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THE RINGS OF FORMAL AND CONVERGENT INVERSE FACTORIAL SERIES

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§ 0. Introduction.

Inverse factorial series with coefficients in $\boldsymbol{C}$ occur naturally in the theory of linear and of non linear difference equations in the complex domain, also in the theory of differential-difference equations in the complex domain, because the basic difference operator acts in a particularly simple way on inverse factorial series (see [1]). It is surprising that inverse factorial series where not used systematically in the theory of finite difference equations and many authors prefer, in this theory of finite difference equations, the use of power series in $1/x$. It is well known that each formal power series in $1/x$ can be written as an inverse factorial serie and the converse is also true but in general the convergence is not preserved. A divergent power series in $1/x$ can be some times written as a convergent factorial series.

With the classical product introduced by N. Nielsen [2] the set of inverse factorial series is a commutative ring with unit which is an algebra over $\boldsymbol{C}$. If we want to introduce linear difference connections it is necessary to know well the structure of the ring of inverse factorial series which has as we shall see later in some senses a different structure as a ring of power series in particular the study of the ring of convergent factorial series cannot be reduced to the study of a ring of convergent power series.

To be complete and general enough for applications we are studying in details the ring of inverse factorial series with coefficients in an arbitrary commutative ring. In this first paper we are proving a **division theorem** and a **preparation theorem** for the rings of formal and convergent factorial series in several variables. It is well known that this two theorems are playing a very important role in the local complex analytic geometry that is in the case of germs of holomorphic functions which is the case of convergent power series. As an application we prove that the ring of germs of holomorphic functions having a factorial series expansion is noetherian.

§ 1. Formal and convergent inverse factorial series.

1. 1. One variable case.

1. 1. 1. Formal factorial series.

Let A be a commutative $\boldsymbol{C}$-algebra with identity. We shall make use of the following notation: for an indeterminate variable x, $(x, m+1) = x(x+1)\cdots(x+m)$ for $m \geq 0$ and $(x, 0) = 1$.

DEFINITION 1. 1. *A formal series of inverse factorials (or simply a formal factorial series) with coefficients in* A *is an infinite series of the form*:

(1)
$$\sum_{m\geq -1} a_m m!/(x, m+1) = a_{-1} + a_0/x + a_1 1!/x(x+1) + a_2 2!/x(x+1)(x+2) + \cdots + a_m m!/(x+1)(x+2)\cdots(x+m) + \cdots$$

where for all m, $a_m \in A$.

Let us **denote by** $\mathscr{F}_A[[x]]$ the set of formal inverse factorial series with coefficients in A.

1. 1. 2. Operations on formal series of inverse factorials.

a) EQUALITY. Two formal factorial series belonging to $\mathscr{F}_A[[x]]$

$$a(x) = \sum_{m\geq -1} a_m m!/(x, m+1) \text{ and } b(x) = \sum_{m\geq -1} b_m m!/(x, m+1)$$

are, by definition, **equal** if and only if for all $m \geq -1$,

$$a_m = b_m$$

b) ADDITION. If $a(x) = \sum_{m\geq -1} a_m m!/(x, m+1)$ and $b(x) = \sum_{m\geq -1} b_m m!/(x, m+1)$ belong to $\mathscr{F}_A[[x]]$, then **by definition**

$$(a+b)(x) = \sum_{m\geq -1} (a_m + b_m) m!/(x, m+1)$$

c) MULTIPLICATION BY AN ELEMENT OF A. Let be given a formal inverse factorial series

$a(x) = \sum_{m \geq -1} a_m m!/(x, m+1)$ and λ an element of A, then **by definition**

$$\lambda a(x) = \sum_{m \geq -1} \lambda a_m m!/(x, m+1)$$

d) MULTIPLICATION OF FORMAL INVERSE FACTORIAL SERIES. Let be given two formal factorial series

$$a(x) = a_{-1} + a'(x) \text{ and } b(x) = b_{-1} + b'(x), \text{ where}$$

$$a'(x) = \sum_{0 \leq 0 \leq \infty} a_s s!/x(x+1)(x+2)\cdots(x+s) \text{ and}$$

$$b'(x) = \sum_{0 \leq 0 \leq \infty} b_s s!/x(x+1)(x+2)\cdots(x+s)$$

Then their formal **product is by definition**:

$$(a.b)(x) = a_{-1}b_{-1} + a_{-1}b(x) + b_{-1}a(x) + \sum_{1 \leq m \leq \infty} c_m m!/x(x+1)(x+2)\cdots(x+m),$$

where

$$m!\, c_m = \sum_{1 \leq k \leq m} (m-k)!(k-1)!\, b_{m-k} C_{m-k,k-1}$$

with

$$C_{m-k,k-1} = \sum_{0 \leq p \leq k-1} \binom{p+m-k}{p} a_{k-1-p}$$

PROPOSITION 1. 1. *With this operations* $\mathcal{F}_A[[x]]$ *is a commutative* $\mathbf{C}$*-algebra.*

Let us denote by A^* the subset of invertible elements of A.

PROPOSITION 1.2. *The ring* $\mathcal{F}_A[[x]]$ *is a local ring with maximal ideal* $\mathcal{M}^* = \{a(x) = \sum_{m \geq -1} a_m m!/(x, m+1)$ *such that* $a_{-1} \notin A^*\}$; *moreover* $\bigcap_{p>0} (M^*)^p = \{0\}$ *and* $\mathcal{F}_A[[x]]$ *is complete with respect to the* $\mathcal{M}^*$*-adic topology.*

A very easy formal computation shows that every inverse factorial series $a(x) = \sum_{m \geq -1} a_m m!/(x, m+1)$ such that $a_{-1} \in A^*$ is invertible in $\mathcal{F}_A[[x]]$.

Now let $\mathcal{A}$ be an ideal in $\mathcal{F}_A[[x]]$ then if $\mathcal{A}$ is not contained in $\mathcal{M}^*$ it does exist in $\mathcal{A}$ an element $a(x) = \sum_{m \geq -1} a_m m!/(x, m+1)$ with $a_{-1} \in A^*$ which implies that $a(x)$ is invertible in $\mathcal{F}_A[[x]]$ and $1 \in \mathcal{A}$, as a consequence $\mathcal{A} = \mathcal{F}_A[[x]]$ and $\mathcal{M}^*$ is the unique maximal ideal of

$\mathcal{T}_A[[x]]$.

1. 1. 3. **Convergence of formal factorial series.**

CASE I. Assume that $A = C$, and let us recall the following results: (see [2] and [3] for details).

1) If a factorial series converges for $x = x_0$, the series converges for every x such that $\mathrm{Re}(x) > \mathrm{Re}(x_0)$;
2) If a factorial series converges for $x = x_0$, the series converges absolutely for every x such that $\mathrm{Re}(x) > \mathrm{Re}(x_0+1)$;
3) If a factorial series converges absolutely for $x = x_0$, the series converges absolutely for every x such that $\mathrm{Re}(x) > \mathrm{Re}(x_0)$.
4) If a factorial series converges for $x = x_0$, then it converges uniformly for $\mathrm{Re}(x) \geq \mathrm{Re}(x_0) + \varepsilon$ for any $\varepsilon > 0$.
5) If there exists a real number λ such that the factorial series converges for $\mathrm{Re}(x) > \lambda + \varepsilon$ for all $\varepsilon > 0$ and diverges, for all $\varepsilon < 0$, then λ is called the **abscissa of convergence** of the factorial series. It can be calculated from the following result of Landau:

   Let

$$\alpha = \operatorname*{Lim\,sup}_{n\to\infty} (\mathrm{Log}(|\sum_{0\leq s\leq n} a_s|)/\mathrm{Log}\, n)$$

and

$$\beta = \operatorname*{Lim\,sup}_{n\to\infty} (\mathrm{Log}(|\sum_{n\leq s\leq\infty} a_s|)/\mathrm{Log}\, n).$$

Then the abscissa of convergence λ is equal to α if $\lambda \geq 0$ and is equal to β if $\lambda < 0$. (In the case $\lambda < 0$, we have to exclude from the convergence domain the points $0, -1, -2, \cdots$)

Let be given a factorial series

$$a(x) = \sum_{m\geq 0} a_m m!/x(x+1)\cdots(x+m)$$

with abscissa of convergence equal to λ and generating function $\hat{a}(1-t)$. Then the series $t^{x-1}\hat{a}(1-t)$ is uniformly convergent in the interval $0 \leq t \leq 1$ provided that $\mathrm{Re}(x) > \mathrm{Max.}\ \{1, \lambda\}$. Conversely, if the series for $a(1-t)$ converges near $t = 1$, then the factorial series

$$a(x) = \sum_{m\geq 0} a_m m!/x(x+1)\cdots(x+m) = \int_{[0,1]} t^{x-1}\hat{a}(t)dt$$

is convergent and defines a holomorphic function in the half plane of convergence. It is easy to see using the Mellin transform that the *addition and the multiplication of convergent factorial series is again a convergent factorial series.* This means that we can speak about the ring of germs of convergent inverse factorial series which will be denoted by $\mathscr{T}_{\boldsymbol{C}}\{x\}$ and in the same way as in the formal case we can prove,

PROPOSITION 1. 2. *The ring* $\mathscr{T}_{\boldsymbol{C}}\{x\}$ *is a local ring with maximal ideal* $\mathscr{M} = \{ a(x) = \sum_{m\geq -1} a_m m!/(x, m+1)$ *such that* $a_{-1}=0 \}$. *Moreover* $\dot{\bigcap}_{p>0}(\mathscr{M})^p = \{0\}$.

REMARK.1. 1. The ring $\mathscr{T}_{\boldsymbol{C}}\{x\}$ is not complete for the $\mathscr{M}$-adic topology. It's completion is $\mathscr{T}_{\boldsymbol{C}}[[x]]$.

CASE II. Assume that $\boldsymbol{A}$ is a normed $\boldsymbol{C}$-algebra. A formal inverse factorial series $a\in\mathscr{T}_A[[x]]$, $a(x) = \sum_{m\geq -1} a_m m!/(x, m+1)$ is said to be normally or absolutely convergent if the inverse factorial series $\sum_{m\geq -1}\|a_m\| m!/(x, m+1)$ is convergent. Let us denote by $\mathscr{T}_A\{x\}$ the set of normally convergent inverse factorial series. The domain of normal or absolut convergence is always a half plane in $\boldsymbol{C}$. Then the results we have in the case $\boldsymbol{A} = \boldsymbol{C}$ can be extented verbating to $\mathscr{T}_A\{x\}$ which *is now a commutative* $\boldsymbol{C}$-algebra *with identity and also a local ring with maximal ideal*

$$\mathscr{M} = \{ a(x) = \sum_{m\geq -1} a_m m!/(x, m+1) \text{ such that } a_{-1}\notin \boldsymbol{A}^* \}.$$

1. 2. **Several variables case.**

1. 2. 1. **Factorial series in several variables.**

Let us introduce the notations: $x = (x_1, \ldots, x_n)$; $m = (m_1, \ldots, m_n)$ *a* multi-index where for all i, $m_i\in \boldsymbol{N}\cup\{-1\}$; $|m| = m_1+\cdots+m_n$;

$\Gamma(m+1) = \Gamma(m_1)\cdots\Gamma(m_n)$; $p = (p_1, \ldots, p_n)$;

$(x, m+p) = (x_1, m_1+p_1)(x_2, m_2+p_2)\cdots(x_n, m_n+p_n)$; $\{p\} = (p_1, p_2, \cdots, p_n)$ with $p_i = p$ for all $i = 1, 2, \ldots, n$, then

$(x, m+\{p\}) = (x_1, m_1+p)(x_2, m_2+p)\cdots(x_n, m_n+p)$

in particular,

$$\begin{aligned}(x,m+\{1\}) &= (x_1, m_1+1)(x_2,m_2+1)\cdots(x_n, m_n+1)\\ &= (x_1)(x_1+1)\cdots(x_1+m_1)(x_2)(x_2+1)\cdots(x_2+m_2)\cdots(x_n)(x_n+1)\cdots(x_n+m_n)\end{aligned}$$

DEFINITION 1. 2. *A formal series of inverse factorials (or simply a formal factorial series) in* $x = (x_1, x_2,\ldots,x_n)$ *with coefficients in a* $\boldsymbol{C}$*-algebra* $\boldsymbol{A}$ *is an infinite series of the following form*:

$$\sum_{|m|\geq -n} a_m m!/(x, m+\{1\}),$$

where for all $m\in(N\cup\{-1\})^n$, $a_m\in A$.

Let us denote this set of formal inverse factorial series by $\mathscr{T}_A[[x]]$.

1. 2. 1. **Operations on formal series of inverse factorials.**

a) EQUALITY. Two formal factorial series belonging to $\mathscr{T}_A[[x]]$

$a(x) = \sum_{|m|\geq -n} b_m m!/(x, m+1)$ and $b(x) = \sum_{|m|\geq -n} a_m m!/(x, m+1)$ are, by definition, **equal** if and only if for all m such that $|m|\geq -n$,

$$a_m = b_m.$$

b) ADDITION. If $a(x) = \sum_{|m|\geq -n} a_m m!/(x, m+1)$ and $b(x) = \sum_{|m|\geq -n} b_m m!/(x, m+1)$ are two inverse factorial series belonging to $\mathscr{T}_A[[x]]$ then by definition

$$(a+b)(x) = \sum_{|m|\geq -n} (a_m+b_m)m!/(x, m+1)$$

c) MULTIPLICATION BY AN ELEMENT OF A. Let be fiven a formal inverse factorial series $a(x) = \sum_{|m|\geq -n} a_m m!/(x, m+1)$ and λ an element of A then **by definition**

$$\lambda a(x) = \sum_{|m|\geq -n} \lambda a_m m!/(x, m+1)$$

d) MULTIPLICATION OF FORMAL INVERSE FACTORIAL SERIES. We define the product of two formal factorial series in several variables by induction on the number of variables. Let us assume that we have a product formula for formal factorial series in n-1 variables. Then write a factorial series in n variables as

$$\sum_{m\geq -1} a_m(x_1, x_2, \ldots, x_{n-1})m!/(x_n, m+1),$$

where for all $m\in N$, $a_m(x_1, x_2, \ldots, x_{n-1})\in\mathscr{T}_A[[x_1, x_2, \ldots, x_{n-1}]]$. Let be given two factorial series $a(x)$ and $b(x)$ in $\mathscr{T}_A[[x_1, x_2, \ldots, x_n]]$ and write it as,

$$a(x) = \sum_{m\geq -1} a_m(x_1, x_2, \ldots, x_{n-1})m!/(x_n, m+1)$$

and

$$b(x) = \sum_{m\geq -1} b_m(x_1, x_2, \cdots, x_{n-1})m!/(x_n, m+1)$$

then we **define the product** of $a(x)$ and $b(x)$ by

$$(a.b)(x) = a_{-1}(x_1, \ldots, x_{n-1})b_{-1}(x_1, \ldots, x_{n-1})+$$

$$\sum_{m\geq 0} a_{-1}(x_1, \ldots, x_{n-1})b_m(x_1, x_2, \ldots, x_{n-1})m!/(x_n, m+1)+$$

$$\sum_{m\geq 0} b_{-1}(x_1, \ldots, x_{n-1})a_m(x_1, x_2, \ldots, x_{n-1})m!/(x_n, m+1)+$$

$$\sum_{0\leq m\leq\infty} c_m(x_1, \ldots, x_{n-1})m!/x_n(x_n+1)\cdots(x_n+m),$$

where

$$m!c_m = \sum_{1\leq k\leq m}(m-k)!(k-1)!b_{m-k}C_{m-k,k-1}$$

with

$$C_{m-k,k-1} = \sum_{0\leq p\leq k}\binom{p+m-k}{p}a_{k-1-p}$$

then by the induction hypothesis we have the product formula for factorial series in several variables. This product is associative and commutative.

PROPOSITION 1. 2. *With the operations defined before the set* $\mathcal{F}_A[[x]]$ *is a commutative* $\boldsymbol{C}$*-algebra with unit.*

Moreover we have also

PROPOSITION 1. 3. *The ring* $\mathcal{F}_A[[x]]$ *is a local ring with maximal ideal* $\mathcal{M}^* = \{a(x) = \sum_{|m|\geq -n} a_m m!/(x, m+1)$ *such that* $a_{-n}\notin \boldsymbol{A}^*\}$. *Moreover* $\bigcap_{p>0}(\mathcal{M}^*)^p = \{0\}$.

1. 2. 2. **Convergence of formal factorial series.**

Using the following notations $x = (x_1, \ldots, x_n)$ and $x_0 = (x_{0,1}, \ldots, x_{0,n})$, and the results of the one variable case we can see that in

CASE I. $\boldsymbol{A} = \boldsymbol{C}$.

1) If a factorial series converges for

$$x = (x_1, x_2, \ldots, x_n) = x_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n})$$

the series converges for every x such that

$$\mathrm{Re}(x)>\mathrm{Re}(x_0) \quad (\mathrm{Re}(x_i)>\mathrm{Re}(x_{0,i}) \text{ all } i = 1, 2, \dots, n);$$

2) If a factorial series converges for $x = x_0$, the series converges absolutely for every x such that $\mathrm{Re}(x)>\mathrm{Re}(x_0+\{1\})$;

3) If a factorial series converges absolutely for $x = x_0$, the series converges absolutely for every $x\in \boldsymbol{R}^n$ such that $\mathrm{Re}(x)>\mathrm{Re}(x_0)$.

4) If a factorial series converges for $x = x_0$, then it converges uniformly for $\mathrm{Re}(x)\geq \mathrm{Re}(x_0)+\varepsilon$ for any $\varepsilon>0$.

5) If there exists $\lambda\in \boldsymbol{R}^n$ such that the factorial series converges for $\mathrm{Re}(x_i)>\lambda_i+\varepsilon$ for all $\varepsilon>0$ and diverges, for all $\varepsilon<0$, then $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is called a **multi-abscissa of convergence** of the factorial series. From all this remarks we see that a factorial series in several variables when it is convergent then the domain of convergence is the product of half planes.

CASE II. Assume that $\boldsymbol{A}$ is a normed $\boldsymbol{C}$-algebra. A formal inverse factorial series $a\in\mathscr{F}_A[[x]]$, $a(x) = \sum_{|m|\geq -n} a_m m!/(x, m+1)$ is said to normally or absolutely convergent if the inverse factorial series $\sum_{|m|\geq -n} \|a_m\| m!/(x, m+1)$ is convergent. Let us denote by $\mathscr{F}_A\{x\}$ the set of normally convergent inverse factorial series. The domain of normal or absolute convergence is always a product of half planes in $\boldsymbol{C}$. Then the results that we have in the case $\boldsymbol{A}=\boldsymbol{C}$ can be extented verbating to $\mathscr{F}_A\{x\}$. It is easy to see that with the formal operation the set $\mathscr{F}_A\{x\}$ *is a commutative ring with identity and also a local ring with maximal ideal*

$$\mathscr{M} = \{ a(x) = \sum_{|m|\geq -n} a_m m!/(x, m+1) \text{ such that } a_{-n}\notin \boldsymbol{A}^* \}.$$

Let us denote by $\mathscr{F}_A\{x\} = \mathscr{F}_A\{x_1, x_2, \dots, x_n\}$ the set of inverse factorial series which have a domain of convergence. It is easy to see that the sum and the product of convergent factorial series is always a convergent inverse factorial series and them it is easy to see that $\mathscr{F}_A\{x\}$ is a commutative $\boldsymbol{C}$-algebra with identity. A factorial series is invertible in the ring $\mathscr{F}_A\{x\}$ if and only if this factorial series has an invertible constant term. As a corollary we have

PROPOSITION 1. 4. *The ring* $\mathscr{F}_A\{x\}$ *is a local ring with maximal ideal* $\mathscr{M}^* = \{ a(x) = \sum_{|m|\geq -n} a_m m!/(x, m+1)$ *such that* $a_{-n}\notin \boldsymbol{A}^* \}$. *Moreover* $\bigcap_{p>0}(\mathscr{M}^*)^p = \{0\}$.

REMARK. 1. 2. As in the case of one variable the ring $\mathscr{T}_A\{x\}$ is not complete for the $\mathscr{M}$-adic topology but it's completion is the ring $\mathscr{T}_A[[x]]$.

REMARK. 1. 3. We have defined $\mathscr{T}_A[[x_1, x_2, \ldots, x_n]]$ (resp. $\mathscr{T}_A\{x_1, x_2, \ldots, x_n\}$) where A is a commutative $\boldsymbol{C}$-algebra with identity (resp. a commutative normed $\boldsymbol{C}$-algebra with identity).

In particular $\mathscr{T}_{\boldsymbol{C}}$ $[[x_1, x_2, \ldots, x_n]] = \mathscr{T}_A[[x_n]]$ with $A = \mathscr{T}_{\boldsymbol{C}}$ $[[x_1, x_2, \ldots, x_{n-1}]]$.

This fact will be used later. We have also $\mathscr{T}_{\boldsymbol{C}}$ $\{x_1, x_2, \ldots, x_n\} = \mathscr{T}_A\{x_n\}$ with $A = \mathscr{T}_{\boldsymbol{C}}$ $\{x_1, x_2, \ldots, x_{n-1}\}$ equiped with a norm that will be defined later.

§ 2. Division theorem and preparation theorem.

2. 1. **Formal case.** Assume first that x is one variable. Let us denote by $\mathscr{T}_{\boldsymbol{C}}$ $[x]$ the subset of $\mathscr{T}_{\boldsymbol{C}}$ $[[x]]$ of inverse factorials series having only a finite numbers of terms.

This means that $a \in \mathscr{T}_{\boldsymbol{C}}$ $[[x]]$,

$$a = \sum_{m \geq -1} a_m m!/(x, m+1)$$

belongs to $\mathscr{T}_{\boldsymbol{C}}$ $[x]$ if and only if it does exist $p \in \boldsymbol{N}$ such that $a_m = 0$ for all $m > p$ and $a_p \neq 0$. Such a factorial series is called a **factorial polynomial**. One of the important difference with power series is that $\mathscr{T}_{\boldsymbol{C}}$ $[x]$ is not a ring. In fact the product of two factorials polynomials is not a factorial polynomial as is shown by the following example.

Example 2. 1.
$(1/x)(1/x)$ is not a factorial polynomial. In fact,

$$1/x^2 = 1/x(x+1) + 1/x(x+1)(x+2) + 2/x(x+1)(x+2)(x+3) + \cdots$$

As a consequence we cannot have a division theorem in $\mathscr{T}_{\boldsymbol{C}}$ $[x]$ and also not in $\mathscr{T}_A[x]$ for an arbitrary commutative $\boldsymbol{C}$-algebra A with unit.
Now consider the ring $\mathscr{T}_{\boldsymbol{C}}$ $[[x_1, x_2, \ldots, x_n]]$ of formal factorial series in $x = (x_1, x_2, \ldots, x_n)$ and let us denote it simply by $\mathscr{T}_{\boldsymbol{C}}$ $[[x]]$.

An element $a \in \mathscr{T}_{\boldsymbol{C}}$ $[[x]]$,

$$a = \sum_{|m| \geq -n} a_m m!/(x, m+1)$$

is invertible in $\mathscr{T}\boldsymbol{c}$ $[[x]]$ if and only if $a_{-1,-1,\dots,-1} \neq 0$.

Let us denote $a_{-1,-1,\dots,-1}$ by $a(\infty) = a(\infty, \infty, \dots, \infty)$.

Such an element $a(x)$ belonging to $\mathscr{T}\boldsymbol{c}$ $[[x_1, x_2, \dots, x_n]]$ can also be written in the form

$$a = \sum_{|m| \geq -n} b_m m!/(x_n, m+1),$$

where for all m, $b_m \in \mathscr{T}\boldsymbol{c}$ $[[x_1, x_2, \dots, x_{n-1}]]$. In the following we write y instead of x_n and x' for $(x_1, x_2, \dots, x_{n-1})$.

DEFINITION 2. 1. *An element* $a \in \mathscr{T}\boldsymbol{c}$ $[[x]]$ *is called of order* $q+1$ *with respect to* y *if it can be written in the form*: $a = \sum_{|m| \geq -n} b_m(x')_m!/(y, m+1)$ *where* $b_m(\infty') = 0$ *for all* $m < q$ *and* $b_q(\infty') \neq 0$.

This means that $b_q(x')$ is a unit in $\mathscr{T}\boldsymbol{c}$ $[[x']]$ and for all $m < q$ $b_m(x')$ is not invertible in $\mathscr{T}\boldsymbol{c}$ $[[x']]$.

Then

$$\begin{aligned} a &= b_{-1}(x') + \cdots + b_{q-1}(x')(q-1)!/(y, q) + b_q(x')(q)!/(y, q+1) + \cdots \\ &= b_{-}1(x') + \cdots + b_{q-1}(x')(q-1)!/(y, q) + b_q(x')c_q(x), \end{aligned}$$

where $c_q(x)$ is a factorial series in y with coefficients in $\mathscr{T}\boldsymbol{c}$ $[[x']]$ and moreover $c_q(x)$ is invertible and $a(\infty', y) = b_q(\infty')c_q(\infty', y)$ where $c_q(\infty', y)$ is also invertible.

DEFINITION 2. 2. *A factorial series* $a \in \mathscr{T}\boldsymbol{c}$ $[[x]]$ *is* ***a factorial polynomial in*** y *if*

$$a = \sum_{|m| \geq -n} b^m(x')m!/(y, m+1),$$

where $b_m(x') \neq 0$ *only for a finite number of index* m.
If $b_q(x') \neq 0$ *and* $b_m(x') = 0$ *for all* $m > q$, *we say that the factorial polynomial* a *is of degree*

$q+1$.

For a given integer $q \geq 0$ we can write an element $a \in \mathcal{T}\mathbf{c}\ [[x]]$ in the form

$$a = (a_q)' + (q!/(y,\ q+1))(a_q)'',$$

where $(a_q)'$ is a factorial polynomial of degree q.

DIVISION THEOREM. *Let* $b \in \mathcal{T}\mathbf{c}\ [[x]]$ *of order* $q+1$ *with respect to* y. *Then for every* $a \in \mathcal{T}\mathbf{c}\ [[x]]$ *it does exist* $g \in \mathcal{T}\mathbf{c}\ [[x]]$ *and a factorial polynomial* $r \in \mathcal{T}\mathbf{c}\ [[x']][y]$ *of degree at least* q *such that*

$$a = bg + r.$$

Moreover g *and* r *are uniquely determined.*

Proof. Define recursively

$$v_0 = a,\ \ldots,\quad v_{j+1} = (q!/(x,\ q+1) - b\{(b'')_q\}^{-1})((v_j)_q)''$$

we have

$$\begin{aligned} a &= \sum_{0 \leq j < \infty} (v_j - v_{j+1}) \\ &= \sum_{0 \leq j < \infty} ((v_j)_q)' + b\{(b_q)''\}^{-1} \sum_{0 \leq j < \infty} ((v_j)_q)''. \end{aligned}$$

Setting $r = \sum_{0 \leq j < \infty} ((v_j)_q)'$ and $g = \{(b_q)''\}^{-1} \sum_{0 \leq j < \infty} ((v_j)_q)''$ we get the existence part of the division theorem:

$$a = bg + r.$$

Uniqueness of the decomposition. It is enough to prove that $bg + r = 0$ implies that $b = 0$ and $r = 0$. We have

$$b(\infty',\ y)g(\infty',\ y) + r(\infty',\ y) = 0$$

which is

$$(q!/(y,\ q+1))g(\infty',\ y)+r(\infty',y)=0,$$

as $r(x',\ y)$ is a factorial polynomial of degree strictly smaller than $q+1$, we see that $g(\infty',\ y)=0$ and this means that the coefficients of the factorial series in y $g(x',\ y)$ are in the maximal ideal $\mathcal{M}'^*$ of $\mathcal{T}\boldsymbol{c}$ $[[x']]$.

We have now

$$(b_q)'g+(b_q)''(q!/(y,\ q+1))g=-r(x',\ y)$$

and this implies that the coefficients of the factorial polynomial $r(x',\ y)$ are in $(\mathcal{M}'^*)^2$. Then using the identity

$$(q!/(y,\ q+1))g(x',\ y)=\{(b_q)''\}^{-1}[r(x',\ y)-(b_q)'g(x',\ y)],$$

we see that the coefficients of $g(x',\ y)$ as a factorial series in y are in $(\mathcal{M}'^*)^2$ and then we deduce that the coefficients of the factorial polynomial in y, $r(x',\ y)$ are in $(\mathcal{M}'^*)^3$ then by induction the coefficients of $g(x',\ y)$ and $r(x',\ y)$ are in $(\mathcal{M}'^*)^p$ for all p which implies that

$$g(x',\ y)=r(x',\ y)=0,$$

and the unicity in the division theorem is proved.

DEFINITION 2. 3. *A monic factorial polynomial* $g\in\mathcal{T}\boldsymbol{c}$ $[[x']][y]$, $g(x',\ y)=g_{-1}(x')+g_0(x')/(y,\ 1)+\cdots+g_{m-1}(x')(m-1)!/(y,\ m)+(m)!/(y,\ m+1)$ *of degree* $m+1$ *is called* ***Weierstrass*** *factorial polynomial in* y *if* $g_j(\infty')=0$ *for all* $j=-1,\ 0,\ 1,\ldots,m-1$.

PREPARATION THEOREM. *Let* $g\in\mathcal{T}\boldsymbol{c}$ $[[x]]$ *be of order* $q+1$ *with respect to* $y=x_n$. *Then there exist a uniquely determined Weierstrass polynomial* $Q\in\mathcal{T}\boldsymbol{c}$ $[[x']][y]$ *of degree* $q+1$ *and a unit* $e\in\mathcal{T}\boldsymbol{c}$ $[[x]]$ *such that*

$$g=eQ.$$

Proof. By the division theorem, $q!/(y,\ q+1)=bg+r$ where $b\in\mathcal{T}\boldsymbol{c}$ $[[x]]$ and $r\in\mathcal{T}\boldsymbol{c}$ $[[x']][y]$ is a factorial polynomial of degree smaller than $q+1$.
Then $g(\infty',\ y)=q!/(y,\ q+1)e^*(y)$ where $e^*(\infty)\neq 0$. Thus $r(\infty',\ y)=0$ and $b(\infty',\ y)=(e^*(y))^{-1}$. This implies that $b(x''',\ y)$ is a unit in $\mathcal{T}\boldsymbol{c}$ $[[x]]$.

Now, $g = eQ$ with $e = (b)^{-1}$ and $Q(x', y) = q!/(y, q+1) - r(x', y)$; clearly $Q(x', y)$ is a Weierstrass polynomial in y.

REMARK 2. 1. It is well known (see [4]) that, in the ring of formal power series $\boldsymbol{C}[[1/x_1, 1/x_2, \dots, 1/x_n]]$ we have a division theorem. that is if $g(1/x_1, 1/x_2, \dots, 1/x_n)$ is of order q with respect to $1/x_n$ for every $f \in \boldsymbol{C}[[1/x_1, 1/x_2, \dots, 1/x_n]]$ we have $f = bg + r$ where $r(1/x', 1/y)$ is a polynomial in $1/y$ of degree strictly smaller than q, moreover b and r are uniquely determined. It is also well known that each formal power series in $1/x_1, 1/x_2, \dots, 1/x_n$ can be written as a formal factorial series and vice versa each formal factorial series in $x_1, x_2, \dots, x_n$ can be written as a formal power series in $1/x_1, 1/x_2, \dots, 1/x_n$. The fact that a polynomial in $1/y$ written as a factorial series is not a factorial polynomial means that the division theorem in $\boldsymbol{C}[[1/x_1, 1/x_2, \dots, 1/x_n]]$ does not imply the division theorem in $\mathscr{T}_{\boldsymbol{C}}[[x]]$. Conversely the division theorem in the ring $\mathscr{T}_{\boldsymbol{C}}[[x]]$ does not imply the division theorem in $\boldsymbol{C}[[1/x_1, 1/x_2, \dots, 1/x_n]]$ the reason is that a factorial polynomial written as a power series in the $1/x_i$ is not in general a polynomial.

2. 2. ***Convergent case.*** We are here following the ideas of [5].

Let $f = \sum_{|m| \geq -n} f_m m!/(x, m+1)$ be a formal factorial series in $x = (x_1, x_2, \dots, x_n)$ with coefficients in $\boldsymbol{C}$. We are introducing the following notation for $\rho = (\rho_1, \rho_2, \dots, \rho_n) \in (\boldsymbol{R}^+)^n$,

$$\|f\|_\rho = \sum_{|m| \geq -n} |f_m| m!/(\rho, m+1)$$

and

$$\mathscr{T}_\rho = \{f \in \mathscr{T}_{\boldsymbol{C}}[[x]] \mid \|f\|_\rho < \infty\}.$$

The set $\mathscr{T}_\rho$ with the usual operations on factorial series is clearly a normed $\boldsymbol{C}$-algebra which is a subalgebra of the $\boldsymbol{C}$-algebra $\mathscr{H}_\rho$ of functions holomorphic in the product of the half planes $\mathrm{Re}(x_i) > \rho_i$.

If $f \in \mathscr{H}_\rho$ and if f has an expansion into a factorial series such that it does exist ρ with $\|f\|_\rho < +\infty$, then

$$\lim_{\rho \to +\infty} (\|f\|_\rho) = |f(\infty)| = a_{-1,-1,\dots,-1}$$

LEMMA 2. 1. *$\mathscr{T}_\rho$ is a **Banach** algebra.*

Proof. If $f=\sum_{|m|\geq -n} f_m m!/(x,\ m+1)$ is in $\mathscr{T}_\rho$, then we have for all m

$$|f_m|m!/(\rho,\ m+1)\leq\|f\|_\rho.$$

Now let $f_j=\sum_{|m|\geq -n} f_{m,j}m!/(x,\ m+1)$ be a Cauchy sequence in $\mathscr{T}_\rho$ then by the inequality given above for each m fixed $f_{m,j}$ is a Cauchy sequence in $\boldsymbol{C}$. We put $f_m=\lim_{j\to+\infty} f_{m,j}$ and $f=\sum_{|m|\geq -n} f_m m!/(x,\ m+1)$.

We claim that the sequence f_j converges toward $f\in\mathscr{T}_\rho$. The argument is standard. For given $\varepsilon>0$, there is an N such that

$$\sum_{|m|\geq -n}|f_{j+i,m}-f_{j,m}|m!/(\rho,\ m+1)=\|f_{j+i}-f_j\|_\rho\leq\varepsilon$$

for all $i>0$ and $j\geq N$;
From

$$|f_m-f_{j,m}|\leq|f_m-f_{j+i,m}|+|f_{j+i,m}-f_{j,m}|$$

we conclude

$$\sum_{-n\leq|m|<s}|f_m-f_{j,m}|m!/(\rho,\ m+1)$$
$$\leq\sum_{-n\leq|m|<s}|f_m-f_{j+i,m}|m!/(\rho,\ m+1)+\varepsilon$$

for all $s>0$, $i>0$ and $j\geq N$. Choosing i large enough we see that

$$\sum_{-n\leq|m|<s}|f_m-f_{j,m}|m!/(\rho,\ m+1)\leq 2\varepsilon$$

for all $j\geq N$ and $s\geq 0$. This implies that

$$\sum_{|m|\geq -n}|f_m-f_{j,m}|m!/(\rho,\ m+1)=\|f-f_j\|_\rho\leq 2\varepsilon$$

for $j\geq N$.

Hence,

$$\|f\|_\rho \leq \|f_N\|_\rho + \|f-f_N\|_\rho \leq +\infty$$

so $f \in \mathcal{T}_\rho$ and moreover $\|f-f_j\|_\rho \leq 2\varepsilon$ for $j \geq N$ implies that $\lim f_j = f$ in $\mathcal{T}_\rho$ amd proves that $\mathcal{T}_\rho$ is a Banach algebra.

Each $f \in \mathcal{T}_\rho$ can be written as a factorial series in $x_n = y$

$$f(x) = \sum_{m \geq +1} f_m(x_1, x_2, \ldots, x_{n-1}) m!/(y, m+1)$$

and

$$\|f\|_\rho = \sum_{m \geq -1} \|f_m\|_{\rho'} \; m!/(\rho_n, m+1)$$

with $\rho' = (\rho_1, \rho_2, \ldots, \rho_{n-1})$. Using this notation we define for a given integer $q \geq 0$,

$$(f_q)' := \sum_{-1 \leq m < q-1} f_m m!/(y, m+1)$$

which is a factorial polynomial of degree $< q+1$. Then

$$f = (f_q)' + b!/(y, q+1)(f_q)''.$$

It is easy to see that we have

$$\|(f_q)'\|_\rho \leq \|f\|_\rho \quad \text{and} \quad \|(f_q)''\|_\rho \leq \{(p_n, q+1)/q!\}\|f\|_\rho.$$

Let us recall that an element $g \in \mathcal{T}\mathbf{c}\ \{x\}$ has order q with respect to $x_n = y$, if $g(\infty', y) = \{q!/(y, q+1)\} e(y)$ where $e(y)$ is a unit in $\mathcal{T}\mathbf{c}\ \{y\}$ that is $e(\infty) \neq 0$. Expressing

$$g(x) = \sum_{m \geq -1} g_m(x') m!/(y, m+1)$$

as a factorial series in y with coefficients in $\mathcal{T}\mathbf{c}\ \{x'\}$ where $x' = (x_1, \ldots, x_{n-1})$ the condition given before is equivalent to $g_m(\infty') = 0$ for all $m < q$ and $g_q(\infty') \neq 0$.

DIVISION THEOREM *Let $g \in \mathcal{T}\mathbf{c}\ \{x\}$ of order $q+1$ with respect to $x_n = y$, then for every $f \in \mathcal{T}\mathbf{c}\ \{x\}$ there exist $b \in \mathcal{T}\mathbf{c}\ \{x\}$ and a factorial polynomial $r \in \mathcal{T}\mathbf{c}\ \{x'\}[y]$ of degree $< q+1$*

such that

$$f = bg + r\,;$$

the elements b and r are uniquely determined.

Proof. Let $\varepsilon \in \boldsymbol{R}$, $0<\varepsilon<1$ be given and

$$g = (g_q)' + \{q!/(y,\ q+1)\}(g_q)''.$$

In the following we drop the index q. We first choose $p=(p_1, p_2,\dots, p_n)$ such that,

1) $g \in \mathscr{T}_\rho$,
2) g'' is a unit and $(g'')^{-1} \in \mathscr{T}_\rho$,
3) $\|q!/(y,\ q+1) - g(g'')^{-1}\| \le \{q!/(y,\ q+1)\}\varepsilon$,
4) $f \in \mathscr{T}_\rho$.

We define $v_j \in \mathscr{T}_\rho$ recursively by setting
$v_0 := f,\dots,\ v_{j+1} = (q!/(y,\ q+1) - g(g'')^{-1})(v_j)'' = -(g')(g'')^{-1}(v_j)''$.
Since

$$\|(v_j)''\|_\rho \le \{(p_n,\ q+1)/q!\}\|v_j\|_\rho,$$

we conclude that

$$\|v_{j+1}\|_\rho \le \varepsilon\|v_j\|_\rho \le \varepsilon^{j+1}\|v_0\|_\rho$$

and

$$v = \sum_{j\ge 0} v_j \in \mathscr{T}_\rho.$$

We now have

$$\begin{aligned} f &= \sum_{j\ge 0}(v_j - v_{j+1}) \\ &= \sum_{j\ge 0}\{(v_j)' + (q!/(y,\ q+1))(v_j)'' - (q!/(y,\ q+1) - g(g'')^{-1})(v_j)''\} \\ &= \sum_{j\ge 0}\{(v_j)' + g(g'')^{+1}(v_j)'' \\ &= v' + g(g'')^{-1}(v)''\,; \end{aligned}$$

v' is a polynomial $r \in \mathcal{F}_\rho\{x'\}[y]$ of degree strictly smaller than $q+1$. We have

$$(*) \quad f = bg + r \text{ where } b = (g'')^{-1}(v)''$$

which proves the existence part of the division theorem.

UNICITY. If we are looking at the identity (*) as a formal identity replacing all terms by their factorial expansion, we get the formal division theorem and the formal unicity implies the unicity in the convergent division theorem.

A monic factorial polynomial

$$P(x', y) = q!/(y, q+1) + P_1(x')(q-1)!/(y, q) + \cdots + P_q(x')$$

is called a Weierstrass factorial polynomial in y if

$$P_1(\infty') = P_2(\infty') = \cdots = P_q(\infty') = 0.$$

Thus the Weierstrass factorial polynomial $P(x', y)$ is of order $q+1$ with respect to y.

PREPARATION THEOREM. *If $g \in \mathcal{F}_{\boldsymbol{C}}$ has finite order $q+1$ with respect to $x_n = y$, then there exist a uniquely determined Weierstrass factorial polynomial $P(x', y) \in \mathcal{F}_{\boldsymbol{C}}\{x'\}[y]$ of degree $q+1$ and a unit $e(x) \in \mathcal{F}_{\boldsymbol{C}}\{x\}$ such that*

$$g = eP.$$

The proof is exactly the same as in the formal case.

REMARK. 2. 2. In the classical Weierstrass theory in ring of germs of holomorphic functions at the origin of $\boldsymbol{C}^n$ we have moreover the following result: *if g is a polynomial then e is a polynomial.*

This result is *not longer true for the ring of germs of factorial series.* In fact the product of two factorial polynomials is not in general a factorial polynomial.

Observation. Let $g \in \mathcal{F}_{\boldsymbol{C}}\{x\}$ (or $\mathcal{F}_{\boldsymbol{C}}[[x]]$)

$$g(x)=g(x_1,\ x_2,\dots,\ x_{n-1},\ y)=\sum_{m\geq -1} g_m(x')m!/(y,\ m+1).$$

Then it does exist a linear change of coordinates (or parameters),

$$1/x'_\nu = 1/x_\nu - c_\nu/y \quad \nu=1,\ 2,\dots,\ n-1$$

such that g is of finite order in y with respect to the new coordinates (or parameters).

First let us remark that the formal order is also the convergent order and vice versa. So, it is enough to prove the result in the formal case. Moreover if a formal power series in $1/x_1$, $1/x_2$, ..., $1/x_{n-1}$, $1/y$ is of order q with respect to $1/y$ then written as an factorial series it is also of order q as a factorial series. The converse is also true. Let us prove the result for a power series in $1/x_1$, $1/x_2,\dots, 1/x_{n-1}$, $1/y$, let $g\in \boldsymbol{C}[[1/x_1,\ 1/x_2,\dots,\ 1/x_{n-1},\ 1/y]]$ and write

$$g=\sum_{q\leq j<\infty} P_j(1/x_1,\ 1/x_2,\dots,\ 1/x_{n-1},\ 1/y)$$

where for all j, P_j is a homogenous polynomial in $1/x_1$, $1/x_2,\dots, 1/x_{n-1}$, $1/y$ and moreover $P_q \neq 0$. Now make the change

$$1/x'_\nu = 1/x_\nu - c_\nu/y \quad \nu=1,\ 2,\dots,\ n-1,$$

we see easily that

$$\begin{aligned} g(0,\ 0,\dots,\ 0,\ 1/y) &= \sum_{q\leq j\leq\infty} P_j(c_1/y,\ c_2/y,\dots,\ c_{n-1}/y,\ 1/y) \\ &= P_q(c_1,\ c_2,\dots,\ c_{n-1},\ 1)1/y^q+\cdots \end{aligned}$$

since $P_q \neq 0$ we can choose c_1, $c_2,\dots, c_{n-1}$ such that

$$P_q(c_1,\ c_2,\dots,\ c_{n-1},\ 1)\neq 0$$

and the result is proved.

§ 3. Properties of the rings of formal and convergent factorial series.

Consider now the set $\mathscr{F}\boldsymbol{c}$ of functions f for which it does exist a coordinate system $(x_1,\ x_2,\dots,\ x_n)$ in $\boldsymbol{C}^n$ and an m-tuple of positif real numbers

$$\omega'_0 = (\omega'_{0,1}, \omega'_{0,2}, \dots, \omega'_{0,n}) \geq (1, 1, \dots, 1)$$

such that for all $\omega > \omega'_0$ it does exist $\lambda \in (\boldsymbol{R}^+)^n$ such that

$$f(x) = \sum_{|m| \geq -n} f_m m! / (x/\omega, m+1)$$
$$= \sum_{|m| \geq -n} f_{m_1, m_2, \dots, m_n} \{m_1! m_2! \cdots m_n!\} / (x_1/\omega_1, m_1+1) \cdots (x_n/\omega_n, m_n+1)$$

the factorial series being uniformely and absolutely convergent in $\mathrm{Re}\, x_i > \lambda_i$. It is easy to see with the usual operations on factorial series that $\mathscr{T}\boldsymbol{c}$ is a commutative ring with identity. This ring is called **the ring of holomorphic functions having convergent factorial expansions**. This ring is also a local ring the maximal ideal is the ideal of non invertible functions in $\mathscr{T}\boldsymbol{c}$. It is also clear that we have a division theorem in $\mathscr{T}\boldsymbol{c}$ and also the preparation theorem.

THEOREM 3. 1. *The ring* $\mathscr{T}\boldsymbol{c}$ *is noetherian.*

Proof. We proceed by induction on the numbers of variables n. For $n = 0$ the result is clear. Let $g \in \mathscr{T}\boldsymbol{c}$, $g \neq 0$ arbitrary, we choose the parameters $x_1, x_2, \dots, x_{n-1}, y$ such that g is of order $q+1$ with respect to y. In order to show that $\mathscr{T}\boldsymbol{c}$ is noetherian it is enough to show that the residue ring $\mathscr{T}\boldsymbol{c}/g\mathscr{T}\boldsymbol{c}$ is always noetherian. Let $f \in \mathscr{T}\boldsymbol{c}$, we have $f = gb + r$ where $r = r_0 + r_1 1/y + \cdots + r_{q-1}(q-1)!/(y, q)$ where for all i, $r_i \in \mathscr{T}'\boldsymbol{c}$ (the same ring but in the variables $x' = (x_1, x_2, \dots, x_{n-1})$).

The map

$$\mathscr{T}\boldsymbol{c} \longrightarrow (\mathscr{T}'\boldsymbol{c})^q$$
$$f \longrightarrow (r_0, r_1, \dots, r_{q-1})$$

is a $\mathscr{T}'\boldsymbol{c}$-module epimorphism with kernel $g\mathscr{T}\boldsymbol{c}$ and as consequence we have an $\mathscr{T}'\boldsymbol{c}$-module isomorphism,

$$\mathscr{T}\boldsymbol{c}/b\mathscr{T}\boldsymbol{c} \longrightarrow (\mathscr{T}'\boldsymbol{c})^q.$$

By induction $\mathscr{T}'\boldsymbol{c}$ is a noetherian ring; hence $\mathscr{T}\boldsymbol{c}/b\mathscr{T}\boldsymbol{c}$ is a noetherian $\mathscr{T}'\boldsymbol{c}$-module and therefore a noetherian ring.

REMARK 3. 1. The ring $\mathscr{T}\boldsymbol{c}$ is not a factorial ring always because of the fact that we cannot consider the ring of factorial polynomials and therefore we don't have a Hensel's lemma. As a conclusion the ring $\mathscr{T}\boldsymbol{c}$ is not factorial and also not henselian.

REMARK 3. 2. The fact that the ring $\mathscr{T}\boldsymbol{c}$ is noetherian will play an important role in the theory of linear difference connexions that we will develop in an other joint paper.

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