_3 - 2/3$. The polyhedron $\Delta^* = \Delta(\langle \mu_i, \vartheta_u f^W(u)\rangle)$, $i = 1,2,3$, gives rise to the face Γ (2.17). A direct calculation shows

$$\langle \mu_3, \vartheta_u f^W(u)\rangle = \frac{u_1}{16} - 2U_2 + 2U_3 + h.o.t.$$

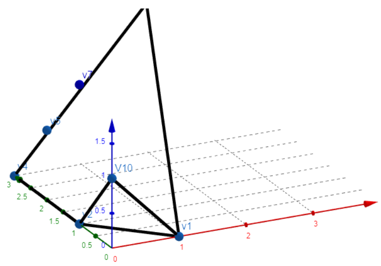

Figure 5.1: The facet Γ

Thus we find the face Γ located on the plane containing $(1,0,0),(0,0,1),(0,1,0)$ and $q = (1,1,1)$, $(q',0) = (1,0,0)$. Thereby the condition (μ) of Lemma 3.1 is satisfied as $\langle (q',0), w_2\rangle = -1$.

We calculate $L_0 = 3$ and $\rho = 1$. A curve $Q(t)$ (3.8) with real coefficients of parametric length 4, namely

$$u_1 = \sum_{j=0}^{3} c_1(j)t^{j+1}, u_2 = -1/3 + \sum_{j=0}^{3} c_2(j)t^{j+1}, u_3 = 2/3 + \sum_{j=0}^{3} c_3(j)t^{j+1}$$

that satisfies

$$-3 + ord\,\langle \mu_j, \vartheta_u f^W(Q(t))\rangle > 0$$

for $j \in \mathbb{J} = \{1, 3\}$ can be constructed.

We remark that after the method of [5, Theorem 3.5.], the real curve with required property has parametric length $16 \times 15^2 + 1 = 3601$.

In fact, if we plug these expressions into $\left\langle \mu_3, \vartheta_u f^W(u) \right\rangle$, we get an expansion with initial term proportional to t^1, ($\langle q, \alpha \rangle = 1$ for $\alpha \in \Gamma$);

$\left\langle \mu_3, \vartheta_u f^W(u) \right\rangle (Q(t)) = \{c_1(0)/2 - 2c_2(0) + 2c_3(0))\}t+$
$1/4\{2c_1(1) + 6c_1(0)c_2(0) - 4c_2(0)^2 - 8c_2(1) + 3c_1(0)c_3(0) + 8c_2(0)c_3(0) - 4c_3(0)^2 + 8c_3(1)\}t^2+$
$1/8\{4c_1(2) + 12c_1(1)c_2(0) + 24c_2(0)^3 + 12c_1(0)c_2(1) - 16c_2(0)c_2(1) - 16c_2(2) + 6c_1(1)c_3(0) - 18c_1(0)c_2(0)c_3(0) - 72c_2(0)^2c_3(0) + 16c_2(1)c_3(0) - 9c_1(0)c_3(0)^2 + 72c_2(0)c_3(0)^2 - 24c_3(0)^3 + 6c_1(0)c_3(1) + 16c_2(0)c_3(1) - 16c_3(0)c_3(1) + 16c_3(2)\}t^3 + \cdots$

The case $\left\langle \mu_1, \vartheta_u f^W(Q(t)) \right\rangle$ also admits a similar expression. In both cases $j \in \mathbb{J} = \{1, 3\}$, coefficient of t depends on $(c_1(0), c_2(0), c_3(0))$, that of t^2 depends on $(c_1(0)$, $c_2(0)$, $c_3(0)$, $c_1(1)$, $c_2(1)$, $c_3(1))$, that of t^3 depends on $(c_i(j))_{i=1,2,3,j=0,1,2}$. Thus the system of algebraic equations imposed on $(c_i(j))_{i=1,2,3,j=0,1,2} \in \mathbb{C}^9$ to make the coefficients of t, t^2, t^3 vanish has non-trivial solutions. In fact, this system can be solved in $\mathbb{R}^9$. As for the construction of a real curve $Q(t)$, i.e. a real curve $X(t)$, see [20].

We can choose as $(c_1(3), c_2(3), c_3(3)) \in \mathbb{C}^3$ arbitrary non-zero vector.

The image of the curve $Q(t)$ by the map

$$x_1 = u_2 u_3^{-1}, x_2 = u_1^{-1} u_2, x_3 = u_1^2 u_2^{-3} u_3^2$$

satisfies (3.2) , (3.3) of Definition 3.1 and $lim_{t \to 0} f(X(t)) = -2 \in \mathcal{K}_\infty(f)$.

As it can be seen in this example, the curve $X(t)$ approaches the surface $\{x; f(x) = -2\}$ as $t \to 0$.

We obtain a curve $X(t) = (x_1(t), x_2(t), x_3(t))$ asymptotically approaching the surface $\{x; f(x) = -2\}$ as follows:

$$x_1(t) = \frac{t^4 + t^3 + t^2 + t - \frac{1}{3}}{t^4 + \frac{131t^3}{256} - \frac{t^2}{4} + \frac{3t}{4} + \frac{2}{3}},$$

$$x_2(t) = \frac{t^4 + t^3 + t^2 + t - \frac{1}{3}}{t^4 + t^3 + t^2 + t},$$

$$x_3(t) = \frac{\left(t^4 + \frac{131t^3}{256} - \frac{t^2}{4} + \frac{3t}{4} + \frac{2}{3}\right)^2 \left(t^4 + t^3 + t^2 + t\right)^2}{\left(t^4 + t^3 + t^2 + t - \frac{1}{3}\right)^3}.$$

On the Figure 5.2, we see two branches of the curve that correspond to the asymptotes $t \to 0$ from $t > 0$ and $t < 0$ respectively. In Example 5.1, the figure illustrating algebraic surface and rational parametric curves are prepared with the aid of the computer programme MATLAB.

Example 5.2. (Isolated singularities at infinity)

Consider a polynomial $f(x) = -3x^{v_0} + x^{v_1} + x^{v_2} + x^{3v_0}$ with $v_0 = (2, 2, 1), v_1 = (1, 0, 1), v_2 = (0, 1, 1)$.

$$W = ({a_1}^T, {a_2}^T, {a_3}^T) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -1 & -1 \\ 2 & 2 & 1 \end{pmatrix}.$$

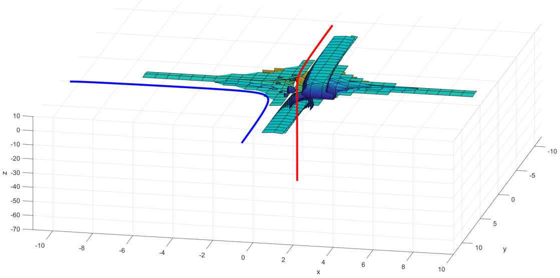

Figure 5.2: Branches of the curve $X(t)$

$$M = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = ({\mu_1}^T, {\mu_2}^T, {\mu_3}^T) = \begin{pmatrix} -1 & -2 & -1 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

The only bad face γ of $\Delta(f)$ is located on the cone $\{t.v_0; t > 0\}$. We have

$$f^W = {u_1}^3 {u_2}^2 {u_3}^2 + u_2 + {u_3}^3 - 3u_3,$$

and $f^W_\gamma(u) = {u_3}^3 - 3u_3$ has singular points $u_3^* = \pm 1$ and critical values ∓ 2 respectively. After [19] we see that in this case the bifurcation set $\mathcal{B}(f) \subset \mathcal{K}_\infty(f)$ contains $\{\pm 2\}$.

We shall construct a curve $X(t)$ that satisfies (3.2) , (3.3) of Definition 3.1 and also the limit condition $lim_{t\to 0} f(X(t)) = -2$. A curve satisfying $lim_{t\to 0} f(X(t)) = 2$ can be also constructed in a parallel way.

First of all we find the face Γ as in Proposition 2.1. The face Γ is on the plane containing $(3,2,0), (0,1,0), (0,0,1)$ and $q = (-1,3,3)$, i.e. $(q',0) = (-1,3,0)$. The condition (μ) of Lemma 3.1 is thus satisfied.

As we see

$$\langle (q',0), w_1 \rangle = -1, \langle (q',0), w_2 \rangle = -1, \langle (q',0), w_3 \rangle = 4,$$

$\mathbb{J} = \{3\}$ and $L_0 = max_{i\neq j} \langle (q',0), w_i - w_j \rangle = 5$.

We remark also that the equality $\Delta_{u^*}(\langle \mu_3, f^W \rangle) = \Delta^*$ holds without the assumption $M \in (\mathbb{Z}_{>0})^{3\times 3}$ imposed in Corollary 3.1.

The curve (3.8) has the expansion

$$u_1 = c_1(0)t^{-1} + c_1(1) + h.o.t., u_2 = c_2(0)t^3 + c_2(1)t^4 + h.o.t,$$

$$u_3 = 1 + c_3(0)t^3 + c_3(1)t^4 + h.o.t.$$

If we plug these expressions into $\langle \mu_3, \vartheta_u f^W(u) \rangle$, we get an expansion with initial term t^3 ($\langle q, \alpha \rangle = 3$ for $\alpha \in \Gamma$);

$$\{c_2(0) + c_1(0)^3 c_2(0)^2 + 6c_3(0)\}t^3 + \{3c_1(0)^2 c_1(1) c_2(0)^2 + c_2(1) + 2c_1(0)^3 c_2(0) c_2(1)$$

$$+6c_3(1)\}t^4 + \{3c_1(0)c_1(1)^2 c_2(0)^2 + 3c_1(0)^2 c_1(2) c_2(0)^2 + 6c_1(0)^2 c_1(1) c_2(0) c_2(1) +$$

$$+c_1(0)^3c_2(1)^2 + c_2(2) + 2c_1(0)^3c_2(0)c_2(2) + 6c_3(2)\}t^5 + h.o.t.$$

The coefficient of t^3 depends on $(c_1(0), c_2(0), c_3(0))$ that of t^4 depends on

$$(c_1(0), c_2(0), c_3(0), c_1(1), c_2(1), c_3(1))$$

that of t^5 depends on $(c_1(0), c_2(0), c_1(1), c_2(1), c_1(2), c_2(2), c_2(3))$. Thus we can construct a curve such that $-5 + ord \left\langle \mu_3, \vartheta_u f^W \right\rangle (Q(t)) > 0$. The minimum parametric length of such a curve $Q(t)$ (3.8) is 4. Here coefficients can be chosen to be real. As for the construction of a real curve $Q(t)$, i.e. a real curve $X(t)$, see [20]. We remark that after the method of [5, Theorem 3.5.], the rational curve with required properties has length 3601.

We get the desired curve $X(t)$ as the image of the curve $Q(t)$ by the map

$$x_1 = u_1u_3, x_2 = (u_1^2u_2u_3)^{-1}, x_3 = u_1^2u_2^2u_3.$$

We obtain a curve $X(t) = (x_1(t), x_2(t), x_3(t))$ asymptotically approaching the surface $\{x; f(x) = -2\}$ as follows:

$$x_1(t) = \left(t^2 + t + \frac{1}{t} + 1\right)\left(t^6 - \frac{8t^5}{3} - t^4 - \frac{t^3}{3} + 1\right),$$

$$x_2(t) = \frac{1}{\left(t^2 + t + \frac{1}{t} + 1\right)^2 \left(t^6 - \frac{8t^5}{3} - t^4 - \frac{t^3}{3} + 1\right)\left(t^6 + t^5 + t^4 + t^3\right)},$$

$$x_3(t) = \left(t^2 + t + \frac{1}{t} + 1\right)^2 \left(t^6 - \frac{8t^5}{3} - t^4 - \frac{t^3}{3} + 1\right)\left(t^6 + t^5 + t^4 + t^3\right)^2.$$

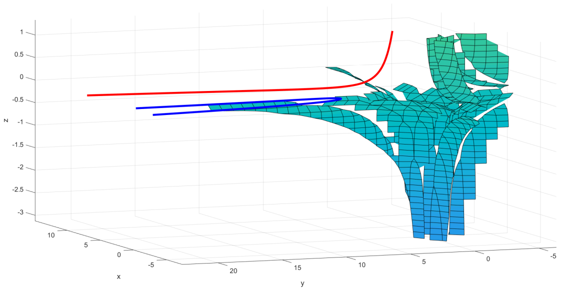

Figure 5.3: Branches of the curve $X(t)$

On the Figure 5.3, we see two branches of the curve that correspond to the asymptotes $t \to 0$ from $t > 0$ and $t < 0$ respectively.

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