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A NOTE ON ERGODIC STATES ON C*-DYNAMICS

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1. Introduction.

In this note we will discuss an ergodicity of invariant states on C*-dynamics. We will present a subspace $\mathscr{H}_\varphi'$ to get a characterization of the ergodicity of an invariant state φ on a C*-dynamics (A, G, α) (Theorem 1), which makes the same role as $L^2(\varphi)$ in the case the C*-algebra A is commutative. Also, an application of the subspace $\mathscr{H}_\varphi'$ will be given for some property of $\pi_\varphi(A)'$ and of the set $\mathscr{S}_G$ of all invariant states (Theorem 2).

Let A be a C*-algebra with unit element and α an action of a group G on A. We say that (A, G, α) is a *C*-dynamics*. A state φ on A is said to be *invariant* if $\varphi(\alpha_g(x)) = \varphi(x)$ for $x \in A$ and $g \in G$. Let $(\pi_\varphi, \mathscr{H}_\varphi, \xi_\varphi)$ be the cyclic representation of A induced by φ. If φ is an invariant state on (A, G, α), then it induces a unitary representation u^φ (or simply u) of G on the Hilbert space $\mathscr{H}_\varphi$ such that $u_g\pi_\varphi(x)u_g^* = \pi_\varphi \circ \alpha_g(x)$ for $x \in A$ and $g \in G$ and $u_g\xi_\varphi = \xi_\varphi$ for $g \in G$. In fact, it is defiend by $u_g^\varphi \pi_\varphi(x)\xi_\varphi = \pi_\varphi(\alpha_g(x))\xi_\varphi$, $x \in A$, $g \in G$. A non-commutative version of ergodicity of an invariant state φ on the C*-dynamics (A, G, α) is that φ is an extreme point in the set of invariant states. We say φ to be *ergodic* in this case. We are concerned with the following properties of the C*-dynamics (A, G, α): (I) $\dim \{\xi \in \mathscr{H}_\varphi : u_g\xi = \xi (g \in G)\} = 1$; (II) $\{\pi_\varphi(A), u_G\}' = \mathbf{C}1_{\mathscr{H}_\varphi}$. The following implications are known ([1], [4], etc.). (I) implies (II) and that φ is ergodic, and (II) is equivalent to the ergodicity of φ. If the C*-algebra A is commutative, that is, $A = C(X)$ and X a compact space, then the action α_g, $g \in G$, is realized from homeomorphisms on X. φ is considered to be an invariant probability measure under the group of these homeomorphisms. $\mathscr{H}_\varphi$ is in fact the Hilbert space $L^2(\varphi)$ and the condition (I) is equivalent to the ergodicity of φ. However if the C*-algebra A is not commutative, then the ergodicity of φ does not imply the condition (I) ([1] p. 395). So, we would like to present the subspace $\mathscr{H}_\varphi'$ (see Notation 1) in place of $\mathscr{H}_\varphi$ in order to get a non-commutative version of the above characterization of the ergodicty of φ.

2. Ergodic states.

NOTATION 1. By $\mathscr{H}_\varphi'$, we denote the closed subspace $[\pi_\varphi(A)'\xi_\varphi]$ of $\mathscr{H}_\varphi$, which is the closed linear span of the subset $\pi_\varphi(A)'\xi_\varphi$.

THEOREM 1. *Let φ be an invariant state on a C^*-dynamics (A, G, α). Then φ is ergodic if and only if*

$$\dim\{\eta\in\mathscr{H}_\varphi' : u_g\eta = \eta(g\in G)\} = 1.$$

PROOF. Let e_φ be the projection of $\mathscr{H}_\varphi$ onto $\mathscr{H}_\varphi'$. It is clear that e_φ belongs to $\pi_\varphi(A)''$. Since the central support of e_φ is the identity, the induction of $\pi_\varphi(A)'$ onto the induced von Neumann algebra $\pi_\varphi(A)'_{e_\varphi}$ is an isomorphism, which also gives the equivalence between the W*-dynamics $\{\pi_\varphi(A)', G, Adu\}$ and $\{\pi_\varphi(A)'_{e_\varphi}, G, Adv\}$, where $v_g = u_g|_{\mathscr{H}_\varphi'}$. Hence the above induction induces an isomorphism of $[\pi_\varphi(A)']^{Adu}$ onto $[\pi_\varphi(A)'_{e_\varphi}]^{Adv}$, and in particular we have that $[\pi_\varphi(A)']^{Adu} = \mathbf{C}1_{\mathscr{H}_\varphi}$, that is, φ is ergodic, if and only if $[\pi_\varphi(A)'_{e_\varphi}]^{Adv} = \mathbf{C}1_{\mathscr{H}_\varphi'}$. Since $\pi_\varphi(A)'_{e_\varphi}$ is a von Neumann algebra acting on $\mathscr{H}_\varphi'$ and ξ_φ is a vector in $\mathscr{H}_\varphi'$ such that $v_g\xi_\varphi = \xi_\varphi$ for $g\in G$ and moreover ξ_φ is cyclic and separating for $\pi_\varphi(A)'_{e_\varphi}$, we have that $[\pi_\varphi(A)'_{e_\varphi}]^{Adv} = \mathbf{C}1_{\mathscr{H}_\varphi'}$ if and only if

$$\dim\{\eta\in\mathscr{H}_\varphi' : v_g\eta = \eta(g\in G)\} = 1$$

([2] Th. 2. 4). Thus we have that φ is ergodic if and only if

$$\dim\{\eta\in\mathscr{H}_\varphi' : u_g\eta = \eta(g\in G)\} = 1$$

This completes the proof.

As immediate consequences of Theorem 1, we obtain the following corollary, which is known ([1] Th. 4. 3. 20, [4], etc.).

COROLLARY. (1) *If* $\dim\{\xi\in\mathscr{H}_\varphi : u_g\xi = \xi(g\in G)\} = 1$, *then φ is ergodic.*
(2) *Suppose that ξ_φ is separating for $\pi_\varphi(A)''$. If φ is ergodic, then*

$$\dim\{\xi\in\mathscr{H}_\varphi : u_g\xi = \xi(g\in G)\} = 1.$$

NOTATION 2. By $\mathscr{S}_G$, we denote the set of all invariant states on the C*-dynamics (A, G, α). For $\varphi\in\mathscr{S}_G$, we denote q_φ be the projection onto the subspace $\{\eta\in\mathscr{H}_\varphi' : u_g\eta = \eta(g\in G)\}$.

One can get the following facts by replacing the projection onto the subspace $\{\xi\in\mathscr{H}_\varphi : u_g\xi$

$= \xi(g \in G)\}$ by q_φ in Prop. 3. 1. 13, Th 3. 1. 14 ([4]):

FACT 1. *For* $\varphi \in \mathcal{S}_G$, *if* $q_\varphi \pi_\varphi(A) q_\varphi$ *is commutative, then* $\{\pi_\varphi(A),\ u_G\}'$ *is commutative.*

FACT 2. *If* $q_\varphi \pi_\varphi(A) q_\varphi$ *is commutative for all* $\varphi \in \mathcal{S}_G$, *then* $\mathcal{S}_G$ *is a simplex.*

The proof of these facts are almost similar with those of Prop. 3. 1. 13. and Th. 3. 1. 14 ([4]), so are omitted.

Thanks to the use of the subspace $\mathscr{H}_\varphi'$, in fact we will show that the condition of Fact 1 is a necessary and sufficient condition.

THEOREM 2. *For* $\varphi \in \mathcal{S}_G$, *if* $\{\pi_\varphi(A),\ u_G\}'$ *is commutative, then* $q_\varphi \pi_\varphi(A) q_\varphi$ *is commutative.*

PROOF. Let p be the projection onto the subspace $[\{\pi_\varphi(A),\ u_G\}' \xi_\varphi]$. Then we have that $u_g p = p$ and $p\pi_\varphi(A)p \subseteq \{p\pi_\varphi(A)p\}'$([1] Th. 4. 1. 25, Prop. 4. 3. 1). In particular, we have $p \leqq q_\varphi$. Hence it suffices to show that $p = q_\varphi$. Since ξ_φ is cyclic and separating for $\pi_\varphi(A)'_{e_\varphi}$ in $\mathscr{H}_\varphi'$, there exists an $E(a) \in \pi_\varphi(A)'_{e_\varphi} \cap v_G'$ such that

$$E(a)\xi_\varphi = q_\varphi a \xi_\varphi,\ (a \in \pi_\varphi(A)'_{e_\varphi}),$$

where e_φ is the projection onto $\mathscr{H}_\varphi'$ and $v_g = u_g|_{\mathscr{H}_\varphi'}$ ([1] Prop. 4. 3. 8, [3]). Hence we have

$$\begin{aligned} q_\varphi \mathscr{H}_\varphi &= q_\varphi \mathscr{H}_\varphi' \subseteq [E(a)\xi_\varphi :\ a \in \pi_\varphi(A)'_{e_\varphi}] \\ &\subseteq [(\pi_\varphi(A)'_{e_\varphi} \cap v_G')\xi_\varphi] = [(\pi_\varphi(A)' \cap u_G')\xi_\varphi] = p\mathscr{H}_\varphi \end{aligned}$$

Thus we have $q_\varphi = p$.

This completes the proof.

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