ON A 2-PARAMETER FAMILY OF SOLUTIONS TO A NONLINEAR 2-SYSTEM WITH AN IRREGULAR SINGULARITY OF SIGNATURE (1, 1)

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NOTATION.
$\mathbb{C}$: the field of complex numbers;
$\mathbb{Z}$: the ring of integers;

$$\mathbb{Z}_{\geq 0} = \{ k \in \mathbb{Z} ; k \geq 0 \};$$

For $z \in \mathbb{C}$,
$\text{Re}(z)$: the real part of $z$,
$\text{Im}(z)$: the imaginary part of $z$;

For a 2-vector $a = (a_1, a_2) \in \mathbb{C}^2$,

$$1(a) = 1(a_1, a_2) = \begin{pmatrix} a_1 \\ 0 \\ \alpha_2 \end{pmatrix} \text{ and } 1 = 1(1, 1);$$

For 2-vectors $a = (a_1, a_2), \beta = (\beta_1, \beta_2) \in \mathbb{C}^2$,

$$(a, \beta) = a_1\beta_1 + a_2\beta_2;$$

For a 2-vector $x = (x_1, x_2) \in \mathbb{C}^2$ and a 2-vector $k = (k_1, k_2) \in \mathbb{Z}_{\geq 0}$,

$$|x| = \max_{i=1,2} |x_i|, |k| = k_1 + k_2, \ x^k = x_1^{k_1}x_2^{k_2};$$

For a domain $M$ in $\mathbb{C}$ or $\mathbb{C}^2$,

$\mathcal{C}(M)$: the set of holomorphic functions from $M$ into $\mathbb{C}^2$;

$\mathcal{B}(M)$: the set of holomorphic and bounded functions from $M$ into $\mathbb{C}^2$;

For constants $\theta_1, \theta_2, r$ and $\delta$,

$$S(\theta_1, \theta_2; r) = \{ t \in \mathbb{C} ; \theta_1 < \text{arg } t < \theta_2, |t| < r \},$$
\[ D(\theta_1, \theta_2; r, \delta) = \{(t, x) \in \mathbb{C} \times \mathbb{C}^2; t \in S(\theta_1, \theta_2; r), |x| < \delta\}, \]

\[ \mathcal{A}(\theta_1, \theta_2; r) : \text{the set of holomorphic functions from } S(\theta_1, \theta_2; r) \text{ into } \mathbb{C}^2 \text{ admitting asymptotic expansions in powers of } t \text{ as } t \to 0, t \in S(\theta_1, \theta_2; r). \]

Further, for simplicity, we use

\[ \mathcal{A}(\theta_1, \theta_2; r) = \mathcal{A}(S(\theta_1, \theta_2; r)), \mathcal{A}(\theta_1, \theta_2; r, \delta) = \mathcal{A}(D(\theta_1, \theta_2; r, \delta)). \]

1. Introduction.

In the study of nonlinear ordinary differential equations, it is important to obtain analytic expressions of general solutions near the fixed singularities. Concerning the problem, we have a general theorem by J. Malmquist ([6], [7], [8]) under the so-called Poincaré condition. We notice that although the Poincaré condition is a generic one, many interesting equations such as Painlevé equations do not satisfy the condition.

As far as the author knows, the first work on the construction of general solutions in the case where the Poincaré condition is completely violated was done by M. Iwano ([4]). He constructed 2-parameter families of solutions to certain 2-systems of signature (1, 1) near the irregular singularity. After the work by M. Iwano, S. Yoshida established a general theory for 2-system in which the ratio of the characteristic exponents is equal to -1 in order to investigate the irregular singularity of Painlevé equations ([12], [13]).

In this paper, we shall study a 2-system with an irregular singularity where the ratio of the characteristic exponents is a negative irrational number.

The system which is studied is of the form:

\[ \Sigma : \quad t^{\sigma+1} \frac{dx}{dt} = (\lambda(t)1(\mu) + t^\sigma1(\sigma))x + f(t, x), \]

where

(i) \( t \) is a coordinate of \( \mathbb{C} \);
(ii) \( x \) is a 2-vector in \( \mathbb{C}^2 \) with coordinates \( x_1, x_2 \);
(iii) \( \sigma \) is a positive integer ;
(iv) \( \mu \) is a 2-vector with entries \( \mu_1, \mu_2 \) with \( \mu_1 > 0, \mu_2 < 0 \) such that the ratio \( \mu_1/\mu_2 \) is an irrational number ;
(v) \( \alpha \) is a 2-vector with entries \( \alpha_1, \alpha_2 \) satisfying the inequality

\[ (1.1) \quad \text{Re} (\mu_1 \alpha_2 - \mu_2 \alpha_1) > 0 ; \]

(vi) \( \lambda(t) \) is a polynomial in \( t \) of degree \( \sigma - 1 \) with \( \lambda(0) = 1 ; \)
(ω) \( f \) is a 2-vector function belonging to \( \mathcal{A}(\theta, \overline{\theta}; r, \delta) \) with \( f(t, x) = O(|x|^r) \) such that the coefficients \( f_d(t) \) in the expansion of \( f(t, x) \) as \( f(t, x) = \sum_{d=0}^{\infty} f_d(t)x^d \) are in \( \mathcal{A}(\theta, \overline{\theta}; r) \). Here \( \theta, \overline{\theta} \) and \( r, \delta > 0 \) are constants and \( O \) denotes Landau's symbol.

Set

\[
(1.2) \quad \Lambda(t) = \int_0^t \lambda(t) t^{-\sigma-1} dt.
\]

We say that \( \theta \) is a singular direction of \( \Lambda(t) \) if

\[
(1.3) \quad \cos(a\theta + \arg \lambda_0) = 0, \quad \lambda_0 = \lambda(0).
\]

A half line passing through \( t = 0 \) with argument \( \theta \) satisfying (1.3) is called a singular line of \( \Lambda(t) \).

We suppose moreover the following assumptions:

(A_1) There exist positive constants \( C \) and \( m \) such that, for each \( i = 1, 2 \), the following inequalities

\[
(1.4) \quad |(k, \mu) - \mu_i| > C|k|^{-m}
\]

hold, for all 2-vectors \( k = (k_1, k_2) \in \mathbb{Z}_{\geq 0}^2 \) with \( |k| \geq 2 \). This condition is usually called the Siegel condition.

(A_2) \( S(\theta, \overline{\theta}; r) \) contains one and only one singular direction of \( \Lambda(t) \) and neither \( \theta \) nor \( \overline{\theta} \) is a singular direction of \( \Lambda(t) \).

We state our main theorem.

**Theorem.** Under the assumptions (A_1) and (A_2), there exists a unique 2-vector function \( p \) holomorphic and bounded in \( D(\theta, \overline{\theta}; r', \delta') \) with \( p(t, y) = O(|y|^r) \) such that a change of coordinates \( (t, x) \rightarrow (t, y) \) of the form

\[
T : \quad x = y + p(t, y)
\]

transforms the system \( \Sigma \) to

\[
\Sigma' : \quad t^{\sigma+1} dy/dt = (\lambda(t) \mathfrak{1}(\mu) + t^{\sigma} \mathfrak{1}(\alpha)) y,
\]

provided that \( r' \) and \( \delta' > 0 \) are sufficiently small. Here the coefficients \( \mathfrak{p}_d(t) \) in the power series expansion of \( p(t, y) \) as \( p(t, y) = \sum_{d=0}^{\infty} \mathfrak{p}_d(t)y^d \) are in \( \mathcal{A}(\theta, \overline{\theta}; r') \).
By virtue of the theorem, we can construct a 2-parameter family of solutions to the system $\Sigma$. In fact, since the general solution to the reduced system $\Sigma'$ is

\[(1.5)\quad x(t; C) = \exp\left[-A(t)\lambda(\mu) + \log t \cdot 1(a)\right]C,\]

a 2-parameter family of solutions to the original system $\Sigma$ is given by

\[x(t; C) = y(t; C) + p(t, y(t; C)).\]

Here $C = (C_1, C_2)$ is an arbitrary constant 2-vector.

The transformation $T$ is given as a composition of three transformations, say $T_1$, $T_2$ and $T_3$, namely $T = T_3 \circ T_2 \circ T_1$.

By the transformation $T_1$ we make the nonlinear term $f = (f_1, f_2)$ of $\Sigma$ to be of order $O(t^\sigma)$. The transformation $T_2$ changes the system $\Sigma_{\tau_1}$ obtained by $T_1$ to a system $\Sigma_{\tau_2}$ so that each $f_i$, $i = 1, 2$, has $x_i$ as a factor. Then we construct the transformation $T_3$ which eliminates the nonlinear term $f = (f_1, f_2)$ of $\Sigma_{\tau_2}$.

We remark that the preliminary reduction of $\Sigma$ by $T_1$ and $T_2$ is needed for proving the convergence of the formal power series $\Sigma_{\lambda^{k_1}p_k(t)}y^k$.

We use the following abbreviation:

\[S(r) = S(\theta, \bar{\theta}; r), \quad D(r, \delta) = D(\theta, \bar{\theta}; r, \delta), \]
\[\mathcal{A}(r) = \mathcal{A}(\theta, \bar{\theta}; r), \quad \mathcal{R}(r, \delta) = \mathcal{R}(D(\theta, \bar{\theta}; r, \delta)),\]

since $\theta$ and $\bar{\theta}$ are fixed, and

\[(1.6)\quad A(t) = \lambda(t)\lambda(\mu) + t^\sigma 1(\delta).\]

2. The construction of preliminary transformations $T_1$ and $T_2$.

2.1. The construction of $T_1$. In this part, it will be shown that we can find a 2-vector function $u$ holomorphic at $(t, \eta) = (0, 0) \in \mathbb{C} \times \mathbb{C}^2$ with $u = O(|\eta|^2)$ so that a change of coordinates $(t, x) \rightarrow (t, \eta)$ of the form

\[T_1: \quad x = \eta + u(t, \eta)\]

changes the system $\Sigma$ to a system $\Sigma_{\tau_1}$ of which the nonlinear terms are of order $O(t^\sigma)$.

This is proved by making use of the following lemma successively.
LEMMA 2.1. Given a system of the form

\[ t^{\sigma+1} \frac{dx}{dt} = A(t)x + t^{\sigma-1} f(t, x), \]

where \( A(t) \) is the matrix of (1.6) and \( f \in \mathcal{A}(r, \delta) \). Then there exists a change of coordinates \( (t, x) \rightarrow (t, \eta) \) of the form

\[ x = \eta + t^{\sigma-1} u(\eta) \]

which transforms (2.1) into a system of the form

\[ t^{\sigma+1} \frac{d\eta}{dt} = A(t)\eta + t^{\sigma-1} g(t, \eta). \]

Here \( u \in \mathcal{C}((\eta \in \mathbb{C} ; |\eta| < \delta')) \) and \( g \in \mathcal{A}(r', \delta') \), provided that \( r', \delta' > 0 \) are sufficiently small.

PROOF. We see that the change of coordinates (2.2) transforms (2.1) into a system of the form

\[ t^{\sigma+1} \frac{d\eta}{dt} = A(t)\eta + t^{\sigma-1} h(t, \eta), \]

where

\[ h(t, \eta) = (1 + t^{\sigma-1} \partial u/\partial \eta)^{-1} \times [-(j-1)t^{\sigma} u - (\partial u/\partial \eta) A(t)\eta + A(t)u - f(t, \eta + t^{\sigma-1} u)]. \]

Therefore, in order that (2.4) is of the form (2.3), it is necessary and sufficient that \( h(0, \eta) \) vanishes identically, that is,

\[ (\partial u/\partial \eta) A(0)\eta - A(0)u - f(0, \eta + u(\eta)) = 0 \]

if \( j = 1 \), or

\[ (\partial u/\partial \eta) A(0)\eta - A(0)u - f(0, \eta) = 0 \]

if \( 2 \leq j \leq \sigma \). Hence, we have only to show that a system of partial differential equations (2.6) or (2.7) has a solution \( u(\eta) \) with \( u = O(|\eta|^p) \) holomorphic at \( \eta = 0 \). Here we notice

\[ A(0) = 1(\mu). \]
The existence of a holomorphic solution to (2.7) will be shown in Appendix. On the other hand, the existence of \( u \) to (2.6) is an immediate consequence of a famous Siegel's theorem.

**Lemma 2.2 (Siegel).** Let there be given an autonomous \( n \)-system

\[(2.9) \quad \frac{dx}{dt} = 1_n(\mu)x + f(x)\]

where \( \mu = (\mu_1, \cdots, \mu_n) \in \mathbb{C}^n, 1_n(\mu) = \text{diag}(\mu_1, \cdots, \mu_n) \), and an \( n \)-vector function \( f \) is holomorphic at \( x = 0 \) with \( f = O(|x|^2) \). Suppose that there exist positive constants \( C \) and \( m \) such that the following inequalities hold:

\[|k, \mu| - \mu_i < C|k|^m, \quad (1 \leq i \leq n)\]

for all \( n \)-vector \( k = (k_1, \cdots, k_n) \in \mathbb{Z}^n \) with \( |k| = k_1 + \cdots + k_n \geq 2 \).

Then there exists an \( n \)-vector function \( u \) holomorphic at \( \eta = 0 \) with \( u = O(|\eta|^2) \) such that a change of coordinates \( (t, x) \rightarrow (t, \eta) \) of the form

\[(2.10) \quad x = \eta + u(\eta)\]

transforms (2.9) into

\[(2.11) \quad \frac{d\eta}{dt} = 1_n(\mu)\eta.\]

In fact, in order that (2.10) transforms (2.9) into (2.11), it is necessary and sufficient that \( u = u(\eta) \) is a solution to

\[(\partial u/\partial \eta)1_n(\mu)\eta - 1_n(\mu)u - f(\eta + u) = 0,\]

which is just of the same form as (2.6) in case of \( n = 2 \). Q.E.D.

2.2. The construction of \( T_2 \). Let \( \Sigma^{T_2} \) be the system transformed from \( \Sigma \) by the \( T_1 \). Then it is written as follows:

\[\Sigma^{T_1} : \quad t^{r_1} \frac{d\eta}{dt} = A(t)\eta + t^{r}g(t, \eta),\]

where \( g \in \mathcal{B}(r', \delta'), r' \) and \( \delta' \) being sufficiently small.

We obtain:

**Lemma 2.3.** The system \( \Sigma^{T_1} \) can be reduced to a system
\[ \Sigma^{T_1} : \quad t^{\sigma+1} d\xi/dt = [A(t) + t^{\sigma} h(t, \xi)] \xi \]

by a suitable transformation of coordinates \((t, \eta) \rightarrow (t, \xi)\) of the form

\[ T_2 : \quad \eta = \xi + t^{\sigma}(w^1(t, \xi) + w^2(t, \xi)) \]

where two 2-vector functions \(w^i, i = 1, 2, h \in \mathcal{A}(r), \delta'\) and that \(h = O(|\xi|)\), provided that \(0 < r'' < r', \quad 0 < \delta'' < \delta'\) are sufficiently small.

**Proof.** Let

\[ (2.12) \quad t^{\sigma+1} d\xi/dt = A(t) \xi + t^{\sigma} G(t, \xi) \]

be the system transformed from \(\Sigma^{T_1}\) by \(T_2\). We have, then,

\[
G(t, \xi, \zeta) = -\left[t^{\sigma+1} \partial w^1/\partial t + \partial w^1/\partial \xi_1 \cdot t^{\sigma+1} \partial \xi_1/\partial t - (A(t) - t^{\sigma})w^1\right] \\
-\left[t^{\sigma+1} \partial w^2/\partial t + \partial w^2/\partial \xi_2 \cdot t^{\sigma+1} \partial \xi_2/\partial t - (A(t) - t^{\sigma})w^2\right] \\
+ f(t, \xi + t^{\sigma}w^1 + t^{\sigma}w^2)
\]

We consider the following partial differential equations:

\[ (2.13) \quad t^{\sigma+1} \partial w^i/\partial t + (\mu_i t^\sigma(t) + \alpha_i t^\sigma) \xi \partial w^i/\partial \xi_i - (A(t) - t^{\sigma})w^i - f(t, (\delta_n \xi_i, \delta_n \zeta_i) + t^{\sigma}w^i) = 0 \]

\(i = 1, 2, \delta_n\) being Kronecker's delta. Notice that (2.13), is the condition that \(G(t, \xi, \zeta)\) has \(\zeta_i\) as factor.

We first obtain a formal solution to (2.13), of the form

\[ (2.14) \quad w^i(t, \xi) = \Sigma_{j=2} w_j^i(t) \xi^j \]

Inserting (2.14), into (2.13), and equating the coefficients of powers in \(\xi\), we have

\[ (2.15) \quad t^{\sigma+1} \partial w_j^i/\partial t + [\lambda(t)1(\mu_i - \mu_0) + t^{\sigma}1(j \alpha_i - \alpha_0) + j \alpha_i - \alpha_0 + \sigma)]w_j^i = \Sigma_{j=2} \left[t^{\sigma}f_{j,n} + w_1^i w_{1,n} \cdots w_{i-1,n} w_{i+1,n} \cdots w_{n,n}\right] + j \geq 2 \]

where \(w_1^i, j\) and \(w_k^i, j\) are the first and the second entries of \(w_j^i\), respectively. It is easy to see that we can uniquely determine \(w_j^i \in \mathcal{A}(r)\), \(j \geq 2\), recursively, by using the following lemma.
LEMMA 2.4 (Hukuhara). For preassigned constants $\theta_1$, $\theta_2$ and $r > 0$, we consider a differential equation of the form:

$$t^{\sigma + 1} dw/dt = c(t)w + f(t),$$

where $\sigma$ is a positive integer, $c(t)$ is a polynomial in $t$ of degree $\sigma - 1$ with $c(0) \neq 0$ and $f \in \mathcal{A}((\theta_1, \theta_2; r))$. Suppose that $S(\theta_1, \theta_2; r)$ contains one and only one singular line of $-\int_0^t c(t) t^{-\sigma - 1} dt$, then the equation has a unique solution $w \in \mathcal{A}((\theta_1, \theta_2; r))$.

It should be remarked that the domain $S(\theta_1, \theta_2; r')$ contains only one singular direction of the system (2.15).

We next prove the convergence of (2.14).

Let $Z_i(t)$ be the general solution to

$$t^{\sigma + 1} dZ_i/dt = (\mu_i(t) + \alpha_i t^\sigma) Z_i, \quad i = 1, 2.$$

Then it follows that the formal series $w' = w'(t, Z_i(t))$ formally satisfies

$$t^{\sigma + 1} dw'/dt = (A(t) - \sigma t^\sigma) w' + f(t, '(\delta \alpha_i Z_i, \delta \alpha Z_2) + t^\sigma w').$$

Therefore, by utilizing the following lemma, we see that $w'(t, \xi_i)$ converges absolutely and uniformly in $D(r', \delta')$, provided that $r'$ and $\delta'$ are taken in a suitable way.

LEMMA 2.5 (Iwano). Let

$$W(t, Z) = \sum_{i=1}^n W_i(t) Z^i$$

be a formal power series in $Z$, where $W_i$ belong to $\mathcal{A}((\theta_1, \theta_2; r))$. Let $c(t)$ be a polynomial in $t$ of degree $\sigma - 1$ with $c(0) \neq 0$. Suppose that $S(\theta_1, \theta_2; r)$ contains one and only one singular line of $C(t) = -\int_0^t c(t) t^{-\sigma - 1} dt$ and that $W(t, Z(t))$ formally satisfies

$$t^{\sigma + 1} dW(t, Z(t))/dt = f(t, Z(t), W(t, Z(t))),$$

where $Z(t)$ is the general solution to
\[ t^{\sigma_1} dZ/dt = c(t)Z \]

and \( f(t, Z, W) \) is holomorphic and bounded for

\[ (t, Z) \in D(\theta_1, \theta_2; r, \delta), \quad |W| < \rho. \]

Then \( W(t, Z) \) converges absolutely and uniformly for

\[ (t, Z) \in D(\theta_1, \theta_2; r', \delta'), \]

provided that \( r', \delta' > 0 \) are sufficiently small.

It is not difficult to show that, for \( w' \), determined above, \( \Sigma^r \) is reduced to a system of the form \( \Sigma^r \) by \( T_3 \). Q. E. D.

3. Construction of the formal transformation \( T_3 \).

We rewrite the system \( \Sigma^r \) as follows:

\[ \Sigma^r : \quad t^{\sigma_1} dx/dt = [A(t) + t^\sigma 1(f(t, x))]x. \]

We can verify that, by a transformation of coordinates \((t, x) \mapsto (t, y)\) of the form

\[ T_3 : \quad x = 1(y)[(1, 1) + \rho(t, y)], \]

\( \Sigma^r \) is changed to a system of the form

\[ t^{\sigma_1} dy/dt = A(t)y + t^\sigma F(t, y), \]

where

\[
F(t, y) = [1 + 1(\rho) + 1(y)\partial p/\partial y]^{-1} \\
\times 1(y)[t^{\sigma_1} \partial p/\partial t + \partial p/\partial y \cdot A(t)y - t^\sigma 1(f(t, 1(y)((1, 1) + \rho)))((1, 1) + \rho)].
\]

Hence, in order that \( \Sigma^r \) is linearized by \( T_3 \), it is necessary and sufficient that \( \rho(t, y) \) satisfies the following system of partial differential equations:

(3.1) \[ t^{\sigma_1} \partial p/\partial t + \partial p/\partial y \cdot A(t)y = t^\sigma 1(f(t, 1(y)((1, 1) + \rho)))((1, 1) + \rho). \]
We want to obtain a formal solution to (3.1) of the form

\[(3.2)\]

\[p(t, y) = \sum_{\alpha \in \mathbb{A}} p_k(t)y^k.\]

Substituting (3.2) to (3.1) and equating the coefficients of like powers in \(y\), we have

\[(3.3)\]

\[t^{a+1}b_{i\alpha}dt + [\lambda(\mu, k) + t^{a}a, k]p_k\]

\[= \sum_{\delta=1+2+\cdots, \delta=\beta_{a+1}+\cdots+\beta, \gamma \geq 0} f_{\delta} b_{i\alpha}^1 \cdots b_{i\alpha}^\gamma p_{\delta}, \]

where \(p_k = ^{(1)}(p_{k1}, p_{k2}), |k| \geq 0, (p_{0,0}) = (1, 1), \alpha', \beta' \in \mathbb{Z}^2, 1 \leq i \leq \gamma + 1, 1 \leq j \leq \nu + 1\) and for \(f(t, x) = \sum_{\alpha \in \mathbb{A}} f_{\alpha}(t)x^\alpha, f_i = ^{(0)}(f_{i1}, f_{i2}).\)

We can uniquely determine \(p_{k\alpha} \in \mathcal{A}(r')\) recursively by the Lemma 2.4. Thus we have constructed a formal transformation \(T_3\) given by \(x = 1(y)[(1, 1) + \sum_{\alpha \in \mathbb{A}} p_k(t)y^k].\)

4. Sketch of the proof of Theorem.

4.1. **Truncated system to formal power series** \(\sum p_k(t)y^k\). Let \(\sum p_k(t)y^k\) be the formal power series constructed in 3 and set, for each \(N \geq 1,\)

\[(4.1)\]

\[p_{(N)}(t, y) = (1, 1) + \sum_{\alpha \in \mathbb{A}} p_k(t)y^k.\]

Then, in order that a change of coordinates \((t, x) \rightarrow (t, y)\) of the form

\[(4.2)\]

\[x = 1(y)(p_{(N)}(t, y) + \varphi(t, y)),\]

transforms the system \(\Sigma^r\) into the system \(\Sigma',\) it is necessary and sufficient that the 2-vector function \(\varphi\) satisfies the following system of partial differential equations:

\[(4.3)\]

\[t(d/dt)\varphi = f_{(N)}(t, y, \varphi),\]

where \(t(d/dt)\) is the differential operator defined by

\[t(d/dt)\varphi = t\partial\varphi/\partial t + \partial\varphi/\partial y \cdot t^{-\alpha}A(t)y,\]

and
(4. 4) \[ f_{N}(t, y, \varphi) = -t(d/dt)\rho_{N} + 1(f(t, 1(y)(\rho_{N} + \varphi))(\rho_{N} + \varphi). \]

We can verify

**Lemma 4.1.** There exist positive constants \( C_{N} \) and \( M \) such that

(4. 5) \[ |f_{N}(t, y, 0)| \leq C_{N}|y|^{n+1}, \]

(4. 6) \[ |f_{N}(t, y, \varphi) - f_{N}(t, y, \psi)| \leq M|y||\varphi - \psi| \]

hold for \((t, y) \in D(r_{N}, \delta_{N}), |\varphi|, |\psi| < \rho_{N}, \) provided \( r_{N}, \delta_{N}, \rho_{N} > 0 \) are sufficiently small.

We note that the inequality

(4. 7) \[ |f_{N}(t, y, \varphi)| \leq C_{N}|y|^{n+1} + M|y||\varphi|, \]

for \((t, y) \in D(r_{N}, \delta_{N}), |\varphi| < \rho_{N}, \) is derived from (4. 5) and (4. 6).

4. 2. **Fundamental lemma.** We now state the lemma which is important in proving the convergence of \( \sum \rho_{n}(t)y^{n} \).

**Fundamental Lemma.** Suppose that the assumptions given in Theorem hold. Then, for each \( N \geq 1, \) the system (4. 3) \( \rho_{n} \) possesses one and only one solution \( \varphi = \varphi_{N}(t, y) \in \mathcal{D}(r_{N}, \delta_{N}) \) satisfying

(4. 8) \[ \varphi_{N}(t, y) = O(|y|^{n+1}) \]

for \((t, y) \in D(r_{N}, \delta_{N}), \) provided that \( r_{N}, \delta_{N} > 0 \) are sufficiently small.

By this lemma, we can prove the convergence of \( \sum \rho_{n}(t)y^{n} \). Indeed, for each \( N \geq 1, \) set

\[ \varphi^{N}(t, y) = \rho_{N}(t, y) + \varphi_{N}(t, y), \]

where \( \varphi_{N}(t, y) \) is the unique solution to (4. 3) \( \rho_{n} \) with (4. 8). Then a change of coordinates \((t, x) \to (t, x) \)

takes the system \( \Sigma_{T_{1}} \) into the system \( \Sigma' \). We see that \( \varphi^{N} \) does not depend on \( N. \) We can assume
that both the constants \( r_N \) and \( \delta_N \) are monotone decreasing in \( N \). For any \( N' \) and \( N \) with \( N' > N \),

\[(p_{(N')} - p_{(N)}) + \varphi_{(N')}\]

is a solution to the system \((4.3)_N \) in \( S(r_N, \delta_N) \), of order \( O(|y|^N)\). From the uniqueness assertion, in Fundamental lemma, it follows that

\[\varphi_{(N)} = (p_{(N')} - p_{(N)}) + \varphi_{(N')}\]

which implies that \( \varphi^* \) is independent of \( N \). Now put

\[\varphi = \varphi^*, \quad r' = r, \quad \delta' = \delta_i.\]

Then \( \varphi \) is in \( S(r', \delta') \) satisfying

\[\varphi(t, y) - \sum_{\lambda \in \Lambda} p_\lambda(t)y^\lambda = O(|y|^{r_{++}}),\]

for each \( N \geq 1 \), which yields the convergence of \( \sum p_\lambda(t)y^\lambda \).

Therefore we have only to prove the fundamental lemma. It should be remarked that \( \varphi \) is the solution to \((4.3)_N \) if and only if \( \varphi = \varphi(t, y(t)) \), satisfies the following system:

\[(4.9)_N \quad td\varphi/dt = f_{(N)}(t, y(t), \varphi),\]

where \( d/dt \) is the usual ordinary differential operator and \( y(t) \) is the general solution to \( \Sigma' \). Further \((4.9)_N \) is equivalent to the system of integral equations:

\[(4.10) \quad \varphi(t_0, y^0) = \int_{t_0}^{t_1} f_{(N)}(t, y(t), \varphi(t, y(t))) t^{-1} dt,\]

where \( y(t) \) is a solution to \( \Sigma' \) with \( y^0 = y(t_0) \) and \( \gamma \) is a path of integration joining \( t = 0 \) to \( t = t_0 \) which will be specified later.

5. Determination of sectorial domain \( \mathcal{S} \) and path of integration \( \gamma \).

To prove the fundamental lemma, we have to define a sectorial domain \( \mathcal{S} \) in \( t \)-plane and a path of integration \( \gamma \) joining \( t = 0 \) to \( t = t_0 \) in \( \mathcal{S} \). For this purpose, we begin with determining positive constants \( x \) and \( \nu_i \), \( i = 1, 2 \). Recalling the assumptions \( \Re(\mu_i \alpha_k - \mu_k \alpha_i) > 0, \mu_i > 0 \) and \( \mu_k < 0 \), we have inequalities either
\( \text{Re} (\alpha_2) > 0 \) or \( \text{Re} (\alpha_1) > 0 \).

(i) In the case \( \text{Re} (\alpha_2) > 0 \), we choose \( \lambda \) in such a way that

\[
-\mu_2 \sigma/ \text{Re} (\alpha_2) < \lambda,
-\mu_2 \sigma/ \text{Re} (\alpha_2) < \lambda < -\mu_2 \sigma/ \text{Re} (\alpha_1)
\]

if \( \text{Re}(\alpha_1) \geq 0 \),

and we put

\( \nu_i = \mu_1 \sigma + \lambda \text{Re} (\alpha_i), \ i = 1, 2. \)

(ii) In the case \( \text{Re} (\alpha_2) < 0 \), then \( \text{Re} (\alpha_1) > 0 \), and we take \( \lambda \) such that

\[
\mu_1 \sigma/ \text{Re} (\alpha_1) < \lambda < -\mu_2 \sigma/ \text{Re} (\alpha_2)
\]

and we set

\( \nu_i = -\mu_2 \sigma + \lambda \text{Re} (\alpha_i), \ i = 1, 2. \)

In either case \( \nu_i, \ i = 1, 2 \), are positive.

We next determine \( \Omega \) with \( \pi/4 < \Omega < \pi/2 \) by

\[
\tan \Omega = [\sigma \max \{\mu_1, -\mu_2\} + (3x + 4) \max |\text{Re} (\alpha_1)|, |\text{Re} (\alpha_2)| + \min \{\nu_1, \nu_2\}] / \min \{\nu_1, \nu_2\}
\]

Note that

\( \tan \Omega > 1. \)

We now pass to define a sectorial domain \( \mathcal{D} \). For this purpose, for a set \( E \) in \( t \)-plane, we denote by \( \Lambda(E)^{-1} \) the set in the \( s \)-plane defined by \( \{s \in \mathbb{C}; s = 1/\Lambda(t), \ t \in E \} \). By assumption, \( \Lambda(S)\)^{-1} contains a half line \( \arg s = \pi/2 \) or \( \arg s = -\pi/2 \). In the following, we only consider the case where \( \Lambda(S)\)^{-1} contains a half line \( \arg s = \pi/2 \). The other case be treated by the same way. In our case, we can choose a small number \( \epsilon > 0 \) so that

\[
\Lambda(S)\)^{-1} \subseteq \{s \in \mathbb{C}; |s| < (1 + \epsilon) \sigma, |s - \pi/2| < \pi - \epsilon\}
\]

for every small \( r > 0. \)
Let $\mathcal{L}$ and $\mathcal{T}$ be half lines starting from the origin defined by $\{t \; ; \; \arg t = 0\}$ and $\{t \; ; \; \arg t = \pi\}$, respectively, and let

$$c = \Lambda(\mathcal{L})^{-1}, \quad \bar{c} = \Lambda(\mathcal{T})^{-1}$$

We denote by $\mathcal{S}(\theta, \bar{\theta} ; r)$ the domain in $s$-plane bounded by $c, \bar{c}$ and the curve

$$|s| = \begin{cases} \alpha r^{\theta} & \text{if } |\theta - \pi/2| < \pi/2 - \Omega \\ \alpha r^{\bar{\theta}} |\cos \theta/ \cos \Omega| & \text{if } \pi/2 - \Omega \leq |\theta - \pi/2| \leq \pi - \epsilon, \end{cases}$$

where $\theta = \arg s$. We define a sectorial domain $\mathcal{S}(\theta, \bar{\theta} ; r)$ by

$$\Lambda(\mathcal{S}(\theta, \bar{\theta} ; r))^{-1} = \mathcal{S}(\theta, \bar{\theta} ; r).$$

We now divide $\mathcal{S}$ into three parts:

$$\mathcal{S}_1 = \mathcal{S} \cap \{s \; ; \; |\arg s - \pi/2| \leq \pi/2 - \Omega\}$$
$$\mathcal{S}_2 = \mathcal{S} \cap \{s \; ; \; -\pi/2 < \arg s < \Omega\}$$
$$\mathcal{S}_3 = \mathcal{S} \cap \{s \; ; \; \pi - \Omega < \arg s < 3\pi/2\}$$

and define the domains $\mathcal{S}_i, i = 1, 2, 3$ in $t$-plane by

$$\Lambda(\mathcal{S}_i)^{-1} = \mathcal{S}_i, \; i = 1, 2, 3.$$ 

Then it is evident that

$$\mathcal{S} = \bigcup_{i=1}^3 \mathcal{S}_i.$$ 

For a given $s_0 \in \mathcal{S}$, we define, in $\mathcal{S}$, a path $\Gamma$ joining $s = 0$ to $s = s_0$ which generally consists of two paths $\Gamma'$ and $\Gamma''$:

1. The case of $\Re(\alpha_1) > 0$.
2. When $s_0$ is in $\mathcal{S}_i$, $\Gamma$ consists of only $\Gamma'$. The coordinate $s$, on $\Gamma'$, is parametrized by $r$ as follows:

$$1/s(r) = r + a - \sqrt{-1}be^{\alpha r}, \; r \in [0, +\infty),$$
where

\[(5.10) \quad 1/s_0 = a - \sqrt{-1} b, \ s_0 = s(0).\]

(ii) When \(s_0\) is in \(S_2\) (or \(S_3\)), \(I'\) consists of two paths \(I'''\) and \(I''\). The coordinate \(s\) on \(I''\), is parametrized by \(\Theta\) as follows:

\[(5.11) \quad s(\Theta) = (|s_0|/ \cos \Theta_0) \cos \Theta e^{\xi \Theta},\]

for \(\Theta \in [\Theta_0, \Omega]\) (or \(\Theta \in [\pi - \Omega, \Theta_0]\)) respectively, where

\[(5.12) \quad \Theta_0 = \text{arg } s_0.\]

Then \(s(\Omega)\) (or \(s(\pi - \Omega)\)) is located in \(S_1\). The coordinate \(s\) on \(I''\) is defined by the same way as (5.9), where starting point is \(s(\Omega)\) (or \(s(\pi - \Omega)\)).

(i) The case of \(\text{Re}(\alpha_0) < 0\).

(i) When \(s_0\) is in \(S_1\), \(I'\) consists of only \(I''\). The coordinate \(s\), on \(I''\), is parametrized by \(r\) as follows:

\[(5.13) \quad 1/s(r) = -r + a - \sqrt{-1} b e^{\xi r}, \ r \in [0, +\infty),\]

where

\[(5.14) \quad 1/s_0 = a - \sqrt{-1} b, \ s_0 = s(0),\]

(ii) When \(s_0\) is in \(S_2\) (or \(S_3\)), we define \(I'''\) by the same way as (5.11) and \(I''\) as (5.13).

For a given \(t_0 \in \mathcal{S}\), we define a path \(\gamma\) in \(\mathcal{S}\) by

\[(5.15) \quad \Lambda(\gamma)^{-1} = I',\]

where \(s_0 = \Lambda(t_0)^{-1} \in S\). Then we can verify

**Lemma 5.1.** We can choose a sufficiently small positive constant \(r\) in such a way that

(i) When \(t_0 \in \mathcal{S}_1\), then \(\gamma \subset \mathcal{S}_1\);

(ii) When \(t_0 \in \mathcal{S}\), then \(\gamma \subset \mathcal{S}\).
6. The stability of the domain $\mathcal{D}$ which is a deformation of $D$.

We define a domain $\mathcal{D} = \mathcal{D}(r, \delta) = \mathcal{D}(\theta, \overline{\theta}; r, \delta)$ in $(t, y)$-space by

$$\mathcal{D} = \{(t, y) \in \mathbb{C} \times \mathbb{C}^2; t \in \mathcal{D}(r), |y_i| < \delta d_i(t) e_i(t)\},$$

where

$$d_i(t) = \begin{cases} t^\sigma \Lambda(t)^{\text{Im}(a_i) \sigma} & \text{on } \mathcal{D}_1 \\ \left[\left(\cos \theta / \cos \Omega\right) t^\sigma \Lambda(t)^{\text{Im}(a_i) \sigma} \right] & \text{on } \mathcal{D}_2 \cup \mathcal{D}_3 \end{cases}$$

and

$$e_i(t) = \exp[-\text{Im}(a_i) \cdot \theta]$$

with $\theta = \arg t$.

We note that the domain $\mathcal{D}(r, \delta)$ is equivalent to domain $D(r, \delta)$, namely, for given $r$ and $\delta > 0$, we can choose constants $r'$ and $\delta' > 0$ so that $\mathcal{D}(r', \delta') \subset D(r, \delta)$, and conversely, for given $r'$ and $\delta' > 0$, we can take constants $r$ and $\delta > 0$ so that $D(r, \delta) \subset \mathcal{D}(r', \delta')$.

We now give some properties for the $\mathcal{D}$.

**Lemma 6.1 (Stability of $\mathcal{D}$).** Let $(t_0, y_0) \in \mathcal{D}(r, \delta)$ and let $y(t)$ be the solution to $\Sigma'$ with $y(t_0) = y_0$, then $(t, y(t)) \in \mathcal{D}(r, \delta)$ for every $t \in \gamma$, provided that $r$ and $\delta > 0$ are sufficiently small.

In order to prove lemma 6.1, we use the following lemma.

**Lemma 6.2.** Let $y(t)$ be the function given in lemma 6.1 and let

$$u_i(t) = C_i e^{-\mu(t)} \Lambda(t)^{-\text{Re}(a_i) \sigma}, i = 1, 2.$$

Then, for $t \in \gamma$ with $t_0 \in \mathcal{D}_1$ the pull back $u_i(t)$ of $u_i$ by (5.9) satisfy

$$d \log |u_i(t)|/dt < -3\nu_i/4 \sigma.$$

It should be remarked that, in particular, $|u_i(t)|$ are monotone decreasing functions with respect to $r$.

**Proof of Lemma 6.1.** By the definition of $u_i(t)$, we have
(6. 6) 
\[ y_i(t) = u_i(t)A(t)^{\sigma_0}e_{\sigma_0} \]
\[ = u_i(t)(t^\sigma A(t)^{\sigma e_{\sigma_0}})^{t^\sigma e_{\sigma_0}}. \]

Let us suppose that \((\lambda_0, \gamma_0) = \lambda_0, \gamma(\lambda_0)\) \(\in S_\lambda.\)

(1) In case that \(\lambda_0 \in S_1;\) Because of (6. 6), we have
\[ |y_i(t)| = |u_i(t)||t^\sigma A(t)^{\sigma e_{\sigma_0}}|e_{\sigma_0}(t) \]
which yield the inequalities
\[ |y_i(t)| = |y_i(\lambda_0)| |[|u_i(\lambda_0)||t^\sigma A(t)^{\sigma e_{\sigma_0}}|e_{\sigma_0}(t_0)|] \times |u_i(t)||t^\sigma A(t)^{\sigma e_{\sigma_0}}|e_{\sigma_0}(t) \]
\[ = |y_i(\lambda_0)||u_i(\lambda_0)||d(t)e_{\sigma_0}(t)|d(t)e_{\sigma_0}(t_0) \]
\[ < \delta d(t)e_{\sigma_0}(t), \text{ } i = 1, 2, \]
\[ \delta \text{ being a small constant.} \]

(2) In case that \(\lambda_0 \in S_2 \text{ (or } S_3);\) Noting that we have
\[ \text{Re } A(t) = \text{Re } A(\lambda_0) \]
on \(\gamma'\) and
\[ C_i = y_i(\lambda_0)e^{\mu_\lambda(\lambda_0)}\lambda_0^{-\sigma_0}, \text{ } i = 1, 2, \]
where \(C_i, \text{ } i = 1, 2, \) are arbitrary constants in (1. 5), we have the inequalities:
\[ |y_i(t)| = |y_i(\lambda_0)e^{\mu_\lambda(\lambda_0)}\lambda_0^{-\sigma_0}e^{\mu_\lambda(t)}|t^\sigma| \]
\[ = |y_i(\lambda_0)||t^\sigma|\text{Re}^{\sigma_0}(e_{\sigma_0}(t)/e_{\sigma_0}(\lambda_0)) \]
\[ = |y_i(\lambda_0)||(|\cos \theta|/\cos \Omega)|t^\sigma A(t)|\text{Re}^{\sigma_0}(e_{\sigma_0}(t)) \]
\[ /(|\cos \theta|/\cos \Omega)|t^\sigma A(\lambda_0)|\text{Re}^{\sigma_0}(e_{\sigma_0}(\lambda_0)) \]
\[ < \delta d(t)e_{\sigma_0}(t), \text{ } i = 1, 2, \]
\[ \delta \text{ being a small constant.} \]

Thus we have proved Lemma 6. 1. Q. E. D.

In order to prove Lemma 6. 2, we make use of the following lemma.

**Lemma 6. 3.** We have
\[(6.7) \quad |a/b| \leq 1, \tan \Omega < 1.\]

\[(6.8) \quad b \geq (\tan \Omega \sin \epsilon)/r > \tan \Omega,\]

\[(6.9) \quad xb \geq \tan \Omega\]

for any \(t_0 \in \mathcal{S}_1.\)

The proof of this Lemma is omitted (see Proposition 2 in [12]).

We now proceed to prove Lemma 6.2.

**Proof of Lemma 6.2.** We consider only the case that \(\text{Re}(\alpha_i) > 0\), because the other case is shown in the similar way.

We have

\[(6.10) \quad d \log u_i(t)/dt = -(u_i + \sigma^{-1} \text{Re}(\alpha_i) \Lambda(t^{-1}) d\Lambda(t)/dt)\]

from (6.4) and

\[(6.11) \quad dt/dt = (d\Lambda(t)/dt)^{-1}(1 - \sqrt{-1} x e^{\pi i})\]

from (5.9). Consequently, we have

\[
d \log |u_i(\tau)|/d\tau = d \text{Re} [\log u_i(\tau)]/d\tau
= \text{Re} [d \log u_i(\tau)/dt \cdot dt/d\tau]
= -\text{Re} [(u_i + \sigma^{-1} \text{Re}(\alpha_i) \Lambda(\tau^{-1})) (1 - \sqrt{-1} x e^{\pi i})], \quad i = 1, 2.\]

So, we prove the following inequalities,

\[
\text{Re} [(u_i + \sigma^{-1} \text{Re}(\alpha_i) \Lambda(\tau^{-1})) (1 - \sqrt{-1} x e^{\pi i})] > 3 \nu_i / 4 \sigma
\]

or equivalently,

\[(6.12) \quad \nu_i \delta^2 e^{2\pi r} - ( - \mu_i \sigma + 3x \text{Re} \alpha_i(\tau + a)^2 + 4 \text{Re} \alpha_i(\tau + a) > 0.\]

Define a function in \(r, \mathcal{A}_i(\tau)\), by the left hand side of (6.12), then we can see that, for \(r \in [0,\]

\( + \infty \),

\[ \mathcal{A}_i(0) > 0, \mathcal{A}_i(0) > 0, \mathcal{A}_i''(\tau) > 0, \ i = 1, 2. \]

In fact, we have the first inequalities in the following way:

\[ \mathcal{A}_i(0) = \nu_i b^2 - (-\mu_\sigma + 3x \Re a_i)a^2 + 4 \Re (a_i)a \]
\[ \geq \nu_i b^2[1 - ((-1)^{-1}\mu_\sigma + 3x \Re a_i)/\nu_i a^2/|a|^2 - 4|\Re a_i||a|/\nu_i b^2] \]
\[ > \nu_i b^2[1 - ((-1)^{-1}\mu_\sigma + 3x \Re a_i)/\nu_i a^2/|a|^2 - 4|\Re a_i||a|/\nu_i b^2] \]
\[ > (\nu_i b^2/\tan \Omega)[\tan \Omega - ((-1)^{-1}\mu_\sigma + (3x + 4)|\Re a_i|)/\nu_i] \]
\[ > 0. \]

Next since

\[ \mathcal{A}_i'(\tau) = 2\nu_i x b^2 \exp \tau - 2(-\mu_\sigma + 3x \Re a_i)(\tau + a) + 4 \Re a_i, \]

we have the second inequalities

\[ \mathcal{A}_i'(0) = 2\nu_i x b^2 - 2(-\mu_\sigma + 3x \Re a_i)a + 4 \Re a_i \]
\[ > 2\nu_i b^2[1 - ((-1)^{-1}\mu_\sigma + 3x \Re a_i)/\nu_i a^2/|a|^2 - 2|\Re a_i||a|/\nu_i b^2] \]
\[ > 2\nu_i b^2[1 - ((-1)^{-1}\mu_\sigma + 3x \Re a_i)/\nu_i a^2/|a|^2 - 2|\Re a_i||a|/\nu_i b^2] \]
\[ = 2\nu_i b^2/\tan \Omega[\tan \Omega - ((-1)^{-1}\mu_\sigma + (3x + 2)|\Re a_i|)/\nu_i] \]
\[ > 0. \]

We have, further, the last inequalities

\[ \mathcal{A}_i''(\tau) = 4 \nu_i x^2 b^2 \exp 2\tau - 2(-\mu_\sigma + 3x \Re a_i) \]
\[ \geq 4 \nu_i x^2 b^2 - 2((-1)^{-1}\mu_\sigma + 3x \Re a_i) \]
\[ = 2\nu_i x^2 b^2[2 - ((-1)^{-1}\mu_\sigma + 3x \Re a_i)/\nu_i a^2/|a|^2 b^2] \]
\[ \geq 2\nu_i x^2 b^2/\tan \Omega[2\tan \Omega - ((-1)^{-1}\mu_\sigma + 3x \Re a_i)/\nu_i] \]
\[ > 2\nu_i/\tan \Omega[2((-1)^{-1}\mu_\sigma + (3x + 4)|\Re a_i|)/\nu_i - ((-1)^{-1}\mu_\sigma + 3x \Re a_i)/\nu_i] \]
\[ = 2/\tan \Omega[((-1)^{-1}\mu_\sigma + (3x + 8)|\Re a_i|)/\nu_i] \]
\[ > 0. \]

Thus we have obtained (6.13), which yield (6.5). Q. E. D.
7. Proof of Fundamental lemma.

7. 1. In this part, we show the following lemma.

LEMMA 7. 1. Let \( y(t) \) be the general solution to the reduced system \( \Sigma' \). Then we can find a positive constant \( f_n \) depending on \( N \) so that we have the inequality:

\[
(7.1) \quad \int_{t_0}^{t_f} |y(t)|^{n+1} |t|^{-1} |dt| \leq f_n |y(t_0)|^{n+1}
\]

for any \( t_0 \in \mathcal{S}(\theta, \theta; r) \), \( r > 0 \) being sufficiently small.

PROOF. Recalling (6.6), we have

\[
|y(t)| = |u_i(t)||(|t^\sigma \Lambda(t))^{\text{real}} e_i(t)|, \quad i = 1, 2.
\]

Notice that \( |(|t^\sigma \Lambda(t))^{\text{real}} e_i(t)| \) is bounded from below and above by constants \( B_i, B_3 \), respectively. Hence we have

\[
\int_{t_0}^{t_f} |y(t)|^{n+1} |t|^{-1} |dt| \leq B_i^{n+1} \int_{t_0}^{t_f} |u_i(t)|^{n+1} |t|^{-1} |dt|
\]

or \( y' \). Moreover we have

\[
|\mathcal{L}(t)|/|t^\sigma \Lambda(t)| > B
\]

for a suitable positive constant \( B \), and hence, we have

\[
(7.2) \quad |dt|/|t| < B^{-1} |ds|/|s|.
\]

Further we see that the pull back \( u_i(s) \) of \( u_i \) by (5.9) satisfy

\[
(7.3) \quad d \log |u_i(s)| > 3 \nu_i/(4 \sqrt{2} \kappa_0) |s|^{-1} |ds|. \]

Indeed, by (6.5), we have

\[
d \log |u_i(s)|/|ds| = d \log |u_i(r)|/|dr|/|dt|/|ds| = d \log |u_i(r)|/|dr|(-|dt|/|ds|)
\]
\[ > 3 \nu_1 / (4 \sigma) \cdot |dt/ds|, \]

and we have

(7. 4) \[ |s| |dt/ds| > (\sqrt{2} \nu)^{-1}, \]

because

the left hand side of (7. 4)

\[ = \left[ (\tau + \alpha)^2 + b^2 e^{2\varepsilon r} \right]^{1/2} \left( 1 + \lambda^2 b^2 e^{2\varepsilon r} \right)^{-1/2} \]

\[ > \left[ b^2 e^{2\varepsilon r} \right]^{1/2} \left( 2 \lambda^2 b^2 e^{2\varepsilon r} \right)^{-1/2} = (\sqrt{2} \nu)^{-1}. \]

Hence (7. 3) holds.

Now, by (7. 2) and (7. 3), we have

\[
\int_{\Gamma} |u_i(t)|^{n+1} |t|^{-1} |dt| \\
< B^{-1} \int_{\Gamma} |u_i(s)|^{n+1} |s|^{-1} |ds| \\
< 4 \sqrt{2} \sigma (3 \nu_i (N+1))^{-1} \int_{\Gamma} |u_i(s)|^{n+1} d \log |u_i(s)|^{n+1} \\
= 4 \sqrt{2} \sigma (3 \nu_i (N+1))^{-1} \int_{\Gamma} |u_i(s)|^{n+1} \\
< 4 \sqrt{2} \sigma (3 \nu_i (N+1))^{-1} |u_i(s_0)|^{n+1},
\]

and hence

\[
\int_{\Gamma} |y_i(t)|^{n+1} |t|^{-1} |dt| \\
\leq 4 \sqrt{2} \sigma (3 \nu_i B (N+1))^{-1} B_i^{n+1} |u_i(s_0)|^{n+1} \\
< 4 \sqrt{2} \sigma (3 \nu_i B (N+1))^{-1} (B_2/B_1)^{n+1} |y_i(s_0)|^{n+1}.
\]

We obtain, therefore, the inequality (7. 1).

Next we have, on \( \Gamma'' \),

\[ s(\Theta) = |s_0| / \cos \Theta e \cdot \cos \Theta e^{\pi \Theta}. \]

Noting \(|ds|\) is a decreasing of increasing function in \( \Theta = \arg s \) corresponding to \( s_0 \in \mathcal{F}_2 \) or \( \mathcal{F}_3 \) respectively, we have

\[ |ds| = \frac{1}{2} |ds/d\Theta| d\Theta \]
\[ = \int_{s_0}^{s_1} \cos \Theta \, d\Theta \]

and

\[ |s^{-1}|d|s| = \int_{\Theta}^{\Theta} \cos \Theta^{-1}d\Theta. \]

We have, hence,

\[
\int_{s_0}^{s_1} |s^{-1}|d|s| < \int_{\Theta}^{\Theta} \cos \Theta^{-1}d\Theta \\
= \begin{cases} 
\int_{\Theta}^{\Theta} \cos \Theta^{-1}d\Theta < \pi/\min \{ \sin \epsilon, \cos \Omega \} \\
\int_{\Theta}^{\Theta} \cos \Theta^{-1}d\Theta < \pi/\min \{ \sin \epsilon, \cos \Omega \}
\end{cases}
\]

for \( \Theta \in [\Theta_0, \Theta] \) or \( [\pi - \Omega, \Theta_0] \) respectively. We have, then,

\[
\int_{s_0}^{s_1} |s^{-1}|d|s| < B_3
\]

for some positive constant \( B_3 \). Noting that, on \( I'' ' \),

\[
|y_i(s)| = |y_i(s_0)|d_i(s)e_i(s)/(d_i(s_0)e_i(s_0)) < B_i|y_i(s_0)|, \quad i = 1, 2,
\]

for a suitable positive constant \( B_i \) where \( d_i(s) \) (or \( e_i(s) \)) is the pull back of \( d_i \) (or \( e_i \)) by \( s \), we have

\[
\int_{s_0}^{s_1} |y_i(s)|s^{-1}|d|s| < \int_{s_0}^{s_1} B_i|y_i(s_0)|s^{-1}|d|s| \\
\leq B_i s^{-1} B_3|y_i(s_0)|s^{-1}. 
\]

Therefore if we choose \( J_\nu \) such that

\[
J_\nu > 4\sqrt{2}\pi\sigma(3\nu_0B(N+1))^{-1}(B_2/B_1)^{\nu+1} + (B_3/B)B_i^{\nu+1},
\]

then we have the inequality (7. 1). Q. E. D.

7. 2. In this part, we give the proof of fundamental lemma.

First of all, we define the family of functions \( \mathcal{F} \) by
\[ \mathcal{S} = \{ \varphi : \mathcal{D} \to \mathbb{C}^2 ; \varphi \in \mathcal{B}(\mathcal{D}), |\varphi| \leq K_N |y|^{N^*} \} \]

where \( \mathcal{D} = \mathcal{D}(r_N, \delta_N) \) and \( K_N \) is a positive constant depending on \( N \) which will be specified later. It is evident that \( \mathcal{S} \) is non-empty and convex, and further closed and normal with respect to the topology of uniform convergence on every compact subset of \( \mathcal{D} \). Next we define an operator \( \mathcal{S} \) acting on \( \mathcal{S} \).

Let \( (b, y^0) \) be a point in \( \mathcal{D} \) and let \( y(t) \) be the solution to \( \Sigma \) satisfying the initial condition \( y(b) = y^0 \). Moreover we set

\[ (7.5) \quad \Phi(b, y^0) = \int_{f(b)}^{b} \varphi(t, y(t)) |t|^{-1} dt \]

and define an operator \( \mathcal{S} \) from \( \mathcal{S} \) into \( \mathcal{S} \) by

\[ (7.6) \quad \varphi(t, y) \to \Phi(t, y). \]

This is well-defined. Indeed, first, we can take \( r_N \) and \( \delta_N \) so small that \((t, y, \varphi) \in D(r_N, \delta_N) \times \{ w \in \mathbb{C}^2 ; |w| < \rho_N \} \), where \( r_N, \delta_N \) and \( \rho_N \) are specified in Lemma 4.1.

Next let \( \varphi \in \mathcal{S} \), then, by (4.7), we have

\[
|\Phi(b, y^0)| \leq \int_{f(b)}^{b} [C_N |y|^{N^*} + M |\varphi| |t|^{-1}] dt \leq \int_{f(b)}^{b} [C_N + M |\varphi|] |y|^{N^*} |t|^{-1} dt \\
\leq \int_{f(b)}^{b} [C_N + M \delta_N K_N] |y|^{N^*} |t|^{-1} dt \\
\leq [C_N + M \delta_N K_N] J_N |y(b)|^{N^*}.
\]

and hence we obtain

\[ |\Phi(b, y^0)| \leq K_N |y(b)|^{N^*}, \]

provided that \( \delta_N \) is chosen so small that

\[ M \delta_N J_N < 1/2 \]

and that we set \( K_N = 2C_N J_N \). Further we can verify that \( \Phi \in \mathcal{B}(\mathcal{D}) \). So, \( \Phi(b, y^0) \in \mathcal{S} \), and hence
\( \mathcal{F}(\mathcal{F}) \subset \mathcal{F} \). Moreover, it is evident that \( \mathcal{F} \) is continuous on \( \mathcal{F} \) by (4. 6). Thus we see that \( \mathcal{F} \) admits a fixed point by virtue of Schauder-Tikhonov fixed point theorem, which shows the existence of \( \varphi_0 \) in Fundamental lemma.

We now prove the uniqueness of \( \varphi_0 \).

Let \( \varphi^1 \) and \( \varphi^2 \) be two solutions satisfying the lemma and put

\[
\phi = \varphi^1 - \varphi^2.
\]

Then \( \phi \in \mathcal{B}(\mathcal{D}) \) and \( |\phi| = O(|y|^{\nu+1}) \).

We define a positive constant \( H \) by

\[
H = \inf \{ H' \geq 0 : |\phi| \leq H'|y|^{\nu+1}, (t, y) \in \mathcal{D} \}.
\]

Then we have

\[
|\phi(t_0, y_0)| \leq \int y |\phi||t|^{-1}|dt|
\leq \int y |H| |y|^{\nu+1}|t|^{-1}|dt|
\leq M \delta_h H \|y(t_0)\|^\nu
< 2^{-1} H |y(t_0)|^{\nu+1}
\]

by (4. 6), and hence, by the definition of \( H \), we have \( H = 0 \). Thus we have shown the lemma.

Q. E. D.

Appendix.

To prove the existence of a holomorphic solution to (2. 7), we consider the following system:

\[
\Sigma_i : \quad \partial u/\partial \eta \cdot A\eta - Au = f(\eta),
\]

where \( A = A(0), f(\eta) = f(0, \eta) \). Suppose that

\[
f(\eta) = \sum |M_{max} f(\eta)|, \quad f \in \mathcal{C} \left( \{ \eta : |\eta| < r \} \right),
\]

and let \( u(\eta) = \sum u_n \eta^n \) be a formal solution to \( \Sigma_i \). Then, substituting it to \( \Sigma_i \), we obtain
\[ [(\mu, k) - A] u_k = f_k \]

for \(|k| \geq 2\). We have, therefore,

\[ u_k = [(\mu, k) - A]^{-1} f_k, \]

and hence, by (1. 4), we have

\[ |u_k| < C^{-1} |k|^n M r^{-|k|}, \]

where \(M = \sup \{|f|; |\eta| < r\}\). Consequently

\[ U(\eta) = \sum C^{-1} |k|^n M r^{-|k|} \eta^n \]

is a majorant series of \(\sum u_k \eta^n\). Now

\[ U(\eta) = \sum_{n \geq 2} \sum_{|k| = n} C^{-1} |k| |M| |\eta| r^{|k|} = C^{-1} M \sum_{n \geq 2} n^N (N+1) |\eta| r^n, \]

which proves the convergence of \(u(\eta)\) for \(|\eta| < r\). Q.E.D.

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