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ON A 2-PARAMETER FAMILY OF SOLUTIONS TO A NONLINEAR 2-SYSTEM WITH AN IRREGULAR SINGULARITY OF SIGNATURE (1, 1)

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NOTATION.

C: the field of complex numbers;

Z: the ring of integers;

$$\mathbf{Z}_{\geq 0} = \{k \in \mathbf{Z}\,;\, k \geq 0\}\,;$$

For $z \in \mathbf{C}$,

Re (z): the real part of z, Im (z): the imaginary part of z;

For a 2-vector $\alpha = (\alpha_1, \alpha_2) \in \mathbf{C}^2$,

$$1(\alpha) = 1(\alpha_1, \alpha_2) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \text{ and } 1 = 1(1, 1)\,;$$

For 2-vectors $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbf{C}^2$,

$$(\alpha, \beta) = \alpha_1\beta_1 + \alpha_2\beta_2\,;$$

For a 2-vector $x = (x_1, x_2) \in \mathbf{C}^2$ and a 2-vector $k = (k_1, k_2) \in \mathbf{Z}_{\geq 0}{}^2$,

$$|x| = \max_{i=1,2} |x_i|,\ |k| = k_1 + k_2,\ x^k = x_1{}^{k_1} x_2{}^{k_2}\,;$$

For a domain M in **C** or $\mathbf{C}^3$,

$\mathcal{O}(M)$: the set of holomorphic functions from M into $\mathbf{C}^2$,

$\mathscr{B}(M)$: the set of holmorphic and bounded functions from M into $\mathbf{C}^2$;

For constants θ_1, θ_2, r and δ,

$$S(\theta_1, \theta_2\,;\, r) = \{t \in \mathbf{C}\,;\, \theta_1 < \arg t < \theta_2,\ |t| < r\},$$

$$D(\theta_1, \theta_2; r, \delta) = \{(t, x)\in \mathbf{C}\times\mathbf{C}^2; t\in S(\theta_1, \theta_2; r), |x|<\delta\},$$

$\mathscr{A}(\theta_1, \theta_2; r)$: the set of holomorphic functions from $S(\theta_1, \theta_2; r)$into $\mathbf{C}^2$ admitting asymptotic expansions in powers of t as $t\to 0$, $t\in S(\theta_1,, \theta_2; r)$.

Further, for simplicity, we use

$$\mathscr{B}(\theta_1, \theta_2; r) = \mathscr{B}(S(\theta_1, \theta_2; r)), \mathscr{B}(\theta_1, \theta_2; r, \delta) = \mathscr{B}(D(\theta_1, \theta_2; r, \delta)).$$

1. Introduction.

In the study of nonlinear ordinary differential equations, it is important to obtain analytic expressions of general solutions near the fixed singularities. Concering the problem, we have a general theorem by J. Malmquist ([6], [7], [8]) under the so-called *Poincaré condition*. We notice that although the Poincaré condition is a generic one, many interesting equations such as Painlevé equations do not satisfy the condition.

As far as the author knows, the first work on the construction of general solutions in the case where the Poincaré condition is completely violated was done by M. Iwano ([4]). He constructed 2-parameter families of solutions to certain 2-systems of signature (1, 1) near the irregular singularity. After the work by M. Iwano, S. Yoshida established a general theory for 2-system in which the ratio of the characteristic exponents is equal to -1 in order to investigate the irregular singularity of Painlevé equations ([12], [13]).

In this paper, we shall study a 2-system with an irregular singularity where the ratio of the characteristic exponents is a negative irrational number.

The system which is studied is of the form:

$$\Sigma: \qquad t^{\sigma+1}dx/dt = (\lambda(t)\mathbf{1}(\mu)+t^{\sigma}\mathbf{1}(\alpha))x+f(t, x),$$

where

(i) t is a coordinate of $\mathbf{C}$;

(ii) x is a 2-vector in $\mathbf{C}^2$ with coordinates x_1, x_2;

(iii) σ is a positive integer;

(iv) μ is a 2-vector with entries μ_1, μ_2 with $\mu_1>0, \mu_2<0$ such that the ratio μ_1/μ_2 is an irrational number;

(v) α is a 2-vector with entries α_1, α_2 satisfying the inequality

$$(1.1) \qquad \mathrm{Re}\,(\mu_1\alpha_2-\mu_2\alpha_1)>0;$$

(vi) $\lambda(t)$ is a polynomial in t of degree $\sigma-1$ with $\lambda(0)=1$;

(vii) f is a 2-vector function belonging to $\mathscr{B}(\underline{\theta}, \overline{\theta}; r, \delta)$ with $f(t, x) = O(|x|^2)$ such that the coefficients $f_k(t)$ in the expansion of $f(t, x)$ as $f(t, x) = \sum_{|k|\geq 2} f_k(t) x^k$ are in $\mathscr{A}(\underline{\theta}, \overline{\theta}; r)$. Here $\underline{\theta}$, $\overline{\theta}$ and r, $\delta > 0$ are constants and O denotes Landau's symbol.

Set

$$\Lambda(t) = -\int_{\infty}^{t} \lambda(t) t^{-\sigma-1} dt. \tag{1.2}$$

We say that θ is a singular direction of $\Lambda(t)$ if

$$\cos(\alpha\theta + \arg \lambda_0) = 0, \ \lambda_0 = \lambda(0). \tag{1.3}$$

A half line passing through $t = 0$ with argument θ satisfying (1. 3) is called a singular line of $\Lambda(t)$.

We suppose moreover the following assumptions:

(A_1) There exist positive constants C and m such that, for each $i = 1, 2$, the following inequalities

$$|(k, \mu) - \mu_i| > C|k|^{-m} \tag{1.4}$$

hold, for all 2-vectors $k = (k_1, k_2) \in \mathbf{Z}_{\geq 0}{}^2$ with $|k| \geq 2$. This conditon is usually called the *Siegel condition*.

(A_2) $S(\underline{\theta}, \overline{\theta}; r)$ contains one and only one singular direction of $\Lambda(t)$ and neiter $\underline{\theta}$ nor $\overline{\theta}$ is a singular direction of $\Lambda(t)$.

We state our main theorem.

THEOREM. *Under the assumptions (A_1) and (A_2), there exists a unique 2-vector function p holomorphic and bounded in $D(\underline{\theta}, \overline{\theta}; r', \delta')$ with $p(t, y) = O(|y|^2)$ such that a change of coordinates $(t, x) \to (t, y)$ of the form*

$$T: \qquad x = y + p(t, y)$$

transforms the system Σ to

$$\Sigma': \qquad t^{\sigma+1} dy/dt = (\lambda(t)\mathbf{1}(\mu) + t^{\sigma}\mathbf{1}(\alpha))y,$$

provided that r' and $\delta' > 0$ are sufficiently small. Here the coefficients $p_k(t)$ in the power series expansion of $p(t, y)$ as $p(t, y) = \sum_{|k|\geq 2} p_k(t) y^k$ are in $\mathscr{A}(\underline{\theta}, \overline{\theta}; r')$.

By virtue of the theorem, we can construct a 2-parameter family of solutions to the system Σ. In fact, since the general solution to the reduced system Σ' is

$$x(t\,;C)=\exp[-\Lambda(t)1(\mu)+\log t\cdot 1(\alpha)]C, \tag{1.5}$$

a 2-parameter family of solutions to the original system Σ is given by

$$x(t\,;C)=y(t\,;C)+p(t,\,y(t\,;C)).$$

Here $C={}^t(C_1,\,C_2)$ is an arbitrary constant 2-vector.

The transformation T is given as a composition of three transformations, say T_1, T_2 and T_3, namely $T=T_3\circ T_2\circ T_1$.

By the transformation T_1 we make the nonlinear term $f={}^t(f_1,f_2)$ of Σ to be of order $O(t^\sigma)$. The transformation T_2 changes the system Σ^{T_1} obtained by T_1 to a system Σ^{T_2} so that each f_i, $i=1,\,2$, has x_i as a factor. Then we construct the transformation T_3 which eliminates the nonlinear term $f={}^t(f_1,f_2)$ of Σ^{T_2}.

We remark that the preliminary reduction of Σ by T_1 and T_2 is needed for proving the convergence of the formal power series $\sum_{|k|\geq 2}p_k(t)y^k$.

We use the following abbreviation:

$$S(r)=S(\underline{\theta},\,\overline{\theta}\,;r),\qquad D(r,\,\delta)=D(\underline{\theta},\,\overline{\theta}\,;r,\,\delta),$$
$$\mathscr{A}(r)=\mathscr{A}(\underline{\theta},\,\overline{\theta}\,;r),\qquad \mathscr{B}(r,\,\delta)=\mathscr{B}(D(\underline{\theta},\,\overline{\theta}\,;r,\delta)),$$

since $\underline{\theta}$ and $\overline{\theta}$ are fixed, and

$$A(t)=\lambda(t)1(\mu)+t^\sigma 1(\alpha). \tag{1.6}$$

2. The construction of preliminary transformations T_1 and T_2.

2.1. *The construction of* T_1. In this part, it will be shown that we can find a 2-vector function u holomorphic at $(t,\,\eta)=(0,\,0)\in\mathbf{C}\times\mathbf{C}^2$ with $u=O(|\eta|^2)$ so that a change of coordinates $(t,\,x)\to(t,\eta)$ of the form

$$T_1:\qquad x=\eta+u(t,\,\eta)$$

changes the system Σ to a system Σ^{T_1} of which the nonlinear terms are of order $O(t^\sigma)$.

This is proved by making use of the following lemma successively.

LEMMA 2. 1. *Given a system of the form*

(2. 1) $$t^{\sigma+1}dx/dt = A(t)x+t^{j-1}f(t, x),$$

where $A(t)$ *is the matrix of (1. 6) and* $f\in\mathscr{B}(r, \delta)$. *Then there exists a change of coordinates* $(t, x)\to(t, \eta)$ *of the form*

(2. 2) $$x = \eta+t^{j-1}u(\eta)$$

which transforms (2. 1) into a system of the form

(2. 3) $$t^{\sigma+1}d\eta/dt = A(t)\eta+t^{j}g(t, \eta).$$

Here $u\in\mathscr{O}(\{\eta\in\mathbf{C}\,;|\eta|<\delta'\})$ *and* $g\in\mathscr{B}(r', \delta')$, *provided that* $r', \delta'>0$ *are sufficiently small.*

PROOF. We see that the change of coordinates (2. 2) transforms (2. 1) into a system of the form

(2. 4) $$t^{\sigma+1}d\eta/dt = A(t)\eta+t^{j-1}h(t, \eta),$$

where

(2. 5) $$h(t,\eta) = (1+t^{j-1}\partial u/\partial\eta)^{-1}\times[-(j-1)t^{\sigma}u-(\partial u/\partial\eta)A(t)\eta+A(t)u-f(t,\eta+t^{j-1}u)].$$

Therefore, in order that (2. 4) is of the form (2. 3), it is necessary and sufficient that $h(0, \eta)$ vanishes identically, that is,

(2. 6) $$(\partial u/\partial\eta)A(0)\eta-A(0)u-f(0, \eta+u(\eta)) = 0$$

if $j = 1$, or

(2. 7) $$(\partial u/\partial\eta)A(0)\eta-A(0)u-f(0, \eta) = 0$$

if $2\leq j\leq\sigma$. Hence, we have only to show that a system of partial differential equations (2. 6) or (2. 7) has a solution $u(\eta)$ with $u = O(|\eta|^2)$ holomorphic at $\eta = 0$. Here we notice

(2. 8) $$A(0) = 1(\mu).$$

The existence of a holomorphic solution to (2. 7) will be shown in Appendix. On the other hand, the existence of u to (2. 6) is an immediate consequence of a famous Siegel's theorem.

LEMMA 2. 2 (Siegel). *Let there be given an autonoumous n-system*

$$dx/dt = \mathbf{1}_n(\mu)x + f(x) \tag{2. 9}$$

where $\mu = (\mu_1, \cdots, \mu_n) \in \mathbf{C}^n$, $\mathbf{1}_n(\mu) = \mathrm{diag}(\mu_1, \cdots, \mu_n)$, and an n-vector function f is holomorphic at $x = 0$ with $f = O(|x|^2)$. Suppose that there exist positive constants C and m such that the following inequalities hold ;

$$|(k, \mu) - \mu_i| < C|k|^{-m}, \ (1 \leq i \leq n)$$

for all n-vector $k = (k_1, \cdots, k_n) \in \mathbf{Z}_{\geq 0}{}^n$ with $|k| = k_1 + \cdots + k_n \geq 2$.

Then there exists an n-vector function u holomorphic at $\eta = 0$ with $u = O(|\eta|^2)$ such that a change of coordinates $(t, x) \to (t, \eta)$ of the form

$$x = \eta + u(\eta) \tag{2. 10}$$

transforms (2. 9) into

$$d\eta/dt = \mathbf{1}_n(\mu)\eta. \tag{2. 11}$$

In fact, in order that (2. 10) transforms (2. 9) into (2, 11), it is necessary and sufficient that $u = u(\eta)$ is a solution to

$$(\partial u/\partial \eta)\mathbf{1}_n(\mu)\eta - \mathbf{1}_n(\mu)u - f(\eta + u) = 0,$$

which is just of the same form as (2. 6) in case of $n = 2$. Q. E. D.

2. 2. *The construction of T_2.* Let Σ^{T_1} be the system transformed from Σ by the T_1. Then it is written as follows:

$$\Sigma^{T_1} \ : \qquad t^{\sigma+1} d\eta/dt = A(t)\eta + t^{\sigma} g(t, \eta),$$

where $g \in \mathscr{B}(r', \delta')$, r' and δ' being sufficiently small.

We obtain:

LEMMA 2. 3. *The system Σ^{T_1} can be reduced to a system*

$$\Sigma^{T_2} \ : \qquad t^{\sigma+1}d\zeta/dt = [A(t)+t^{\sigma}1(h(t,\ \zeta))]\zeta$$

by a suitable transformation of coordinates $(t,\ \eta)\rightarrow(t,\ \zeta)$ *of the form*

$$T_2 \ : \qquad \eta = \zeta+t^{\sigma}(w^1(t,\ \zeta_1)+w^2(t,\ \zeta_2))$$

where two 2-vector functions $w^i, i = 1,\ 2,\ h\in\mathscr{B}(r'',\ \delta'')$ *and that* $h = O(|\zeta|)$, *provided that* $0< r''\leq r',\ 0<\delta''\leq\delta'$ *are sufficiently small.*

PROOF. Let

$$(2.\ 12) \qquad t^{\sigma+1}d\zeta/dt = A(t)\zeta+t^{\sigma}G(t,\ \zeta)$$

be the system transformed from Σ^{T_1} by T_2. We have, then,

$$\begin{aligned} G(t,\ \zeta_1,\zeta_2) = &-[t^{\sigma+1}\partial w^1/\partial t+\partial w^1/\partial\zeta_1\cdot t^{\sigma+1}\partial\zeta_1/\partial t-(A(t)-\sigma t^{\sigma})w^1] \\ &-[t^{\sigma+1}\partial w^2/\partial t+\partial w^2/\partial\zeta_2\cdot t^{\sigma+1}\partial\zeta_2/\partial t-(A(t)-\sigma t^{\sigma})w^2] \\ &+f(t,\ \zeta+t^{\sigma}w^1+t^{\sigma}w^2) \end{aligned}$$

We consider the following partial differential equations:

$$(2.\ 13)_i \quad t^{\sigma+1}\partial w^i/\partial t+(\mu_i\lambda(t)+\alpha_i t^{\sigma})\zeta_i\partial w^i/\partial\zeta_i-(A(t)-\sigma t^{\sigma})w^i-f(t,\ {}^t(\delta_{i1}\zeta_1,\ \delta_{i2}\zeta_2)+t^{\sigma}w^i) = 0$$

$i = 1,\ 2,\ \delta_{ij}$ being Kronecker's delta. Notice that $(2.\ 13)_i$ is the condidtion that $G(t,\ \zeta_1,\ \zeta_2)$ has ζ_i as factor.

We first obtain a formal solution to $(2.13)_i$ of the form

$$(2.\ 14)_i \qquad w^i(t,\ \zeta_i) = \textstyle\sum_{j\geq2}w^i{}_j(t)\zeta_i^j\cdot$$

Inserting $(2.\ 14)_i$ into $(2.\ 13)_i$ and equating the coefficients of powers in ζ_j, we have

$$\begin{aligned} (2.\ 15)_i \qquad & t^{\sigma+1}dw^i{}_j/dt+[\lambda(t)1(j\mu_i-\mu_1,\ j\mu_i-\mu_2)+t^{\sigma}1(j\alpha_i-\alpha_1+\sigma,\ j\alpha i-\alpha_2+\sigma)]w^i{}_j \\ &= \sum_{\substack{|k|=|\gamma+\nu|\geq2 \\ \alpha_1+\cdots+\alpha_\gamma \\ +\beta_1+\cdots+\beta_\gamma=j}} t^{\sigma j}f_k\cdot w^i_{1,\alpha_1}\cdots\cdots w^i_{1,\alpha_\gamma}w^i_{2,\beta_1}\cdots\cdots w^i_{2,\beta_\nu},\ j\geq2 \end{aligned}$$

where $w^i_{1,\ j}$ and $w^i_{2,\ j}$ are the first and the second entries of $w^i{}_j$ respectively. It is easy to see that we can uniquely determine $w^i{}_j\in\mathscr{A}(r'')$, $j\geq2$, recursively, by using the following lemma.

LEMMA 2. 4 (Hukuhara). *For preassigned constants θ_1, θ_2 and $r>0$, we consider a differential equation of the form :*

$$t^{\sigma+1}dw/dt = c(t)w+f(t),$$

where σ is a positive integer, $c(t)$ is a polynomial in t of degree $\sigma-1$ with $c(0)\neq 0$ and $f\in\mathscr{A}(\theta_1, \theta_2; r)$. Suppose that $S(\theta_1, \theta_2; r)$ contains one and only one singular line of $-\int_\infty^t c(t)t^{-\sigma-1}dt$, then the equation has a unique solution $w\in\mathscr{A}(\theta_1, \theta_2; r)$.

It should be remarked that the domain $S(\underline{\theta}, \bar{\theta}; r'')$ contains only one singular direction of the system $(2.15)_i$.

We next prove the covergence of $(2.14)_i$.

Let $Z_i(t)$ be the general solution to

$$t^{\sigma+1}dZ_i/dt = (\mu_i\lambda(t)+a_it^\sigma)Z_i, \quad i = 1, 2.$$

Then it follows that the formal series $w^i = w^i(t, Z_i(t))$ formally satisfies

$$t^{\sigma+1}dw^i/dt = (A(t)-\sigma t^\sigma)w^i+f(t, {}^t(\delta_{i1}Z_1, \delta_{i2}Z_2)+t^\sigma w^i).$$

Therefore, by utilizing the following lemma, we see that $w^i(t, \zeta_i)$ converges absolutely and uniformly in $D(r', \delta')$, provided that r' and δ' are taken in a suitable way.

LEMMA 2. 5 (Iwano). *Let*

$$W(t, Z) = \Sigma_{j\geq 1} W_j(t)Z^j$$

be a formal power series in Z, where W_j belong to $\mathscr{B}(\theta_1, \theta_2; r)$. Let $c(t)$ be a polynomial in t of degree $\sigma-1$ with $c(0)\neq 0$. Suppose that $S(\theta_1, \theta_2; r)$ contains one and only one singular line of $C(t) = -\int_\infty^t c(t)t^{-\sigma-1}dt$ and that $W(t, Z(t))$ formally satisfies

$$t^{\sigma+1}dW(t, Z(t))/dt = f(t, Z(t), W(t, Z(t))),$$

where $Z(t)$ is the general solution to

$$t^{\sigma+1}dZ/dt = c(t)Z$$

and $f(t, Z, W)$ *is holomorphic and bounded for*

$$(t, Z)\in D(\theta_1, \theta_2; r, \delta), \quad |W|<\rho.$$

Then $W(t,Z)$ *converges absolutely and uniformly for*

$$(t, Z)\in D(\theta_1, \theta_2; r', \delta'),$$

provided that r', $\delta'>0$ *are sufficiently small.*

It is not difficult to show that, for w'_j determined above, Σ^{T_1} is reduced to a system of the form Σ^{T_2} by T_2. Q. E. D.

3. Construction of the formal transformation T_3.

We rewrite the system Σ^{T_2} as follows:

Σ^{T_2}: $$t^{\sigma+1}dx/dt = [A(t)+t^{\sigma}\mathbf{1}(f(t, x))]x.$$

We can verify that, by a transformation of coordinates $(t, x)\to(t, y)$ of the form

T_3: $$x = \mathbf{1}(y)[{}^t(1, 1)+p(t, y)],$$

Σ^{T_2} is changed to a system of the form

$$t^{\sigma+1}dy/dt = A(t)y+t^{\sigma}F(t, y),$$

where

$$F(t, y) = [1+\mathbf{1}(p)+\mathbf{1}(y)\partial p/\partial y]^{-1} \times\mathbf{1}(y)[t^{\sigma+1}\partial p/\partial t+\partial p/\partial y\cdot A(t)y-t^{\sigma}\mathbf{1}(f(t, \mathbf{1}(y)({}^t(1, 1)+p)))({}^t(1, 1)+p)].$$

Hence, in order that Σ^{T_2} is linearized by T_3, it is necessary and sufficicent that $p(t, y)$ satisfies the following system of partial differential equations:

(3. 1) $$t^{\sigma+1}\partial p/\partial t+\partial p/\partial y\cdot A(t)y = t^{\sigma}\mathbf{1}(f(t, \mathbf{1}(y)({}^t(1, 1)+p)))({}^t(1, 1)+p).$$

We want to obtain a formal solution to (3. 1) of the form

$$p(t, y) = \textstyle\sum_{|k|\geq 1} p_k(t) y^k. \tag{3. 2}$$

Substituting (3. 2) to (3. 1) and equating the coefficients of like powers in y, we have

$$\begin{aligned} &t^{\sigma+1} dp_{ik}/dt + [\lambda(t)(\mu, k) + t^{\sigma}(\alpha, k)] p_{ik} \\ &= \sum_{\substack{|l|=|\gamma+\nu|\geq 1 \\ l+\alpha^1+\cdots+\alpha^{\gamma+\delta_{i1}} \\ +\beta+\cdots\beta^{\gamma+\delta_{i2}}=k}} t^{\sigma} f_{il} \cdot p_{1\alpha^1} \cdots p_{1\alpha^{\gamma}} \cdot p_{2\beta^1} \cdots p_{2\beta^{\gamma}}, \end{aligned} \tag{3. 3}$$

where $p_k = {}^t(p_{1k}, p_{2k})$, $|k| \geq 0$ ($p_{(0,0)} = {}^t(1, 1)$), α^i, $\beta^j \in \mathbf{Z}_{\geq 0}{}^2$, $1 \leq i \leq \gamma+1$, $1 \leq j \leq \nu+1$ and for $f(t, x) = \sum_{|l|\geq 1} f_l(t) x^l$, $f_l = {}^t(f_{1l}, f_{2l})$.

We can uniquely determine $p_{ik} \in \mathscr{A}(r')$ recursively by the Lemma 2. 4. Thus we have constructed a formal transformation T_3 given by $x = \mathbf{1}(y)[{}^t(1, 1) + \sum_{|k|\geq 1} p_k(t) y^k]$.

4. Sketch of the proof of Theorem.

4. 1. *Truncated system to formal power series* $\sum p_k(t) y^k$. Let $\sum p_k(t) y^k$ be the formal power series constructed in 3 and set, for each $N \geq 1$,

$$p_{(N)}(t, y) = {}^t(1, 1) + \sum_{1\leq |k| \leq N} p_k(t) y^k. \tag{4. 1}$$

Then, in order that a change of coordinates $(t,x) \to (t,y)$ of the form

$$x = \mathbf{1}(y)(p_{(N)}(t, y) + \varphi(t,y)), \tag{4. 2}$$

transforms the system Σ^{T_2} into the system Σ', it is necessary and sufficient that the 2-vector function φ satisfies the following system of partial differential equations:

$$t(d/dt)\varphi = f_{(N)}(t, y, \varphi), \tag{4. 3$_N$}$$

where $t(d/dt)$ is the differential operator defined by

$$t(d/dt)\varphi = t\partial\varphi/\partial t + \partial\varphi/\partial y \cdot t^{-\sigma} A(t) y,$$

and

(4. 4) $$f_{(N)}(t, y, \varphi) = -t(d/dt)p_{(N)} + \mathbf{1}(f(t, \mathbf{1}(y)(p_{(N)}+\varphi)))(p_{(N)}+\varphi).$$

We can verify

LEMMA 4. 1. *There exist positive constants C_N and M such that*

(4. 5) $$|f_{(N)}(t, y, 0)| \leq C_N|y|^{N+1},$$

(4. 6) $$|f_{(N)}(t, y, \varphi) - f_{(N)}(t, y, \psi)| \leq M|y||\varphi-\psi|$$

hold for $(t, y) \in D(r_N, \delta_N)$, $|\varphi|$, $|\psi| < \rho_N$, provided r_N, δ_N, $\rho_N > 0$ are sufficiently small.

We note that the inequality

(4. 7) $$|f_{(N)}(t, y, \varphi)| \leq C_N|y|^{N+1} + M|y||\varphi|,$$

for $(t,y) \in D(r_N, \delta_N)$, $|\varphi| < \rho_N$, is derived from (4. 5) and (4. 6).

4. 2. *Fundamental lemma.* We now state the lemma which is important in proving the convergence of $\sum p_k(t)y^k$.

FUNDAMENTAL LEMMA. *Suppose that the assumptions given in Theorem hold. Then, for each $N \geq 1$, the system (4. 3)$_N$ possesses one and only one solution $\varphi = \varphi_{(N)}(t, y) \in \mathscr{B}(r_N, \delta_N)$ satisfying*

(4. 8) $$\varphi_{(N)}(t, y) = O(|y|^{N+1})$$

for $(t, y) \in D(f_N, \delta_N)$, provided that r_N, $\delta_N > 0$ are sufficiently small.

By this lemma, we can prove the convergence of $\sum p_k(t)y^k$. Indeed, for each $N \geq 1$, set

$$\varphi^N(t, y) = p_{(N)}(t, y) + \varphi_{(N)}(t, y),$$

where $\varphi_{(N)}(t, y)$ is the unique solution to (4. 3)$_N$ with (4. 8). Then a change of coordinates $(t,x) \to (t, y)$ of the form

$$x = \mathbf{1}(y)\varphi^N(t, y)$$

takes the system Σ^{T_2} into the system Σ'. We see that φ^N does not depend on N. We can assume that both the constants r_N and δ_N are monotone decreasing in N. For any N' and N with $N'>N$,

$$(p_{(N')}-p_{(N)})+\varphi_{(N')}$$

is a solution to the system $(4.3)_N$ in $\mathscr{B}(r_{N'}\delta_{N'})$, of order $O(|y|^{N+1})$. From the uniqueness assertion in Fundamental lemma, it follows that

$$\varphi_{(N)}=(p_{(N')}-p_{(N)})+\varphi_{(N')}$$

which implies that φ^N is independent of N. Now put

$$\varphi=\varphi^1,\ r'=r_1,\ \delta'=\delta_1.$$

Then φ is in $\mathscr{B}(r',\delta')$ satisfying

$$\varphi(t,y)-\sum_{|k|\leq N}p_k(t)y^k=O(|y|^{N+1}),$$

for each $N\geq 1$, which yields the convergence of $\sum p_k(t)y^k$.

Therefore we have only to prove the fundamental lemma. It should be remarked that φ is the solution to $(4.3)_N$ if and only if $\varphi=\varphi(t,y(t))$, satisfies the following system:

$$(4.9)_N \qquad td\varphi/dt=f_{(N)}(t,y(t),\varphi),$$

where d/dt is the usual ordinary differential operator and $y(t)$ is the general solution to Σ'. Further $(4.9)_N$ is equivalent to the system of integral equations:

$$(4.10) \qquad \varphi(t_0,y^0)=\int_\gamma f_{(N)}(t,y(t),\varphi(t,y(t)))t^{-1}dt,$$

where $y(t)$ is a solution to Σ' with $y^0=y(t_0)$ and γ is a path of integration joining $t=0$ to $t=t_0$ which will be specified later.

5. Determination of sectorial domain $\mathscr{S}$ and path of integration γ.

To prove the fundamental lemma, we have to define a sectorial domain $\mathscr{S}$ in t-plane and a path of integration γ joining $t=0$ to $t=t_0$ in $\mathscr{S}$. For this purpose, we begin with determing positive constants κ and ν_i, $i=1,2$. Recalling the assumptions $\mathrm{Re}\,(\mu_1\alpha_2-\mu_2\alpha_1)>0$, $\mu_1>0$ and $\mu_2<0$, we have inequalities either

$$\mathrm{Re}(\alpha_2)>0 \text{ or } \mathrm{Re}(\alpha_1)>0.$$

(i) In the case $\mathrm{Re}(\alpha_2)>0$, we choose $\varkappa$ in such a way that

$$-\mu_2\sigma/\mathrm{Re}(\alpha_2)<\varkappa \qquad \text{if } \mathrm{Re}(\alpha_1)\geq 0,$$
$$-\mu_2\sigma/\mathrm{Re}(\alpha_2)<\varkappa<-\mu_1\sigma/\mathrm{Re}(\alpha_1) \qquad \text{if } \mathrm{Re}(\alpha_1)<0$$

and we put

(5. 1) $$\nu_i=\mu_i\sigma+\varkappa\,\mathrm{Re}(\alpha_i),\ i=1,2.$$

(ii) In the case $\mathrm{Re}(\alpha_2)<0$, then $\mathrm{Re}(\alpha_1)>0$, and we take $\varkappa$ such that

$$\mu_1\sigma/\mathrm{Re}(\alpha_1)<\varkappa<-\mu_2\sigma/\mathrm{Re}(\alpha_2)$$

and we set

(5.2) $$\nu_i=-\mu_i\sigma+\varkappa\,\mathrm{Re}(\alpha_i),\ i=1,2.$$

In either case ν_i, $i=1, 2$, are positive.

We next determine Ω with $\pi/4<\Omega<\pi/2$ by

(5. 3) $$\tan\Omega=[\sigma\max\{\mu_1,\ -\mu_2\}+(3\varkappa+4)\max\{|\mathrm{Re}(\alpha_1)|,\ |\mathrm{Re}(\alpha_2)|\}+\min\{\nu_1,\ \nu_2\}]/\min\{\nu_1,\ \nu_2\}$$

Note that

(5. 4) $$\tan\Omega>1.$$

We now pass to define a sectorial domain $\mathscr{S}$. For this purpose, for a set E in t-plane, we denote by $\Lambda(E)^{-1}$ the set in the s-plane defined by $\{s\in\mathbf{C}\,;\,s=1/\Lambda(t),\ t\in E\}$. By assumption, $\Lambda(S(r))^{-1}$ contains a half line $\arg s=\pi/2$ or $\arg s=-\pi/2$. In the following, we only consider the case where $\Lambda(S(r))^{-1}$ contains a half line $\arg s=\pi/2$. The other case be treated by the same way. In our case, we can choose a small number $\varepsilon>0$ so that

$$\Lambda(S(r))^{-1}\subset\{s\in\mathbf{C}\,;\,|s|<(1+\varepsilon)\sigma r^{\sigma},\ |\arg s-\pi/2|<\pi-\varepsilon\}$$

for every small $r>0$.

Let $\underline{l}$ and $\bar{l}$ be half lines starting from the origin defined by $\{t\,;\arg t=\underline{\theta}\}$ and $\{t\,;\arg t=\bar{\theta}\}$, respectively, and let

$$\underline{c}=\Lambda(\underline{l})^{-1},\ \bar{c}=\Lambda(\bar{l})^{-1}$$

We denote by $\mathfrak{S}(\underline{\theta},\ \bar{\theta}\,;r)$ the domain in s-plane bounded by $\underline{c}$, $\bar{c}$ and the curve

$$|s|=\begin{cases}\sigma r^{\sigma} & \text{if } |\Theta-\pi/2|<\pi/2-\Omega\\ \sigma r^{\sigma}|\cos\Theta/\cos\Omega| & \text{if } \pi/2-\Omega\leq|\Theta-\pi/2|\leq\pi-\varepsilon,\end{cases}\tag{5. 5}$$

where $\Theta=\arg s$. We define a sectorial domain $\mathscr{S}(\underline{\theta},\ \bar{\theta}\,;r)$ by

$$\Lambda(\mathscr{S}(\underline{\theta},\ \bar{\theta}\,;r))^{-1}=\mathfrak{S}(\underline{\theta},\ \bar{\theta}\,;r).\tag{5. 6}$$

We now divide $\mathfrak{S}$ into three parts:

$$\mathfrak{S}_1=\mathfrak{S}\cap\{s\,;|\arg s-\pi/2|\leq\pi/2-\Omega\}$$

$$\mathfrak{S}_2=\mathfrak{S}\cap\{s\,;-\pi/2<\arg s<\Omega\}$$

$$\mathfrak{S}_3=\mathfrak{S}\cap\{s\,;\pi-\Omega<\arg s<3\pi/2\}$$

and define the domains $\mathscr{S}_i$, $i=1,2,3$ in t-plane by

$$\Lambda(\mathscr{S}_i)^{-1}=\mathfrak{S}_i,\ i=1,2,3.\tag{5. 7}$$

Then it is evident that

$$\mathscr{S}=\bigcup_{1\leq i\leq 3}\mathscr{S}_i.\tag{5. 8}$$

For a given $s_0\in\mathfrak{S}$, we define, in $\mathfrak{S}$, a path Γ joining $s=0$ to $s=s_0$ which generally consists of two paths Γ' and Γ''.

(I) The case of $\mathrm{Re}\,(\alpha_2)>0$.

(i) When s_0 is in $\mathfrak{S}_1$, Γ consists of only Γ'. The coordinate s, on Γ', is paramatrized by τ as follows:

$$1/s(\tau)=\tau+a-\sqrt{-1}be^{\kappa\tau},\ \tau\in[0,\ +\infty),\tag{5. 9}$$

where

(5. 10) $$1/s_0 = a - \sqrt{-1}b,\ s_0 = s(0).$$

(ii) When s_0 is in $\mathfrak{S}_2$(or $\mathfrak{S}_3$), Γ consists of two paths Γ''' and Γ'. The coordinate s, on Γ''', is parametrized by Θ as follows:

(5. 11) $$s(\Theta) = (|s_0|/\cos\Theta_0)\cos\Theta e^{\sqrt{-1}\Theta},$$

for $\Theta \in [\Theta_0, \Omega]$ (or $\Theta \in [\pi - \Omega, \Theta_0]$) respectively, where

(5. 12) $$\Theta_0 = \arg s_0.$$

Then $s(\Omega)$ (or $s(\pi - \Omega)$) is located in $\mathfrak{S}_1$. The coordinate s on Γ' is defined by the same way as (5. 9), where starting point is $s(\Omega)$ (or $s(\pi - \Omega)$).

(II) The case of Re $(\alpha_2) < 0$.

(i) When s_0 is in $\mathfrak{S}_1$, Γ consists of only Γ'. The coordinate s, on Γ', is parametrized by τ as follows:

(5. 13) $$1/s(\tau) = -\tau + a - \sqrt{-1}be^{\kappa\tau},\ \tau \in [0, +\infty),$$

where

(5. 14) $$1/s_0 = a - \sqrt{-1}b,\ s_0 = s(0),$$

(ii) When s_0 is in $\mathfrak{S}_2$ (or $\mathfrak{S}_3$), we define Γ''' by the same way as (5. 11) and Γ' as (5. 13).

For a given $t_0 \in \mathscr{S}$, we define a path γ in $\mathscr{S}$ by

(5. 15) $$\Lambda(\gamma)^{-1} = \Gamma,$$

where $s_0 = \Lambda(t_0)^{-1} \in \mathfrak{S}$. Then we can verify

LEMMA 5. 1. *We can choose a sufficiently small positive constant r in such a way that*

(i) *When $t_0 \in \mathscr{S}_1$, then $\gamma \subset \mathscr{S}_1$;*

(ii) *When $t_0 \in \mathscr{S}$, then $\gamma \subset \mathscr{S}$.*

6. The stability of the domain $\mathscr{D}$ which is a deformation of D.

We define a domain $\mathscr{D} = \mathscr{D}(r, \delta) = \mathscr{D}(\underline{\theta}, \overline{\theta}; r, \delta)$ in (t, y) -space by

$$\mathscr{D} = \{(t, y)\in \mathbf{C}\times\mathbf{C}^2; t\in\mathscr{S}(r), |y_i|<\delta d_i(t)e_i(t)\}, \tag{6. 1}$$

where

$$d_i(t) = \begin{cases} |t^\sigma \Lambda(t)|^{\mathrm{Re}(\alpha_i)/\sigma} & \text{on } \mathscr{S}_1 \\ [(|\cos\theta|/\cos\Omega)t^\sigma\Lambda(t)]^{\mathrm{Re}(\alpha_i)/\sigma} & \text{on } \mathscr{S}_2\cup\mathscr{S}_3 \end{cases} \tag{6. 2}$$

and

$$e_i(t) = \exp[-\mathrm{Im}(\alpha_i)\cdot\theta] \tag{6. 3}$$

with $\theta = \arg t$.

We note that the domain $\mathscr{D}(r, \delta)$ is equivalent to domain $D(r, \delta)$, namely, for given r and $\delta>0$, we can choose constants r' and $\delta'>0$ so that $\mathscr{D}(r', \delta')\subset D(r, \delta)$, and conversely, for given r' and $\delta'>0$, we can take constants r and $\delta>0$ so that $D(r, \delta)\subset\mathscr{D}(r', \delta')$.

We now give some properties for the $\mathscr{D}$.

LEMMA 6. 1 *(Stability of $\mathscr{D}$). Let $(t_0, y^0)\in\mathscr{D}(r, \delta)$ and let $y(t)$ be the solution to Σ' with $y(t_0) = y^0$, then $(t, y(t))\in\mathscr{D}(r, \delta)$ for every $t\in\gamma$, provided that r and $\delta>0$ are sufficiently small.*

In order to prove lemma 6. 1, we use following lemma.

LEMMA 6. 2. *Let $y(t)$ be the function given in lemma 6. 1 and let*

$$u_i(t) = C_i e^{-\mu_i\Lambda(t)}\Lambda(t)^{-\mathrm{Re}(\alpha_i)/\sigma},\ i=1, 2. \tag{6. 4}$$

Then, for $t\in\gamma$ with $t_0\in\mathscr{S}_1$ the pull back $u_i(\tau)$ of u_i by (5. 9) satisfy

$$d\log|u_i(\tau)|/d\tau < -3\nu_i/4\sigma. \tag{6. 5}$$

It should be remarked that, in particular, $|u_i(\tau)|$ are monotone decreasing functions with respect to τ.

PROOF OF LEMMA 6. 1. By the definition of $u_i(t)$, we have

(6. 6)
$$y_i(t) = u_i(t)\Lambda(t)^{\mathrm{Re}(a_i)/\sigma}t^{a_i}$$
$$= u_i(t)(t^\sigma\Lambda(t))^{\mathrm{Re}(a_i)/\sigma}t^{-\mathrm{Re}(a_i)}t^{a_i}.$$

Let us suppose that $(t_0, y^0) = (t_0, y(t_0)) \in \mathscr{D}$.

(I) In case that $t_0 \in \mathscr{S}_1$; Because of (6. 6), we have

$$|y_i(t)| = |u_i(t)||t^\sigma\Lambda(t)|^{\mathrm{Re}(a_i)/\sigma}e_i(t)$$

which yield the ineqalities

$$|y_i(t)| = |y_i(t_0)|/[|u_i(t_0)||t_0{}^\sigma\Lambda(t_0)|^{\mathrm{Re}(a_i)/\sigma}e_i(t_0)] \times |u_i(t)||t^\sigma\Lambda(t)|^{\mathrm{Re}(a_i)/\sigma}e_i(t)$$
$$= |y_i(t_0)|(|u_i(t)|/|u_i(t_0)|)(d_i(t)e_i(t)/d_i(t_0)e_i(t_0))$$
$$< \delta d_i(t)e_i(t),\ i = 1,\ 2,$$

δ being a small constant.

(II) In case that $t_0 \in \mathscr{S}_2$ (or $\mathscr{S}_3$); Noting that we have

$$\mathrm{Re}\,\Lambda(t) = \mathrm{Re}\,\Lambda(t_0)$$

on γ' and

$$C_i = y_i(t_0)e^{\mu_i\Lambda(t_0)}t_0{}^{-a_i},\ i = 1,\ 2,$$

where C_i, $i = 1, 2$, are arbitrary constants in (1. 5), we have the inequalities:

$$|y_i(t)| = |y_i(t_0)e^{\mu_i\Lambda(t_0)}t_0{}^{-a_i}e^{-\mu_i\Lambda(t)}t^{a_i}|$$
$$= |y_i(t_0)||t^\sigma/t_0{}^\sigma|^{\mathrm{Re}a_i/\sigma}(e_i(t)/e_i(t_0))$$
$$= |y_i(t_0)|[(|\cos\theta|/\cos\Omega)|t^\sigma\Lambda(t)|]^{\mathrm{Re}a_i/\sigma}e_i(t)$$
$$/[(|\cos\theta|/\cos\Omega)|t_0{}^\sigma\Lambda(t_0)|]^{\mathrm{Re}a_i/\sigma}e_i(t_0)$$
$$< \delta d_i(t)e_i(t),\ i = 1,\ 2,$$

δ being a small constant.

Thus we have proved Lemma 6. 1. Q. E. D.

In order to prove Lemma 6. 2, we make use of the following lemma.

LEMMA 6. 3. *We have*

(6. 7) $$|a/b| \leq 1/\tan\Omega < 1,$$

(6. 8) $$b \geq (\tan\Omega \sin\varepsilon)/r > \tan\Omega,$$

(6. 9) $$\varkappa b \geq \tan\Omega$$

for any $t_0 \in \mathscr{S}_1$.

The proof of this Lemma is omitted (see Proposition 2 in [12]).

We now procced to prove Lemma 6. 2.

PROOF OF LEMMA 6. 2. We consider only the case that $\mathrm{Re}(\alpha_2) > 0$, because the other case is shown in the similar way.

We have

(6. 10) $$d \log u_i(t)/dt = -(u_i + \sigma^{-1}\mathrm{Re}(\alpha_i)\Lambda(t)^{-1})d\Lambda(t)/dt$$

from (6. 4) and

(6. 11) $$dt/d\tau = (d\Lambda(t)/dt)^{-1}(1-\sqrt{-1}\varkappa b e^{\kappa\tau})$$

from (5. 9). Consequently, we have

$$\begin{aligned} d \log |u_i(\tau)|/d\tau &= d\,\mathrm{Re}\,[\log u_i(\tau)]/d\tau \\ &= \mathrm{Re}\,[d \log u_i(t)/dt \cdot dt/d\tau] \\ &= -\mathrm{Re}\,[(\mu_i + \sigma^{-1}\,\mathrm{Re}\,(\alpha_i)\Lambda(t(\tau))^{-1})(1-\sqrt{-1}\varkappa b e^{\kappa\tau})],\ i = 1,\ 2. \end{aligned}$$

So, we prove the following inequalities;

$$\mathrm{Re}\,[(\mu_i + \sigma^{-1}\,\mathrm{Re}\,(\alpha_i)\Lambda(t(\tau))^{-1})(1-\sqrt{-1}\varkappa b e^{\kappa\tau})] > 3\nu_i/4\sigma$$

or equivalently,

(6. 12) $$\nu_i b^2 e^{2\kappa\tau} - (-\mu_i\sigma + 3\varkappa\,\mathrm{Re}\,\alpha_i)(\tau + a)^2 + 4\,\mathrm{Re}\,(\alpha_i)(\tau + a) > 0.$$

Define a function in τ, $\mathfrak{A}_i(\tau)$, by the left hand side of (6. 12), then we can see that, for $\tau \in [0,$

$+\infty$),

(6. 13) $$\mathfrak{A}_i(0)>0,\ \mathfrak{A}_i'(0)>0,\ \mathfrak{A}_i''(\tau)>0,\ i=1,\,2.$$

In fact, we have the first inequalities in the following way:

$$\begin{aligned}\mathfrak{A}_i(0) &= \nu_i b^2-(-\mu_i\sigma+3\varkappa\,\mathrm{Re}\,\alpha_i)a^2+4\,\mathrm{Re}\,(\alpha_i)a\\ &\geq \nu_i b^2[1-((-1)^{i-1}\mu_i\sigma+3\varkappa\,\mathrm{Re}\,\alpha_i)/\nu_i(a/b)^2-4|\mathrm{Re}\,\alpha_i||a|/\nu_i b^2]\\ &> \nu_i b^2[1-((-1)^{i-1}\mu_i\sigma+3\varkappa\,\mathrm{Re}\,\alpha_i)/\nu_i\cdot 1/\tan\Omega-4|\mathrm{Re}\,\alpha_i|/(\nu_i\tan\Omega)]\\ &> (\nu_i b^2/\tan\Omega)[\tan\Omega-((-1)^{i-1}\mu_i\sigma+(3\varkappa+4)|\mathrm{Re}\,\alpha_i|)/\nu_i]\\ &> 0.\end{aligned}$$

Next since

$$\mathfrak{A}_i'(\tau) = 2\nu_i\varkappa b^2e^{2\varkappa\tau}-2(-\mu_i\sigma+3\varkappa\,\mathrm{Re}\,\alpha_i)(\tau+a)+4\,\mathrm{Re}\,\alpha_i,$$

we have the secound inequnlities

$$\begin{aligned}\mathfrak{A}_i'(0) &= 2\nu_i\varkappa b^2-2(-\mu_i\sigma+3\varkappa\,\mathrm{Re}\,\alpha_i)a+4\,\mathrm{Re}\,\alpha_i\\ &> 2\nu_i b[\varkappa b-((-1)^{i-1}\mu_i\sigma+3\varkappa|\mathrm{Re}\alpha_i|)/\nu_i\cdot|a/b|-2|\mathrm{Re}\,\alpha_i|/(\nu_i b)]\\ &> 2\nu_i b[1-((-1)^{i-1}\mu_i\sigma+3\varkappa|\mathrm{Re}\alpha_i|)/\nu_i\cdot 1/\tan\Omega-2|\mathrm{Re}\,\alpha_i|/(\nu_i\tan\Omega)]\\ &= 2\nu_i b/\tan\Omega\cdot[\tan\Omega-((-1)^{i-1}\mu_i\sigma+(3\varkappa+2)|\mathrm{Re}\,\alpha_i|)/\nu_i]\\ &> 0.\end{aligned}$$

We have, further, the last inequalities

$$\begin{aligned}\mathfrak{A}_i''(\tau) &= 4\nu_i\varkappa^2b^2e^{2\varkappa\tau}-2(-\mu_i\sigma+3\varkappa\,\mathrm{Re}\,\alpha_i)\\ &\geq 4\nu_i\varkappa^2b^2-2((-1)^{i-1}\mu_i\sigma+3\varkappa|\mathrm{Re}\,\alpha_i|)\\ &= 2\nu_i\varkappa^2b^2[2-((-1)^{i-1}\mu_i\sigma+3\varkappa|\mathrm{Re}\,\alpha_i|)/\nu_i\cdot 1/\varkappa^2b^2)]\\ &\geq 2\nu_i\varkappa^2b^2/\tan\Omega\cdot[2\tan\Omega-((-1)^{i-1}\mu_i\sigma+3\varkappa|\mathrm{Re}\,\alpha_i|)/\nu_i]\\ &> 2\nu_i/\tan\Omega\cdot[2((-1)^{i-1}\mu_i\sigma+(3\varkappa+4)|\mathrm{Re}\,\alpha_i|/\nu_i-((-1)^{i-1}\mu_i\sigma+3\varkappa|\mathrm{Re}\,\alpha_i|)/\nu_i]\\ &= 2/\tan\Omega\cdot[(-1)^{i-1}\mu_i\sigma+(3\varkappa+8)|\mathrm{Re}\,\alpha_i|]\\ &> 0.\end{aligned}$$

Thus we have obtained (6. 13), which yield (6. 5). Q. E. D.

7. Proof of Fundamental lemma.

7. 1. In this part, we show the following lemma.

LEMMA 7. 1. *Let $y(t)$ be the general solution to the reduced system Σ'. Then we can find a positive constant J_N depending on N so that we have the inequality ;*

$$\int_{\gamma}|y(t)|^{N+1}|t|^{-1}|dt|\leq J_N|y(t_0)|^{N+1} \tag{7. 1}$$

for any $t_0\in\mathscr{S}(\underline{\theta},\ \bar{\theta}\ ;r)$, $r>0$ being sufficiently small.

PROOF. Recalling (6. 6), we have

$$|y(t)|=|u_i(t)||(t^{\sigma}\Lambda(t))^{\mathrm{Re}a_i/\sigma}e_i(t)|,\ i=1,2.$$

Notice that $|(t^{\sigma}\Lambda(t))^{\mathrm{Re}a_i/\sigma}e_i(t)|$ is bounded from below and above by constants B_1, B_2, respectively. Hence We have

$$\int_{\gamma_\cdot}|y_i(t)|^{N+1}|t|^{-1}|dt|\leq B_2^{N+1}\int_{\gamma_\cdot}|u_i(t)|^{N+1}|t|^{-1}|dt|$$

or γ'. Moreover we have

$$|\lambda(t)|/|t^{\sigma}\Lambda(t)|>B$$

for a suitable positive constant B, and hence, we have

$$|dt|/|t|<B^{-1}|ds|/|s|. \tag{7. 2}$$

Further we see that the pull back $u_i(s)$ of u_i by (5. 9) satisfy

$$d\log|u_i(s)|>3\nu_i/(4\sqrt{2}\kappa\sigma)|s|^{-1}|ds|. \tag{7. 3}$$

Indeed, by (6. 5), we have

$$\begin{aligned}d\log|u_i(s)|/|ds|&=d\log|u_i(\tau)|/d\tau\cdot d\tau/|ds|\\&=d\log|u_i(\tau)|/d\tau\cdot(-|d\tau/|ds|)\end{aligned}$$

$$>3\nu_i/(4\sigma)\cdot|d\tau/ds|,$$

and we have

(7. 4) $$|s||d\tau/ds|>(\sqrt{2}\varkappa)^{-1},$$

because

$$\begin{aligned}&\text{the left hand side of (7. 4)}\\&=[(\tau+a)^2+b^2e^{2\varkappa\tau}]^{1/2}(1+\varkappa^2b^2e^{2\varkappa\tau})^{-1/2}\\&>[b^2e^{2\varkappa\tau}]^{1/2}(2\varkappa^2b^2e^{2\varkappa\tau})^{-1/2}=(\sqrt{2}\varkappa)^{-1}.\end{aligned}$$

Hence (7. 3) holds.

Now, by (7. 2) and (7. 3), we have

$$\begin{aligned}&\int_{\gamma'}|u_i(t)|^{N+1}|t|^{-1}|dt|\\&<B^{-1}\int_{\Gamma'}|u_i(s)|^{N+1}|s|^{-1}|ds|\\&<4\sqrt{2}\varkappa\sigma(3\nu_i(N+1))^{-1}\int_{\Gamma'}|u_i(s)|^{N+1}d\log|u_i(s)|^{N+1}\\&=4\sqrt{2}\varkappa\sigma(3\nu_i(N+1))^{-1}\int_{\Gamma'}d|u_i(s)|^{N+1}\\&<4\sqrt{2}\varkappa\sigma(3\nu_i(N+1))^{-1}|u_i(s_0)|^{N+1},\end{aligned}$$

and hence

$$\begin{aligned}&\int_{\gamma'}|y_i(t)|^{N+1}|t|^{-1}|dt|\\&\leq 4\sqrt{2}\varkappa\sigma(3\nu_iB(N+1))^{-1}B_2^{N+1}|u_i(s_0)|^{N+1}\\&<4\sqrt{2}\varkappa\sigma(3\nu_iB(N+1))^{-1}(B_2/B_1)^{N+1}|y_i(s_0)|^{N+1}.\end{aligned}$$

We obtain, therefore, the inequality (7. 1).

Next we have, on Γ'',

$$s(\Theta)=|s_0|/\cos\Theta_0\cdot\cos\Theta e^{\sqrt{-1}\Theta}.$$

Noting $|ds|$ is a decreasing of increasing function in $\Theta=\arg s$ corresponding to $s_0\in\mathscr{S}_2$ or $\mathscr{S}_3$ respectively, we have

$$|ds|=\mp|ds/d\Theta|d\Theta$$

$$= \mp |s_0/\cos \Theta_0| d\Theta$$

and

$$|s|^{-1}|ds| = \mp |\cos \Theta|^{-1} d\Theta.$$

We have, hence,

$$\int_{\Gamma''} |s|^{-1}|ds| < \mp \int_{\Gamma''} |\cos \Theta|^{-1} d\Theta$$
$$= \begin{cases} \int_{\Omega}^{\theta_0} |\cos \Theta|^{-1} d\Theta < \pi / \min\{\sin \varepsilon, \cos \Omega\} \\ \int_{\pi-\Omega}^{\theta_0} |\cos \Theta|^{-1} d\Theta < \pi / \min\{\sin \varepsilon, \cos \Omega\} \end{cases}$$

for $\Theta \in [\Theta_0, \Omega]$ or $[\pi - \Omega, \Theta_0]$ respectively. We have, then,

$$\int_{\Gamma''} |s|^{-1}|ds| < B_3$$

for some positive constant B_3. Noting that, on Γ'',

$$|y_i(s)| = |y_i(s_0)| d_i(s) e_i(s)/(d_i(s_0) e_i(s_0)) < B_4 |y_i(s_0)|, \ i = 1, 2,$$

for a suitable positive constant B_4 where $d_i(s)$ (or $e_i(s)$) is the pull back of d_i (or e_i) by s, we have

$$\int_{\Gamma''} |y_i(s)|^{N+1} |s|^{-1}|ds| \le \int_{\Gamma''} B_4^{N+1} |y_i(s_0)|^{N+1} |s|^{-1}|ds|$$
$$\le B_4^{N+1} B_3 |y_i(s_0)|^{N+1}.$$

Therefore if we choose J_N such that

$$J_N > 4\sqrt{2}\kappa\sigma(3\nu_i B(N+1))^{-1}(B_2/B_1)^{N+1} + (B_3/B) B_4^{N+1},$$

then we have the inequality (7. 1). Q. E. D.

7. 2. In this part, we give the proof of fundamental lemma.

First of all, we define the family of functions $\mathscr{F}$ by

$$\mathscr{F} = \{\varphi : \mathscr{D} \to \mathbf{C}^2 ; \varphi \in \mathscr{B}(\mathscr{D}), |\varphi| \leq K_N |y|^{N+1}\}$$

where $\mathscr{D} = \mathscr{D}(r_N, \delta_N)$ and K_N is a positive constant depending of N which will be specified later. It is evident that $\mathscr{F}$ is non-empty and convex, and further closed and normal with respect to the topology of uniform convergence on every compact subset of $\mathscr{D}$. Next we define an operator $\mathscr{T}$ acting on $\mathscr{F}$.

Let (t_0, y^0) be a point in $\mathscr{D}$ and let $y(t)$ be the solution to Σ' satisfying the initial condition $y(t_0) = y^0$. Moreover we set

$$\Phi(t_0, y^0) = \int_\gamma f_{(N)}(t, y(t), \varphi(t, y(t)))t^{-1}dt \tag{7. 5}$$

and define an operator $\mathscr{T}$ from $\mathscr{F}$ into $\mathscr{F}$ by

$$\varphi(t, y) \to \Phi(t, y). \tag{7. 6}$$

This is well-defined. Indeed, first, we can take r_N and δ_N so small that $(t, y, \varphi) \in D(r_N, \delta_N) \times \{w \in \mathbf{C}^2 ; |w| < \rho_N\}$, where r_N, δ_N and ρ_N are specified in Lemma 4. 1.

Next let $\varphi \in \mathscr{F}$, then, by (4. 7), we have

$$\begin{aligned} |\Phi(t_0, y^0)| &\leq \int_\gamma |f_{(N)}(t, y(t), \varphi(t))||t|^{-1}|dt| \\ &\leq \int_\gamma [C_N|y|^{N+1} + M|y||\varphi|]|t|^{-1}|dt| \\ &\leq \int_\gamma [C_N + M|y|K_N]|y|^{N+1}|t|^{-1}|dt| \\ &\leq \int_\gamma [C_N + M\delta_N K_N]|y|^{N+1}|t|^{-1}|dt| \\ &\leq [C_N + M\delta_N K_N]J_N|y(t_0)|^{N+1}, \end{aligned}$$

and hence we obtain

$$|\Phi(t_0, y^0)| \leq K_N|y(t_0)|^{N+1},$$

provided that δ_N is choosen so small that

$$M\delta_N J_N < 1/2$$

and that we set $K_N = 2C_N J_N$. Further we can verify that $\Phi \in \mathscr{B}(\mathscr{D})$. So, $\Phi(t_0, y^0) \in \mathscr{F}$, and fence

$\mathscr{T}(\mathscr{F})\subset\mathscr{F}$. Moreover, it is evident that $\mathscr{T}$ is continuous on $\mathscr{F}$ by (4. 6). Thus we see that $\mathscr{T}$ admits a fixed point by virtue of Schauder-Tihonov fixed point theorem, which shows the existence of φ_N in Fundamental lemma.

We now prove the uniqueness of φ_N.

Let $\varphi^1$and φ^2 be two solutions satisfying the lemma and put

$$\psi = \varphi^1 - \varphi^2.$$

Then $\psi\in\mathscr{B}(\mathscr{D})$ and $|\psi| = O(|y|^{N+1})$.

We define a positive constant H by

$$H = \inf\{H'\geq 0\,;|\psi|\leq H'|y|^{N+1},\ (t,\ y)\in\mathscr{D}\}.$$

Then we have

$$\begin{aligned}|\psi(t_0,\ y^0)| &\leq \int_\gamma M|y||\psi||t|^{-1}|dt| \\ &\leq \int_\gamma M|y|H|y|^{N+1}|t|^{-1}|dt| \\ &\leq M\delta_N H J_N|y(t_0)|^{N+1} \\ &< 2^{-1}H|y(t_0)|^{N+1}\end{aligned}$$

by (4. 6), and hence, by the definition of H, we have $H = 0$. Thus we have shown the lemma. Q. E. D.

Appendix.

To prove the existence of a holomorphic solution to (2. 7), we consider the following system:

$$\Sigma_1\ :\qquad \partial u/\partial\eta\cdot A\eta - Au = f(\eta),$$

where $A = A(0)$, $f(\eta) = f(0,\ \eta)$. Suppose that

$$f(\eta) = \sum_{|k|\geq 2} f_k\eta^k,\ f\in\mathcal{O}(\{\eta\,;|\eta|<r\},$$

and let $u(\eta) = \sum u_k\eta^k$ be a formal solution to Σ_1. Then, substituting it to Σ_1, we obtain

$$[(\mu, k)-A]u_k = f_k$$

for $|k|\geq 2$. We have, therefore,

$$u_k = [(\mu, k)-A]^{-1}f_k,$$

and hence, by (1. 4), we have

$$|u_k| < C^{-1}|k|^m Mr^{-|k|},$$

where $M = \sup\{|f|;|\eta|<r\}$. Consequently

$$U(\eta) = \sum C^{-1}|k|^m Mr^{-|k|}\eta^k$$

is a majorant series of $\sum u_k\eta^k$. Now

$$U(\eta) = \sum_{N\geq 2}\sum_{|k|=N} C^{-1}|k|M|\eta/r|^{|k|}$$
$$= C^{-1}M\sum_{N\geq 2}N^m(N+1)|\eta/r|^N,$$

which proves the convergence of $u(\eta)$ for $|\eta|<r$. Q. E. D.

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