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SPECIFICATION ERROR IN SEQUENTIAL ESTIMATION

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1. INTRODUCTION

Most of sequential procedures are proposed under the assumption that observations are independent and identically distributed (i. i. d.). Our concern is to examine performances of these procedures when the assumption of independence is violated. This paper deals with sequential estimation with bounded risk.

Let $\{X_i\}$ be a sequence of random variables with common unknown mean μ. Given a sample of size n, μ is estimated by the sample mean $\bar{X}_n = \sum_{i=1}^{n} X_i/n$. Then we want to determine a sample size n such that

(1. 1) $$E(\bar{X}_n-\mu)^2/W \simeq 1$$

for a given small constant $W>0$.

We assume that for every i

(1. 2) $$X_i = \mu+\rho(X_{i-1}-\mu)+\varepsilon_i, \qquad |\rho|<1,$$

where $\{\varepsilon_i\}$ is i. i. d. with $E\varepsilon_i = 0$ and $E\varepsilon_i^2 = \sigma^2(>0)$, and ρ and σ are unknown. That is, the sequence $\{X_i-\mu\}$ is the first order autoregressive process.

It is well known that

(1. 3) $$nE(\bar{X}_n-\mu)^2 \rightarrow \sigma^2(1-\rho)^{-2} \quad \text{as } n\rightarrow\infty$$

(e. g. Brockwell and Davis [1], Theorem 7. 1. 1). Hence if ρ and σ are known, (1. 1) is satisfied by $n\simeq n* = \sigma^2/[(1-\rho)^2 W]$.

If $\rho = 0$, $n* = \sigma^2/W$. Since σ is unknown, Takada [3] considered the following sequential procedure. Let $m(\geqq 2)$ be an initial sample size such that $m = O(W^{-d})$ as $W\rightarrow 0$ $(0<d<1)$. Define the sample size N_1 by

(1. 4) $$N_1 = \min\{n\geqq m; n\geqq S_n/W\},$$

where $S_n = \sum_{i=1}^{n}(X_i-\bar{X}_n)^2/(n-1)$, and estimate μ by $\bar{X}_{N_1}$. Then it was shown that

$$\lim_{W\to 0} E(\bar{X}_{N_1}-\mu)^2/W = 1 \qquad \text{(asymptotic consistency)}$$

and

$$\lim_{W\to 0} EN_1/n* = 1 \qquad \text{(asymptotic efficiency)}$$

if $\rho = 0$, $E|\varepsilon_i|^{2p}<\infty$ $(p>1)$ and $[1+(p-1)p*/p]^{-1}<d<1$, where $p* = p/2$ for $p\geqq 2$ and $p* = p-1$ for $1<p<2$.

We shall examine performances of the above procedure when observations are not independent ($\rho \neq 0$).

2. PRELIMINARIES

Let θ be a prameter depending on ρ and σ, and let $\tilde{n} = \theta/W$. Consider the sample size defined by

(2. 1) $$N = \min\{n \geqq m\,;\, n \geqq Y_n/W\},$$

where Y_n is an estimator of θ based on the sample of size n and $m = O(W^{-d})$ as $W \to 0$. The proof of the following lemma is similar to that of (2. 32) of Sriram [2], so omitted.

LEMMA 1. *For any* $\varepsilon > 0$

$$E\{(\bar{X}_N - \bar{X}_{\tilde{n}})^2 \mathrm{I}_A / W\} \leqq f(\varepsilon),$$

where $f(\varepsilon)$ $(0<\varepsilon<1)$ *does not depend on* W, $f(\varepsilon) \downarrow 0$ *as* $\varepsilon \downarrow 0$, *and* I_A *denotes the indicator function of the set* $A = \{|N - \tilde{n}| \leqq \varepsilon \tilde{n}\}$.

LEMMA 2. *If for* $0<\varepsilon<1$

$$P(|Y_n - \theta| \geqq \varepsilon\theta) = O(n^{-p/2}) \qquad (p>2)$$

as $n \to \infty$, *then*

(2. 2) $$P(N<(1+\varepsilon)\tilde{n}) = O(W^{d(p/2-1)})$$

and

(2. 3) $$P(N>(1+\varepsilon)\tilde{n}) = O(W^{p/2})$$

as $W \to 0$.

PROOF. Let $n_1 = (1-\varepsilon)\tilde{n}$. By (2. 1) we have that

$$\begin{aligned} P(N<n_1) &= P(Y_n \leqq nW \text{ for some } m \leqq n < n_1) \\ &\leqq P(Y_n \leqq n_1 W \text{ for some } m \leqq n < n_1) \\ &= P(Y_n - \theta \leqq -\varepsilon\theta \text{ for some } m \leqq n < n_1) \\ &\leqq \sum_{n=m}^{\infty} P(|Y_n - \theta| \geqq \varepsilon\theta). \end{aligned}$$

It follows from the condition on Y_n that

$$\sum_{n=m}^{\infty} P(|Y_n - \theta| \geqq \varepsilon\theta) = O(m^{-(p/2-1)}).$$

Since $m = O(W^{-d})$, (2. 2) is proved. To prove (2. 3), let $n_2 = (1+\varepsilon)\tilde{n}$. Then

$$\begin{aligned} P(N>n_2) &\leqq P(Y_{n_2} > n_2 W) \\ &= P(Y_{n_2} - \theta > \varepsilon\theta) \\ &\leqq P(|Y_{n_2} - \theta| > \varepsilon\theta) \\ &= O(n_2^{-p/2}), \end{aligned}$$

which proves (2. 3).

Using Lemma 2, the following lemma is proved along the proof of (2. 31) of Sriram [2], so omitted.

LEMMA 3. *If* $E|\varepsilon_i|^{2p}<\infty(p>2)$, $2/(p+1)<d<1$ *and for* $0<\varepsilon<1$

$$P(|Y_n-\theta|\geqq\varepsilon\theta) = O(n^{-p/2})$$

as $n\to\infty$, *then*

$$\lim_{W\to 0} E\{(\bar{X}_N-\mu)^2 I_{\bar{A}}/W\} = 0,$$

where $\bar{A} = \{|N-\tilde{n}|>\varepsilon\tilde{n}\}$.

THEOREM 1. *If* $E|\varepsilon_i|^{2p}<\infty(p>2)$, $2/(p+1)<d<1$ *and for* $0<\varepsilon<1$

$$P(|Y_n-\theta|\geqq\varepsilon\theta) = O(n^{-p/2})$$

as $n\to\infty$, *then*

$$\lim_{W\to 0} E(\bar{X}_N-\mu)^2/W = \sigma^2/[(1-\rho)^2\theta]$$

PROOF. It follows from (1. 3) that

$$\lim_{W\to 0} E(\bar{X}_{\tilde{n}}-\mu)^2/W = \sigma^2/[(1-\rho)^2\theta]$$

Hence to prove the theorem, it is enough to show that

$$\lim_{W\to 0} E(\bar{X}_N-X_{\tilde{n}})^2/W = 0. \tag{2. 4}$$

Similar arguments of the proof of Lemma 3 yield

$$\lim_{W\to 0} E\{(\bar{X}_{\tilde{n}}-\mu)^2 I_{\bar{A}}/W\} = 0.$$

Then (2. 4) follows from Lemmas 1 and 3, so that the proof is completed.

THEOREM 2. *If* $E|\varepsilon_i|^{2p}<\infty(p>2)$ *and for* $0<\varepsilon<1$

$$P(|Y_n-\theta|>\varepsilon\theta) = O(n^{-p/2})$$

as $n\to\infty$, *then*

$$\lim_{W\to 0} E(N)/\tilde{n} = 1$$

PROOF. Let $n_1 = (1-\varepsilon)\tilde{n}$ and $n_2 = (1+\varepsilon)\tilde{n}$. Then we have

$$E(N)/\tilde{n} = \{\sum_{m\leq n<n_1} + \sum_{n_1\leq n\leq n_2} + \sum_{n>n_2} nP(N = n)\}/\tilde{n},$$

so that

$$|E(N)/\tilde{n}-1|\leqq(1-\varepsilon)P(N<n_1)+|\tilde{n}^{-1}\sum_{n_1\leq n\leq n_2} nP(N = n)-1| + \sum_{n>n_2} nP(N = n)/\tilde{n}.$$

Since

$$|\tilde{n}^{-1}\sum_{n_1\leq n\leq n_2} nP(N = n)-1|\leqq\varepsilon\sum_{n_1\leq n\leq n_2} P(N = n)+P(N<n_1)+P(N>n_2),$$

from Lemma 2 the required result is obtained if it is shown that

(2. 5) $$\lim_{W\to 0}\sum_{n>n_2} nP(N=n)/\tilde{n}=0.$$

It is easy to show that

$$\sum_{n>n_2} nP(N=n)=(n_2+1)P(N>n_2)+\sum_{n>n_2}P(N>n).$$

By the same argument as the proof of (2. 3),

$$P(N>n)=O(n^{-p/2}).$$

Since $p>2$,

$$\sum_{n>n_2}P(N>n)=O(n_2^{-(p/2-1)}).$$

Hence (2. 5) is proved.

3. EFFECT OF DEPENDENCY

We examine the effect of dependency on the sequential procedure N_1 given by (1. 3) when the true model is (1. 2).

THEOREM 3. *If* $E|\varepsilon_i|^{2p}<\infty$ $(p>2)$ *and* $2/(p+1)<d<1$, *then*

$$\lim_{W\to 0} E(\bar{X}_{N1}-\mu)^2/W=(1+\rho)/(1-\rho)$$

and

$$\lim_{W\to 0} E(N_1)/n*=(1-\rho)/(1+\rho)$$

PROOF. It follows from Lemma 2 of Sriram [2] that

$$P(|S_n-\sigma^2/(1-\rho^2)|>\varepsilon)=O(n^{-p/2}).$$

Hence the proof is completed from Theorems 1 and 2.

It turns out that the sequential procedure N_1 is asymptotically consistent and efficient if and only if observations are independent, that is, $\rho=0$. A sequential procedure which is asymptotically consistent and efficient is obtained by the following procedure.

Consider the sample size N_2 defined by

$$N_2=\min\{n\geq m;\ n\geq \hat{\sigma}_n^2/[(1-\hat{\rho}_n)^2W]\},$$

where $m=O(W^{-d})$ as $W\to 0$ $(0<d<1)$,

$$\hat{\rho}_n=\sum_{i=1}^{n-1}(X_i-\bar{X}_n)(X_{i+1}-\bar{X}_n)/\sum_{i=1}^{n}(X_i-\bar{X}_n)^2$$

and

$$\hat{\sigma}_n{}^2=n^{-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2(1-\hat{\rho}_n{}^2).$$

Then μ is estimated by $\bar{X}_{N_2}$.

THEOREM 4. *If* $E|\varepsilon_i|^{2p}<\infty(p>2)$ *and* $2/(p+1)<d<1$, *then*

$$\lim_{W\to 0} E(\bar{X}_{N_2}-\mu)^2/W = 1 \quad \textit{(asymptotic consistency)}$$

and

$$\lim_{W\to 0} E(N_2)/n* = 1 \quad \textit{(asymptotic efficiency).}$$

PROOF. Lemma 2 of Sriram [2] shows

$$P(|\hat{\sigma}_n^2/(1-\hat{\rho}_n)^2-\sigma^2/(1-\rho)^2|>\varepsilon) = O(n^{-p/2}).$$

Then the theorem follows from Theorems 1 and 2.

REMARK. Our attention has been devoted to the sequential estimation with bounded risk. Other problems such as sequential estimation with cost per observation and fixed-width interval estimation can be treated by the similar argument and effects of dependency can be examined. Asymptotically optimal procedure for these problems are obtained by Sriram [2] under the model (1. 2).

References

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[3] Takada, Y., Asymptotically bounded regret sequential estimation of the mean, Sequential Analysis, **7** (1988), 253-262.

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