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IRREGULAR SINGULAR POINT OF A LINEAR DIFFERENTIAL SYSTEM – DEPENDENCE ON THE INITIAL MATRICES

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Abstract

We study the dependence of a formal fundamental matrix of a linear differential system, at an irregular singular point, on the initial matrices.

1. Introduction

1-1 Motivations

We study the following differential linear system:

$$x^p Y(x) = A(x) Y(x), \tag{1}$$

where $Y(x)$, $A(x)$ are $n\times n$ matrices with entries in C, $p\in\mathbf{N}^*$, $p>1$ and $A(x)$ has the formal expansion:

$$A(x) = \sum_{k\geq 0} A_k x^k.$$

The behavior at the singular point $x=0$ is well known (see [1], [2]): when A_0 has n distinct eigenvalues, there exists a formal fundamental matrix of (1) of the form:

$$Y(x) = P(x) Z(x),$$

where:

$$P(x) = \sum_{k\geq 0} P_k x^k,\ Z(x) = \exp[Q(x)] x^D,$$

where $Q(x)$ is a diagonal matrix whose diagonal entries are polynomials in x^{-k}, $k\in\{1, 2, \ldots, p-1\}$ and D a constant diagonal matrix.

This result is based on the diagonalization of A_0, so it is very hard to determine exactly how the matrices $Q(x)$, D, $P(x)$ depend on the entries of A_k. ($k\in\mathbf{N}$).

We propose, here, an equivalent form for $Y(x)$, without using a matrix diagonalization. The method gives explicit results, shows that the matrices of the irregular part and the regular part of $Y(x)$ depend rationaly on the entries of A_k.

It gives also, an exact finite (and very simple) numerical algorithm to compute $Y(x)$.

1-2 Results

Theorem 1

When A_0 is diagonalizable, there exists a formal fundamental matrix $Y(x)$, of (1) of the form :

$$Y(x) = P(x)Z(x)\exp\left[A_0\frac{x^{1-p}}{1-p}\right],$$

where :

$$P(x) = \sum_{k\geq 0} P_k x^k,\ P_0 = I,$$

$$x^{p-1}Z'(x) = \left(\sum_{k\geq 0} B_{k+1}x^k\right)Z(x).$$

The P_k and B_k and polynomials of A_i $(0\leq i\leq k)$ whose coefficients are rational fractions of the entries of A_0.

Theorem 2

When A_0 has n distinct eigenvalues, there exists a formal fundamental matrix of the form :

$$Y(x) = P(x)Q(x)Z(x),$$

where :

$$P(x) = \sum_{k\geq 0} P_k x^k,\ P_0 = I,$$

$$Q(x) = \exp\left[\sum_{l\geq 0} B_{p+l}\frac{x^{l+1}}{l+1}\right],$$

$$Z(x) = \exp\left[\sum_{k=0}^{p-2} B_k\frac{x^{k-p+1}}{k-p+1}\right]x^{B_{p-1}}$$

with :

$$B_0 = A_0 \text{ and } [B_i,B_j] = B_iB_j - B_jB_i = 0 \quad \forall i,\ j.$$

The P_k and B_k are polynomials of A_i $(0\leq i\leq k)$ whose coefficients are rational fractions of the entries of A_0.

Property

There exists a finite exact numerical algorithm to compute the P_k, B_k from the $A_i (0\leq i\leq k)$.

2. Linear part

Let A, $B\in M_n(\mathrm{C})$. We study the following system of linear equations:

$$\begin{cases} AX - XA = B - Y, \\ AY - YA = 0 \quad X,\ Y\in M_n(\mathrm{C}). \end{cases} \tag{2}$$

Let f be the endomorphism of $M_n(\mathrm{C})$ defined by:

$$f(X) = AX - XA.$$

The problem (2) is equivalent to:

$$B = f(X) + Y \text{ with } Y\in \mathrm{Ker}\ f.$$

It has solutions for all B in $M_n(\mathrm{C})$ if and only if:

$$M_n(\mathrm{C}) = \mathrm{Ker}\ f + \mathrm{Im}\ f \quad \Leftrightarrow \quad M_n(\mathrm{C}) = \mathrm{Ker}\ f \oplus \mathrm{Im}\ f.$$

But we have:

2-1 Theorem

$$M_n(\mathrm{C}) = \mathrm{Ker}\ f + \mathrm{Im}\ f \quad \Leftrightarrow \quad A \textit{ is diagonalizable in } \mathrm{C}.$$

The (easy) proof of this theorem is given in [4].

2-2 Corollary

If A is diagonalizable in C, *there exists an unique $X \in \mathrm{Im}\ f$ and an unique $Y \in \mathrm{Ker}\ f$ such that :*

$$\begin{cases} AX - XA = B - Y, \\ AY - YA = 0. \end{cases}$$

It suffices to remark that the restriction of f on $\mathrm{Im}\ f$ is an isomorphism.

We will give, now an explicit form for the solutions X and Y. We suppose, hence, that $M_n(\mathrm{C}) = \mathrm{Ker}\ f \oplus \mathrm{Im}\ f$.

Let $p(\lambda)$, be the characteristic polynomial of f. Remark that f depends linearly on A, that the coefficients of $p(\lambda)$ are polynomials of Trace $(f^k)(1 \leq k \leq n^2)$, thus, they are polynomials of the entries of A, which can be explicitly computed (by the Newton or Fadev-Frame formula, see [3]). Put:

$$p(\lambda) = \sum_{j=0}^{k} (-1)^{m-j} a_{m-j} \lambda^{m-j},$$

where:

$$m = n^2,\ a_m = 1,\ a_{m-k} \neq 0,\ k < m \ (\text{as } I \in \mathrm{Ker}\ f,\ \det f = 0,\ \text{so } a_0 = 0).$$

Moreover, we have: $a_{m-k+1} = 0$ (because if λ_1 and λ_2 are two distinct eigenvalues of A, $\lambda_1 - \lambda_2$ and $\lambda_2 - \lambda_1$ are roots of $p(\lambda)$ of the same order, and all non zero roots of $p(\lambda)$ are obtained in such a way).

2-3 Proposition

$$X = \sum_{j=0}^{k-2} (-1)^{k-j+1} \frac{a_{m-j}}{a_{m-k}} f^{k-j-1}(B),$$

$$Y = B - \sum_{j=0}^{k-2} (-1)^{k-j+1} \frac{a_{m-j}}{a_{m-k}} f^{k-j}(B).$$

PROOF

Let us prove first that:

$$\sum_{j=0}^{k} (-1)^{m-j} a_{m-j} f^{k+1-j} = 0.$$

By Hamilton-Cayley, we have:

$$\sum_{j=0}^{k}(-1)^{m-j}a_{m-j}f^{m-j} = 0.$$

As $M_n(\mathrm{C}) = \mathrm{Ker}\ f + \mathrm{Im}\ f$, for all B in $M_n(\mathrm{C})$, we have:

$$B = C + f(D) \text{ with } f(C) = 0.$$

So:

$$f(B) = f^2(D) \text{ and } f^l(B) = f^{l+1}(D) \quad \forall\, l \geq 1.$$

But we have:

$$\sum_{j=0}^{k}(-1)^{m-j}a_{m-j}f^{m-j}(D) = 0 = \sum_{j=0}^{k}(-1)^{m-j}a_{m-j}f^{m-j-1}(B) = 0.$$

Thus:

$$\sum_{j=0}^{k}(-1)^{m-j}a_{m-j}f^{m-1-j} = 0.$$

We apply $m-k-1$ times this reasoning and get:

$$\sum_{j=0}^{k}(-1)^{m-j}a_{m-j}f^{k+1-j} = 0.$$

We have:

$$\sum_{j=0}^{k-2}(-1)^{m-j}a_{m-j}f^{k+1-j}(D) + (-1)^{m-k}a_{m-k}f(D) = 0.$$

So:

$$f(D) = \sum_{j=0}^{k-2}(-1)^{k+j-1}\frac{a_{m-j}}{a_{m-k}}f^{k-j}(B).$$

It suffices to take:

$$X = \sum_{j=0}^{k-2}(-1)^{k-j+1}\frac{a_{m-j}}{a_{m-k}}f^{k-j-1}(B)$$

and

$$Y = B - \sum_{j=0}^{k-2}(-1)^{k-j+1}\frac{a_{m-j}}{a_{m-k}}f^{k-j}(B)$$

to get the result.

2-4 Corollary

X and Y are polynomials of A and B whose coefficients are rational fractions of the entries of A.

PROOF

$f(B) = AB - BA$, $f^2(B) = A^2B + BA^2 - 2ABA$, thus (by induction) $f^k(B)$ is a polynomial of A and B. We have already proved that a_{m-j} is a polynomial in the entries of A, so we get the result.

2-5 Example

Let A be the matrix:

$$A=\begin{pmatrix}1 & 0 & -1 & 2\\ 1 & 3 & 1 & 3\\ 2 & -1 & 0 & 1\\ 1 & 0 & 1 & 0\end{pmatrix}.$$

We have:

$$p(\lambda)=\lambda^{16}-24\lambda^{14}+214\lambda^{12}-852\lambda^{10}+1361\lambda^{8}-396\lambda^{6}+32\lambda^{4}$$

so:

$$X=99/8f(B)-1361/32f^{3}(B)+213/8f^{5}(B)-107/16f^{7}(B)+3/4f^{9}(B)-1/32f^{11}(B)$$

and:

$$Y=B-99/8f^{2}(B)+1361/32f^{4}(B)-213/8f^{6}(B)+107/16f^{8}(B)-3/4f^{10}(B)+1/32f^{12}(B).$$

2-6 An exact finite numerical algorithm

We will describe an easy exact finite algorithm to get a solution X and Y. We identify $M_n(\mathrm{C})$ with C^{m^2}. Let A_1 be the matrix of f in the canonical basis of C^{m^2}. Let B_1 the column of the components of B in this basis. We first determine Y_1 by the two conditions:

$$\begin{cases} A_1Y_1=0, \\ A_1Y_1=B_1-Y_1 \text{ has solutions.}\end{cases}$$

To express these two conditions we write: $A_1=KA_2$, where A_2 is a row echelon form of A_1 and K is invertible (it is a classical decomposition which can be done with the Gauss elimination's method). Let r be the rank of A_1 (i.e. the number of non zero rows of A_2). Put $B_2=K^{-1}B_1$. Let A_3 be the the square matrix of order m^2 formed with the first r rows of A_2 and the last m^2-r rows of K^{-1}. Let B_3 be the column of order m^2 which has the first r entries equal to 0, and the last m^2-r rows equal to the last m^2-r entries of B_2. We now solve the linear system: $A_3Y_1=B_3$. So, we get Y_1 and the corresponding matrix Y.

Let C be the matrix $B-Y$, and C_1 the corresponding column. Put $C_2=K^{-1}C_1$. Let C_3 be the column of order m^2 with the first r entries equal to the first r entries of C_2 and the last m^2-r equal to 0. We solve the linear system: $A_3X_1=C_3$. So, we get X_1 and the corresponding matrix X.

As we use only a finite number of the four operations: $\{+, -, *, /\}$, the above method is an exact finite algorithm.

We can also get a second finite algorithm by using 2-3.

Remark

The first algorithm is based on classical numerical methods: product of matrices, *PLU* decomposition of a matrix, inverse of a matrix, and resolution of a linear system. The study of the errors of such computations is done in any classical book of numerical linear analysis.

Note that, if the entries of A and B are in N, and if n is not too large ($n \leq 10$), we can solve exactly the problem (with Maple, for example).

2-8 An example

$$A = \begin{pmatrix} m & 2 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

We use Maple for the computations and get:

$$Y = \begin{pmatrix} \frac{2m^2+5m+9}{m^2+2m+5} & \frac{2m+6}{m^2+2m+5} & \frac{-m-3}{m^2+2m+5} \\ \frac{m+3}{m^2+2m+5} & \frac{m^2+7}{m^2+2m+5} & \frac{m-1}{m^2+2m+5} \\ \frac{m+3}{m^2+2m+5} & \frac{2-2m}{m^2+2m+5} & \frac{m^2+3m+4}{m^2+2m+5} \end{pmatrix},$$

$$X = \begin{pmatrix} \frac{1}{m^2+2m+5} & \frac{2m^2+4m+8}{m^2+2m+5} & \frac{-2m^2-4m-9}{m^2+2m+5} \\ \frac{m^2+m+5}{m^2+2m+5} & \frac{-4m^3-11m^2-26m-17}{m^2+2m+5} & \frac{3m^3+8m^2+19m+11}{m^2+2m+5} \\ \frac{2m^2+3m+16}{m^2+2m+5} & \frac{-6m^3-17m^2-40m-27}{m^2+2m+5} & \frac{4m^3+11m^2+26m+16}{m^2+2m+5} \end{pmatrix}.$$

3. Application to the linear differential system

3-1 Theorem

When A_0 is diagonalizable, there exists a formal fundamental matrix of (1) *of the type:*

$$Y(x) = P(x)Z(x)\exp[A_0\frac{x^{1-p}}{1-p}],$$

where:

$$P(x) = \sum_{k\geq 0} P_k x^k, \quad P_0 = I_n \ (\textit{formal series})$$

$$x^{p-1}Z'(x) = \Big(\sum_{k\geq 0} B_{k+1}x^k\Big)Z(x).$$

The P_k and B_k are polynomials of A_i $(0 \leq i \leq k)$ whose coefficients are rational fractions of the entries of A_0.

PROOF

Put $Y(x) = P(x)Z(x)$ with: $x^pT'(x) = B(x)T(x)$. If $Y(x)$ is a solution of (1), we have:

$$x^pP'(x) = A(x)P(x) - P(x)B(x). \tag{3}$$

Put:

$$P(x) = \sum_{k\geq 0} P_k x^k \text{ and } B(x) = \sum_{k\geq 0} B_k x^k$$

with: $P_0 = I_n$ and $B_0 = A_0$ in (3). We identify the terms of degree k in x and get:

$$-[A_0, P_k] = H_k + A_k - B_k, \tag{4}$$

where H_k is a polynomial of A_i, P_i, B_i with $0 \leq i \leq k-1$.

We proceed by induction and by using the previous linear part with:

$$A = A_0,\ B = H_k + A_k,\ X = -P_k,\ Y = B_k$$

As A_0 is diagonalizable, there exists an unique B_k in Ker f and an unique P_k in Im f which are solutions of (4). Moreover, B_k and P_k are polynomials of A_i with $0 \leq i \leq k$ which can be explicitely computed.

As $B_k \in \text{Ker } f$, we have: $[A_0, B_k] = 0 \quad \forall k \in \mathbf{N}$. Thus, the linear system: $x^p T'(x) = B(x) T(x)$ has a fundamental matrix of the form:

$$T(x) = Z(x) \exp\left(A_0 \frac{x^{1-p}}{1-p}\right),$$

where:

$$x^{p-1} Z'(x) = \left(\sum_{k \geq 0} B_{k+1} x^k\right) Z(x).$$

We have proved the theorem.

3-2 Theorem

When A_0 has n distinct eigenvalues, there exists a formal fundamental matrix of (1) *of the type:*

$$Y(x) = P(x) Q(x) Z(x),$$

where:

$$P(x) = \sum_{k \geq 0} P_k x^k,\ P_0 = I_n,$$

$$Q(x) = \exp\left(\sum_{l \geq 0} B_{p+l} \frac{X^{l+1}}{l+1}\right),$$

$$Z(x) = \exp\left(\sum_{k \geq 0}^{p-2} B_k \frac{x^{k-p+1}}{k-p+1}\right) x^{B_{p-1}},\ B_0 = A_0$$

and:

$$[B_i, B_j] = 0 \quad \forall i, j.$$

The P_k and B_k are polynomials of A_i $(0 \leq i \leq k)$ whose coefficients are rational fractions of the entries of A_0.

PROOF

As A_0 is diagonalizable, we can use the previous theorem: there exists a fundamental matrix $Y(x) = P(x) T(x)$ with:

$$P(x) = \sum_{k \geq 0} P_k x^k,\ P_0 = I_n,$$

$$x^p T'(x) = B(x) T(x),$$

$$B(x) = \sum_{k \geq 0} B_k x^k,\quad B_0 = A_0 \text{ and } [A_0, B_k] = 0 \quad \forall k.$$

As A_0 has n distinct eigenvalues, $[A_0, B_k] = 0 \quad \forall k$ implies: $[B_i, B_j] = 0 \quad \forall i,j$, and

$T(x) = Q(x)Z(x)$ with:

$$Q(x) = \exp\Big(\sum_{l\geq 0} B_{p+l}\frac{x^{l+1}}{l+1}\Big),$$

$$Z(x) = \exp\Big(\sum_{k\geq 0}^{p-2} B_k \frac{x^{k-p+1}}{k-p+1}\Big)x^{B_{p-1}}.$$

Note that the condition:
"A_0 has n distinct eigenvalues" is equivalent to: rank f = rank $f^2 = n^2-n$ and can also be verified by an exact finite algorithm.

Remark 1

When A_0 has n distinct eigenvalues, $Y(x)$ is uniquely determined: we will call it the natural formal fundamental matrix.

Remark 2

We can use the same method for the system:

$$\varepsilon^q x^p Y'(x) = A(x, \varepsilon)Y(x), \tag{5}$$

where:

$$A(x, \varepsilon) = \sum_{k\geq 0} A_k(x)\varepsilon^k \text{ and } A_k(x) = \sum_{l\geq 0} A_{k,l}x^l$$

and $A_{0,0}$ diagonalizable.

Example

We consider the system:

$$Y'(x) = \Big(\frac{A_0}{x^2}+\frac{A_1}{x}+A_2\Big)Y(x)$$

with:

$$A_0 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & m \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

then:

$$Y(x) = P(x)\exp\Big(B_2 x + B_3\frac{x^2}{2}+\cdots\Big)x^{B_1}\exp\Big(-\frac{A_0}{x}\Big)$$

with:

$$P(x) = (I_3+P_1x+P_2x^2+\cdots)$$

and:

$$B_1 = \begin{pmatrix} \frac{m^2-3m+6}{m^2-3m+9} & \frac{2m-3}{m^2-3m+9} & \frac{3}{m^2-3m+9} \\ \frac{2m-3}{m^2-3m+9} & \frac{-m^2+2m}{m^2-3m+9} & \frac{-2m}{m^2-3m+9} \\ \frac{3}{m^2-3m+9} & \frac{-2m}{m^2-3m+9} & \frac{m^2-2m+3}{m^2-3m+9} \end{pmatrix}$$

$$P_1 = \begin{pmatrix} \frac{2m-3}{m^2-3m+9} & \frac{-2m^2+7m-12}{m^2-3m+9} & \frac{-m}{m^2-3m+9} \\ \frac{-2m+6}{m^2-3m+9} & \frac{-2m}{m^2-3m+9} & \frac{m}{m^2-3m+9} \\ \frac{-m-3}{m^2-3m+9} & \frac{2m-9}{m^2-3m+9} & \frac{3}{m^2-3m+9} \end{pmatrix}$$

$$B_2 = \begin{pmatrix} \frac{m^4-13m^3+51m^2-121m+96}{(m^2-3m+9)^2} & \frac{2m^3-20m^2+62m-111}{(m^2-3m+9)^2} & \frac{-2m^3+19m^2-61m+123}{(m^2-3m+9)^2} \\ \frac{2m^3-20m^2+62m-111}{(m^2-3m+9)^2} & \frac{-m^4+11m^3-50m^2+113m-138}{(m^2-3m+9)^2} & \frac{m^2-m-12}{(m^2-3m+9)^2} \\ \frac{-2m^3+19m^2-61m+123}{(m^2-3m+9)^2} & \frac{m^2-m-12}{(m^2-3m+9)^2} & \frac{-m^4+8m^3-28m^2+62m-39}{(m^2-3m+9)^2} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} \frac{-3m^3+9m^2-22m+32}{(m^2-3m+9)^2} & \frac{3m^4-16m^3+50m^2-83m+71}{(m^2-3m+9)^2} & \frac{3m^3-6m^2+21m+8}{(m^2-3m+9)^2} \\ \frac{3m^3-9m^2+11m-7}{(m^2-3m+9)^2} & \frac{3m^3-11m^2+38m-64}{(m^2-3m+9)^2} & \frac{m^3-8m^2+21m-13}{(m^2-3m+9)^2} \\ \frac{-m^3+5m^2-m-40}{(m^2-3m+9)^2} & \frac{-2m^2+25m-19}{(m^2-3m+9)^2} & \frac{2m^2-16m+32}{(m^2-3m+9)^2} \end{pmatrix}$$

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