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ON RATIONAL POINTS OF k-FORMS OF A HOMOGENEOUS SPACE OF TYPE A OVER AN ALGEBRAIC NUMBER FIELD I

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Introduction.

The study of the rational points of an algebraic variety over an algebraic number field is one of the most fundamental problems in number theory. In this paper we consider one of such problems, so-called the Hasse principle for certain class of rational varieties.

We say that an algebraic variety V defined over an algebraic number field k satisfies the HASSE PRINCIPLE, when the following statement is valid:

If V has a rational point over the completion k_v for every place v of k, then V has a k-rational point.

It is known that the Hasse principle holds for quadratic forms (due to Hasse and Minkowski) and k-forms of $\mathbf{P}^n$ (due to Châtelet, see § 1). A k-form of $\mathbf{P}^n$ is by definition an algebraic variety over k which is isomorphic to $\mathbf{P}^n$ over an algebraic closure of k.

Particularly the last result is a combination of the fundamental theorem of Brauer group over an algebraic number field i.e. class field theory and the result due to Châtelet: *if a k-form V of $\mathbf{P}^n$ has a k-rational point, then V is isomorphic to $\mathbf{P}^n$ over* k.

Let us notice that in these two examples the varieties have a structure of homogeneous spaces over an algebraic closure. We treat the following problems for certain class of homogeneous spaces which contains the flag varieties and the Grassmann varieties.

PROBLEM 1. *Let k be a field of characteristic zero. Does an analogue of Châtelet's theorem hold for a k-form of a homogeneous space? Namely, does a k-form V of a homogeneous space H have a rational point if and only if V is isomrophic to H over k?*

When this statement is true for H, we will say that H has the CHÂTELET PROPERTY.

PROBLEM 2. *Let k be an algebraic number field. Does the Hasse principle hold for a k-form of a homogeneous space?*

In this paper we use a general cohomological method and translate the Problems for homogeneous space of PGL (n) into questions on algebras. To explain our result we need a notation.

NOTATION 1. Let $i_1, i_2, \cdots, i_l$ and n be positive integers such that $i_1+i_2+\cdots+i_l=n$. We set

$$P_{i_1, i_2, \cdots, i_l}=\left\{ \sigma \in GL(n) \;\middle|\; \sigma=\begin{array}{c} \begin{array}{cccc} \overbrace{\;}^{i_1} & \overbrace{\;}^{i_2} & \cdots & \overbrace{\;}^{i_l} \end{array} \\ \left[\begin{array}{c|c|c|c} * & * & * & * \\ \hline 0 & * & * & * \\ 0 & 0 & \ddots & * \\ \hline 0 & 0 & 0 & * \end{array}\right] \begin{array}{l})\,i_1 \\ )\,i_2 \\ \vdots \\ )\,i_l \end{array} \end{array} \right\}$$

modulo the center of GL (n)

$$H_{i_1, i_2, \cdots, i_l}=PGL(n)/P_{i_1, i_2, \cdots, i_l}$$

where PGL (n) =GL (n) modulo the center of GL (n).

$P_{i_1, i_2, \cdots, i_l}$ is a parabolic subgroup so that $H_{i_1, i_2, \cdots, i_l}$ is a projective variety.

We obtained the following results.

Our answer to Problem 1 is as follows.

Let k be either an algebraic number fields or its completion with respect to an arbitrary finite place.

Under the assumption $(i_1, i_2, \cdots, i_l)\neq(i_l, i_{l-1}, \cdots, i_1)$, $H_{i_1, i_2, \cdots, i_l}$ *has the Châtelet property if and only if the greatest common measure of* $(i_1, i_2, \cdots, i_l)$ *is* 1 (§3, *Theorem* 2).

Our answer to Problem 2 is the following result.

Let k be an algebraic number field. Let V be a k-form of $H_{i_1, i_2, \cdots, i_l}$. *The Hasse principle holds for V if one of the following conditions satisfied*

(1) $(i_1, i_2, \cdots, i_l)\neq(i_l, i_{l-1}, \cdots, i_1)$

(2) $(i_1, i_2, \cdots, i_l)=(i_l, i_{l-1}, \cdots, i_1)$ *and the greatest common measure of* $(i_1, i_2, \cdots, i_l)$ *equals to* 1 (§3, *Theorem* 3).

We wish to express our hearty thanks to H.Umemura for his constant interest in our work.

§1. Basic idea

In this section, we assume that k is a field of characteristic zero.

The objective of this paper is to discuss some arithmetic aspects of the k-form homogeneous space $H_{l_1, l_2, \cdots, l_r}$.

Let us first review the case where $H_{1,m}$ is the m-dimensional projective space $\mathbf{P}^m$. Then a k-form is a classical Brauer variety. For this variety we know the fundamental result due to Châtelet (see the Ch.10 of Serre [6]).

Châtelet's theorem : Let k be a perfect field and V be a k-form of $\mathbf{P}^n$. *For V to have a k-rational point, it is necessary and sufficient that V is isomorphic to* $\mathbf{P}^n$ *over k.*

Furthermore, when k is an algebraic number field, this theorem and the Hasse principle for the Brauer group imply the Hasse principle for Brauer varietes (see the Ch. 10 of Serre [6]).

Our study started when we discovered that we could easily prove it by only using some elementary property of Galois cohomology and Hilbert's theorem 90 once we notice that a Brauer variety is a k-form of some homogeneous space.

PROPOSITION 1. *Let V be a k-form of* $H_{l_1, l_2, \cdots, l_r}$. *For V to have a k-rational point, it is necessary and sufficient that V is isomorphic to a homogeneous space G/T over k, where G and T satisfy the following condition* (∗).

(∗) *G is an algebraic group over k and T is an algebraic subgroup over k of G such that is an isomorphism* $G\otimes_k\bar{k}\cong PGL(n)$ *over* $\bar{k}$ *which isomorphism induces an isomorphism* $T\otimes_k\bar{k}\cong P_{l_1, l_2, \cdots, l_r}$.

PROOF . If V=G/T, then V has a k-rational point p, the image of identity of G. Conversely, we suppose that if V has a k-rational point. For brevity we write H, P instead of $H_{l_1, l_2, \cdots, l_r}$, $P_{l_1, l_2, \cdots, l_r}$. It is well known that Aut^oH=PGL (n). Aut^oV is a k-form of Aut^oH. Set G be Aut^oV. Let T be the stabilizer group at p, namely $T=\{A\in G \mid Ap=p\}$ which is an algebraic subgroup of G over k. The variety V is isomorphic to G/T over k.

It is evident that G and T satisfy the condition (∗). Since V is a k-form of H, there is a $\bar{k}$-isomorphism ϕ : $V\otimes_k\bar{k}\cong H\otimes_k\bar{k}$. Let e be the point of H induced from the identity of PGL (n). We may assume that ϕ (P)=e, since H is a homogeneous. If we put $f(\sigma)=\phi\sigma\phi^{-1}$ in Aut^oV for an element $\sigma\in Aut^o$ (V), then f : G → PGL (n) is an isomorhism over $\bar{k}$. It follows from the definition of stabilizer group that maps $T\otimes_k\bar{k}$ onto $P\otimes_k\bar{k}$.

q.e.d.

Let us fix some notation. Let G be an algebraic group over k and T be a subgroup of G. Aut (G,T)=$\{\sigma\in AutG \mid \sigma T=T\}$

Θ (PGL (n), $P_{i_1, i_2, \cdots, i_l})_k$
$= \{$ (G, T) pair of G and T satisfiying the condition $(*)\}/\sim$, where (G, T)$\sim$(G', T') if and only if there is an isomorphism f: G $\rightarrow$ G' with f (T)=T' over k.

PROPOSITION 2. *With notations as above,* Θ (PGL (n), $P_{i_1, i_2, \cdots, i_l})_k$ *is bijective to* H^1 (*k, Aut* (PGL (n), $P_{i_1, i_2, \cdots, i_l}$)).

PROOF. For an class (G, T) of Θ (PGL (n), $P_{i_1, i_2, \cdots, i_l})_k$ there is a mapping f satisfying the condition $(*)$. Then we put $p_s = (f^{-1}) \cdot {}^s f$ for each $s \in$ Gal ($\bar{k}/k$). That is the required 1-cocycle.

Conversely, for any 1-cocycle p_s satisfying the condition $(*)$, we define the action of an element s of the Galois group Gal ($\bar{k}/k$) on (PGL (n), P) with p_s and $1 \otimes s$. Then the quotient object is the required pair.

q.e.d.

For detail on this subject, see Ch.3 of Serre [5].

LEMMA 1. *When* $n \geq 3$, *Aut* (PGL (n)) *is a semi-direct product* PGL $(n) \cdot \langle J \rangle$ *where* PGL (n) *is the group of inner automorphisms and* J : PGL (n) $\rightarrow$ PGL (n) *is the transposition, i.e.* $J(\sigma) = {}^t\sigma^{-1}$ *for* $\sigma \in$ PGL (n).

This is a classical result (see Theorem 2, Ch.3 of Dieudonné [4]).

LEMMA 2. *The normalizer of* $P_{i_1, i_2, \cdots, i_l}$ *in* PGL (n) *coincides with* $P_{i_1, i_2, \cdots, i_l}$ *itself.*

PROOF. Let us denote simply by P the algebraic subgroup $P_{i_1, i_2, \cdots, i_l}$. Let B be the set of all the upper triangular matrices. It is a Borel subgroup of PGL (n). If P is B, the this lemma is well known for Borel subgroup (Normalizer theorem). Let $X \in$ PGL (n) such that $XPX^{-1} \subset P$. B is a Borel subgroup of P. Then $XBX^{-1} \subset P$ and XBX^{-1} is a Borel subgroup of P. There exists an element $Y \in P$ such that $YXBX^{-1}Y^{-1} = B$. Then $YX \in B \subset P$. We conclude that $X \in P$.

q.e.d.

LEMMA 3. (1) *When* $(i_1, i_2, \cdots, i_l) \neq (i_l, i_{l-1}, \cdots, i_1)$, *Aut* (PGL ($n$), $P_{i_1, i_2, \cdots, i_l}$) $= P_{i_1, i_2, \cdots, i_l}$.

(2) *When* $(i_1, i_2, \cdots, i_l) = (i_l, i_{l-1}, \cdots, i_1)$, *Aut* (PGL($n$), $P_{i_1, i_2, \cdots, i_l}) = P_{i_1, i_2, \cdots, i_l} \cdot \langle I \rangle$

$$\textit{where } I = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix} J \textit{ in Aut } (\mathrm{PGL}\ (n)).$$

PROOF. Let σ be a member of PGL (n). If $\sigma P_{i_1, i_2, \cdots, i_l} \sigma^{-1} = P_{i_1, i_2, \cdots, i_l}$, then σ is $P_{i_1, i_2, \cdots, i_l}$ by Lemma 2. Recall that Aut (PGL (n))=PGL (n)•<J>. Assume that σ is an element satisfying the following condition: $\sigma J (P_{i_1, i_2, \cdots, i_l}) = P_{i_1, i_2, \cdots, i_l}$. Then we get $\sigma {}^tP_{i_1, i_2, \cdots, i_l} \sigma^{-1} = P_{i_1, i_2, \cdots, i_l}$. Let B_1 be the group of all the upper-triangular matrices and and B_2 be the group of all the lower-triangular matrices. $\sigma B_2 \sigma^{-1} \in P_{i_1, i_2, \cdots, i_l}$. $\sigma B_2 \sigma^{-1}$is the Borel subgroup of $P_{i_1, i_2, \cdots, i_l}$. There is a $\tau \in P_{i_1, i_2, \cdots, i_l}$ such that $\tau \sigma B_2 \sigma^{-1} \tau^{-1} = B_1$.

On the other hand, $B_2 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} B_1 \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$. So $\tau\sigma \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} B_1 \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \sigma^{-1}\tau^{-1} = B_1$. The Normalizer theorem leads that $\tau\sigma \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \in B_1 \subset P_{i_1, i_2, \cdots, i_l}$, then $\sigma \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \in P_{i_1, i_2, \cdots, i_l}$.

Conversely, let $\sigma = \nu \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \in P_{i_1, i_2, \cdots, i_l} \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$

Then we get that $(\nu \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} J) (P_{i_1, i_2, \cdots, i_l}) = \nu \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} {}^tP_{i_1, i_2, \cdots, i_l} \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \nu^{-1}$. If $(\nu \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix} J) (P_{i_1, i_2, \cdots, i_l}) = P_{i_1, i_2, \cdots, i_l}$, then $P_{i_l, i_{l-1}, \cdots, i_1} = P_{i_1, i_2, \cdots, i_l}$. We get that $(i_1, i_2, \cdots, i_l) = (i_l, i_{l-1}, \cdots, i_1)$.

q.e.d.

We can summarize these results as follows.

PROPOSITION 3. *Under the same situation of proposition 2,*

(1) *when* $(i_1, i_2, \cdots, i_l) \neq (i_l, i_{l-1}, \cdots, i_1)$, $\Theta\,(\mathrm{PGL}(n), P_{i_1, i_2, \cdots, i_l})_k \cong H^1 (k, \mathrm{PGL}\,(n), P_{i_1, i_2, \cdots, i_l})$.

(2) *when* $(i_1, i_2, \cdots, i_l) = (i_l, i_{l-1}, \cdots, i_1)$, $\Theta\,(\mathrm{PGL}\,(n), P_{i_1, i_2, \cdots, i_l})_k \cong H^1 (k, Aut\,(\mathrm{PGL}\,(n), P_{i_1, i_2, \cdots, i_l}.\langle \mathrm{I} \rangle)$

$$\text{where } I = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix} J \text{ in } Aut\ (\mathrm{PGL}\ (n)).$$

THEOREM 1. *Let k be a field of characteristic zero. If* $(i_1, i_2, \cdots, i_l) \cong (i_l, i_{l-1}, \cdots, i_1)$ *and* $1 \in \{i_1, i_2, , i_l\}$, *for a k-form V to have rational points it is necessary and sufficient that V is isomorphic to* $H_{i_1, i_2, \cdots, i_l}$ *over k.*

REMARK. For $H_{1,i}$ $(i \geq 2)$ this is Châtelet's theorem itself.

LEMMA 4. *If* $1 \in \{i_1, i_2, \cdots, i_l\}$, *then* $H^1(K, P_{i_1, i_2, \cdots, i_l}) = 1$.

PROOF. We consider the case $i_1 = 1$.
$P_{i_1, i_2, \cdots, i_l}$ has the following exact sequence.

$$1 \to K = \left\{ \begin{array}{c} \overset{1}{\frown}\ \overset{i_2}{\frown}\ \cdots\ \overset{i_l}{\frown} \\ \left[\begin{array}{c|c|c|c} 1 & * & * & * \\ \hline 0 & 1\ddots 1 & * & * \\ \hline 0 & 0 & \ddots & * \\ \hline 0 & 0 & 0 & 1\ddots 1 \end{array}\right] \begin{array}{l})1 \\ )i_2 \\ \vdots \\ )i_l \end{array} \end{array} \right\} \to P_{i_1, i_2, \cdots, i_l} \to GL(i_2) \times \cdots \times GL(i_l) \to 1$$

Then we get the exact sequnce as pointed sets.

$H^1(k, K) \to H^1(k, P_{1, i_2, \cdots, i_l}) \to H^1(k, GL(i_2) \times \cdots \times GL(i_l))$

The well known result $H^1(k, G_a) = 0$ implies $H^1(k, K) = 1$. And the Hilbert's theorem 90 says that $H^1(k, GL(i_2) \times \cdots \times GL(i_l)) = 1$. Then we conclude $H^1(k, P_{1, i_2, \cdots, i_l}) = 1$.

q.e.d.

PROOF of THEOREM 1. The condition is clearly sufficient.

Conversely, if V has a rational point, by proposition 1, there is a pair (G, T) satisfying the condition (*). Proposition 2 ascertains that under the situation of theorem 1, $H^1(k, P_{1, i_2, \cdots, i_l})$ completely classify the pairs. Then Lemma 4 said $H^1(k, P_{1, i_2, \cdots, i_l}) = 0$. There is only one pair $(PGL(n), P_{1, i_2, \cdots, i_l})$ over k up to k-isomorphism. V=G/T is isomorphic to $PGL(n)/P_{i_1, i_2, \cdots, i_l}$ over k.

q.e.d.

By the fundamental theorem of central simple algebra over an algebraic number field, we get the exact sequence

$$1 \rightarrow H^1(k, PGL(n)) \rightarrow \prod_{v:\text{ place}} H^1(k_v, PGL(n)).$$

By the language of k-form V of $H_{i_1, i_2, \cdots, i_l}$ this exact sequence means that if, for every place v, $V \otimes_k k_v$ is isomorphic to $H_{i_1, i_2, \cdots, i_l}$ over k_v, then V is isomorphic to $H_{i_1, i_2, \cdots, i_l}$ over k.

Therefore by Theorem 1, if, for any place v, $V \otimes_k k_v$ has a k_v-rational point then $V \otimes_k k_v$ is isomorphic to $H_{i_1, i_2, \cdots, i_l}$ over k_v. We conclude that V has a k-rational point. We can get the following statement.

THEOREM 1'. *Let k be an algebraic number field, v be a place of k and k_v be the completion of k relative to v.*

When $(i_1, i_2, \cdots, i_l) \neq (i_l, i_{l-1}, \cdots, i_1)$, $1 \in \{i_1, i_2, \cdots, i_l\}$ and $\underline{Aut(H_{i_1, i_2, \cdots, i_l}) = PGL(n)}$, then any k-form V of $H_{i_1, i_2, \cdots, i_l}$ holds the Hasse principle : If for all places v, $V \otimes_k k_v$ has a k_v-rational point, we conclude that V has a k-rational point.

REMARK. The underlined part can be omitted (See Prop. 4). This was tha starting point. At first we wondered if in general for every k-form of $H_{i_1, i_2, \cdots, i_l}$ the Châtelet property holded. But we could find an counter example.

EXAMPLE. *Let $k=\mathbf{R}$ (real number field) and $H_{4,2}=Gr(4, 2)$ (Grassmann variety). The $\mathbf{C}/\mathbf{R}$-form does not have the Châtelet property.*

PROOF. Gr (4, 2) is tha set of all zero points of $T_0T_1 - T_2T_3 - T_4T_5$ by using Plucker coordinate. Gr (4, 2) is isomorphic to some quadratic hyper surface V : $T_0 + T_1 + T_2 - T_3 - T_4 - T_5 = 0$ over $\mathbf{R}$. V_1 has a $\mathbf{R}$-form V_2 : $T_0 + T_1 + T_2 + T_3 + T_4 - T_5 = 0$ over $\mathbf{R}$. But these two varieties are not isomorphic over $\mathbf{R}$. Indeed, a biregular morphism must be projective since Pic $V_1 = \mathbf{Z}\, O(1)$. Aut V_1 is the set of all the projective transformations which stabilize V_1, the dimension of a maximal kernel space of a projective-linear invariant. The variety V_1 has the dimension 3 and V_2 has 1. Clearly, each of the two has $\mathbf{R}$-rational points.

q.e.d.

REMARK. In this paper, we could not cover the k-forms of the type as $H_{4,2}=Gr(4, 2)$. But, this is a good counter example.

Let us review the relation between forms and Galois cohomology.

(2.1) Let V be a projective varieties (algebraic groups) over a field of characteristic zero.

There is a natural bijection between the set of k-forms of V up to k-isomorphism and H^1 (k, Aut V) (see Prop. 4, Ch. 3 of Serre [5]).

We use the word *k-form* both as an algebraic variety and as an algebraic group. In context we can easily understand which k-form is used.

The previous consideration leads us to the next proposition.

PROPOSITION 4. *Let ψ be the natural mapping from Aut* (PGL (n), $P_{i_1, i_2, \cdots, i_l}$) *into Aut* $(H_{i_1, i_2, \cdots, i_l})$ *and ψ' be the mapping induced from ψ such that ψ' : H^1 (k, Aut* (PGL (n), $P_{i_1, i_2, \cdots, i_l}$) $\rightarrow H^1$ *(k, Aut* $(H_{i_1, i_2, \cdots, i_l})$).

For a k-form V of $H_{i_1, i_2, \cdots, i_l}$ to have rational points, it is necessary and sufficient the corresponding 1-cohomology class $c \in H^1$ (k, Aut $(H_{i_1, i_2, \cdots, i_l})$) is in the image of the ψ'.

It is compatible with the mapping which maps the class of $(G, T) \in \Theta$ (PGL (n), $P_{i_1, i_2, \cdots, i_l})_k$ *to G/T.*

PROOF. Let a k-form V of H has rational points. By (2.1) and Proposition 1 and 2, V is isomorphic to G/T over k. There is a k-isomorphism $f: G \otimes_k \bar{k} \rightarrow PGL(n) \otimes_k \bar{k}$ such that $f \mid T \otimes_k \bar{k}$ is also an isomorphism onto $H_{i_1, i_2, \cdots, i_l} \otimes_k \bar{k}$.

For $s \in Gal(\bar{k}/k)$, we define $p_s = f \cdot {}^s f$. Then (p_s) is a 1-cocycle. And also f induces $f' : (G/T) \otimes_k \bar{k} \rightarrow H_{i_1, i_2, \cdots, i_l} \otimes_k \bar{k}$ and similarly we define 1-cocycle (p_s') such that $f'((p_s)) = (p'_s)$.

Conversely, let 1-cocycle (p'_s) of H^1 (k, Aut $(H_{i_1, i_2, \cdots, i_l})$). Let an pair (G, T) correspond to 1-cocycle (p_s). Then G/T corresponds to (p'_s).

q.e.d.

This proposition can be generalized for other kinds of homogeneous space instead of $H_{i_1, i_2, \cdots, i_l}$. We will use this generalization to attack the similar problem of homogeneous space of other types.

We assume that H=G/T and Aut^0 (H)=G, for k-form of H. Let ψ be the natural mapping Aut (G, T) $\rightarrow$ Aut (H) and ψ' be the mapping induced from ψ such that H^1 (k, Aut (G, T)) $\rightarrow H^1$ (k, Aut (H)). For a k-form V of H to have rational points it is necessary and sufficient that the corresponding 1-cohomology class $c \in H^1$ (k, Aut (H)) is in the image of the ψ.

PPROPOSITION 5. *Let k be a field of characteristics zero.*

(1) *When $(i_1, i_2, \cdots, i_l) \neq (i_l, i_{l-1}, \cdots, i_1)$, Aut $(H_{i_1, i_2, \cdots, i_l}) \cong$* PGL (n).

(2) *When $(i_1, i_2, \cdots, i_l) = (i_l, i_{l-1}, \cdots, i_1)$, Aut $(H_{i_1, i_2, \cdots, i_l}) \cong$* PGL $(n) \cdot \langle$ I $\rangle$,

where I is the automorphism of $H_{i_1, i_2, \cdots, i_t}$ induced from the element $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$ J of Aut (PGL (n), $P_{i_1, i_2, \cdots, i_t}$).

PROOF. Aut^o $(H_{i_1, i_2, \cdots, i_t})$ is a normal subgroup. It is well known that Aut^o $(H_{i_1, i_2, \cdots, i_t})=$ PGL (n). We define the homomorphism from Aut $(H_{i_1, i_2, \cdots, i_t})$ to Aut (Aut^o $(H_{i_1, i_2, \cdots, i_t})$) such that for each $\sigma \in$ Aut $(H_{i_1, i_2, \cdots, i_t})$, $\psi(\sigma)(\tau)=\sigma\tau\sigma^{-1}$ ($\tau \in$ Aut^o $(H_{i_1, i_2, \cdots, i_t})$).

Let us prove its injectivity first.
Assume that $\psi(\sigma)$ is the unit, for any $\tau \in$ Aut^o $(H_{i_1, i_2, \cdots, i_t})$, $\sigma\tau\sigma^{-1}=\tau$, $\sigma\tau=\tau\sigma$ as elements of Aut $(H_{i_1, i_2, \cdots, i_t})$. We represent the points of the homogeneous space $H_{i_1, i_2, \cdots, i_t}$ by using the language of cosets of PGL (n)/$P_{i_1, i_2, \cdots, i_t}$, for example $\gamma P_{i_1, i_2, \cdots, i_t}$. Let us remember that Aut^o $(H_{i_1, i_2, \cdots, i_t})=$PGL (n). We get $\sigma(\tau\gamma P_{i_1, i_2, \cdots, i_t})=\tau\sigma(\gamma P_{i_1, i_2, \cdots, i_t})$. Put γ be the unit e. $\sigma(\tau P_{i_1, i_2, \cdots, i_t})=\tau\sigma(P_{i_1, i_2, \cdots, i_t})$. σ is determined uniquely by $\sigma(P_{i_1, i_2, \cdots, i_t})$. Let $\sigma(P_{i_1, i_2, \cdots, i_t})$ be $\delta P_{i_1, i_2, \cdots, i_t}$ for some $\delta \in$ PGL (n). Since δ is well defined, for each $\zeta \in P_{i_1, i_2, \cdots, i_t}$, $\delta(\zeta P_{i_1, i_2, \cdots, i_t})=\delta(P_{i_1, i_2, \cdots, i_t})$, $\zeta(\sigma P_{i_1, i_2, \cdots, i_t})=\sigma(P_{i_1, i_2, \cdots, i_t})$, $\zeta\delta P_{i_1, i_2, \cdots, i_t}=\delta P_{i_1, i_2, \cdots, i_t}$.

$P_{i_1, i_2, \cdots, i_t}\delta P_{i_1, i_2, \cdots, i_t}=\delta P_{i_1, i_2, \cdots, i_t}$. We reach that $P_{i_1, i_2, \cdots, i_t}\delta=\delta P_{i_1, i_2, \cdots, i_t}$. By Lemma 2, $\delta \in P_{i_1, i_2, \cdots, i_t}$. We find that δ is the identity mapping. We conclude that ψ is injective.

Next, we will show that if ψ is surjective, ${}^tP_{i_1, i_2, \cdots, i_t}=\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$ $P_{i_1, i_2, \cdots, i_t}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$. Lemma 1 says that Aut (Aut^o $(H_{i_1, i_2, \cdots, i_t})$)=Aut (PGL (n))=PGL (n)$\cdot$ <J>. Then ψ be surjective.

q.e.d.

§2. Translasion into the language of central simple algebras

We will review the result of central simple algebras and attack the problem by using the language of them as possible.

NOTATION. We set

A (n,k)=the set of all the isomorphic classes of the central simple algebras over k which are isomorphic to $M_n(\bar{k})$ over $\bar{k}$
=the set of the k-forms of $M_n(k)$ as algebra.

Let us recall the following facts.

(2.2) $\mathrm{Aut}_{\mathrm{algebra}}(M_n(k)) = PGL(n)$

where PGL (n) acts on $M_n(k)$ as an inner automorphism (the theorem of Skolem-Noether, see Prop. 4, Ch. 9 of Weil [7]).

(2.3) $A(n, k) = H^1(k, \mathrm{Aut}_{\mathrm{algebra}}(M_n(k))) = H^1(k, PGL(n))$

where for each class of a central simle algebra A of A (n, k) there is an isomorphism $f: A \otimes_k \bar{k} \to M_n(k) \otimes_k \bar{k}$ (see Prop. 8, Ch. 10 of Serre [5]).

We define 1-cocycle (p_s) such that $p_s = f^{-1} \cdot {}^s f$.

NOTATION. For the natural injection $\psi: P_{i_1, i_2, \cdots, i_t} \to PGL(n)$ it induces the mapping $\psi: H^1(k, P_{i_1, i_2, \cdots, i_t}) \to H^1(k, PGL(n)) = A(n, k)$. Then we denote the image of ψ in A (n, k), $IM_{i_1, i_2, \cdots, i_t}$.

For the general result of central simple algebras, see Ch. 9. of Weil [7]. For k-forms and Galois cohomology, see Ch. 10. of Serre [6].

LEMMA 5. *If* $1 \in \{i_1, i_2, \cdots, i_t\}$, *then* $IM_{i_1, i_2, \cdots, i_t} = 0$.

PROOF. It is only a translation of Lemma 4. Under this condition $H^1(k, P_{i_1, i_2, \cdots, i_t}) = 0$. $IM_{i_1, i_2, \cdots, i_t}$ is the image of it. Therefore we get $IM_{i_1, i_2, \cdots, i_t} = 0$.

q.e.d.

We will review the result of central simple algebra especially over $\mathfrak{z}$-adic number field (the completion relative to a finite place) and attack the problem mainly over these fields. Let k be a $\mathfrak{z}$-adic field in the followimg (2.4) - (2.6). Let Br (k) be the Brauer group of k.

(2.4) The Brauer group Br (k) is isomorphic to $\mathbf{Q}/\mathbf{Z}$ (see Prop. 6, Ch. 13 of Serre [6]).

(2.5) The natural injection $A(n, k) \to Br(k)$ is a natural injection (by the definition).

(2.6) The image of the composite morphism another $A(n, k) \to Br(k) \to \mathbf{Q}/\mathbf{Z}$ coincides with $\frac{1}{n}\mathbf{Z}/\mathbf{Z}$ (see Cor. 3, Prop. 7, Ch. 13 of Serre [6]).

(2.7) For any central simple algebra A to be isomorphic to $M_n(k)$ over an extension field of degree r, it is necessary and sufficient that order of the class of A divides r. Then there is $C \in A(r, k)$ such that $A = M_{\frac{n}{r}}(C)$ (see Cor. 3, Prop. 7, Ch. 13 of Serre [6]).

Let K be a finite extension of k. If the 1-cocycle $(p_s) \in H^1(k, PGL(n))$ corresponds to a central simple algebra A, then 1-cocycle $(p_s) \in H^1(K, PGL(n))$ corresponds to $A \otimes_k K$.

PROPOSITION 6. *Let k be a $\mathfrak{z}$-adic field. for $A \in A$ (n, k) to be in $IM_{i_1, i_2, \cdots, i_l}$, it is necessary and sufficient that there exists an $C \in A$ (d, k), where d=the greatest common measure of $(i_1, i_2, \cdots, i_l)$, such that $A = M_{\frac{n}{d}}(C)$.*

PROOF. We will show that it is the sufficient condition. At first we will prove it for the case $l=2$. Let A be in IM_{i_1, i_2}. We can choose a 1-cocycle (p_s) such that $p_s \in P_{i_1, i_2}$.

$$p_s = \begin{pmatrix} p_{1s} & * \\ 0 & p_{2s} \end{pmatrix} \text{ where } (p_{1s}) \text{ is a 1-cocycle of } H^1(K, PGL(i_1)).$$

There exists an extension field K of degree i_1 such that (p_{1s}) is trivial over K. There is $f \in PGL$ (i_1) that $p_{1s} = f^{-1} \cdot {}^s f$. Then $f p_{1s} f^{-1} = 1$ in PGL (i_1). Further put $p_s' = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} p_s {}^s \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}^{-1}$, then (p_s) is cohomologous to (p_s'). On the other hand, $p_s' \in P_{1, 1, \cdots, i_2}$. Therefore $A \otimes_k K$ is in $IM_{1, 1, \cdots, i_2}$ is trivial. $A \otimes_k K$ is trivial. Then we say that the order of A divides i_1, and also divides n. By (2.7) there is $C \in A$ (d_1, k), where d_1=the greatest common measure of (n, i_1), such that $A = M_{\frac{n}{d}}(C)$.

For $l \geq 3$, we remark that $P_{i_1, i_2, \cdots, i_l} \subset P_{i_1 + \cdots + i_j, i_{j+1} + \cdots + i_l}$ where $1 \leq j \leq l-1$. For each j, we get that the order of $A \in IM_{i_1, i_2, \cdots, i_l}$ divides the greatest common measure of $(n, i_1 + \cdots + i_j)$. By varying j from i to l, we conclude that the order of A divides $(n, i_1, i_2, \cdots, i_l)$. There exists $C \in$ A (d, k), where d is the greatest common measure of $(n, i_1, i_2, \cdots, i_l)$. There exists $C \in A$ (d, k) that A is isomorphic to $M_{\frac{n}{d}}(C)$.

Conversely, let us show the necessity. Assume that $A \in A$ (n, k) such that $A = M_r(C)$ where $C \in A$ (i, k) and $ri = n$. Let (p_{1s}) be the corresponding 1-cocycle to A. We will show that (p_{1s}) can be chose under the condition that $p_{1s} \in P_{i, i, \cdots, i}$. We put $f'_1 : M(\bar{k}) \to C \otimes_k \bar{k}$ and f' naturally induces the isomorphism $f: M_n(k) = M_r(M_i(k)) \to M_r(C \otimes_k \bar{k}) = M_r(C) \otimes_k \bar{k}$. The $f^{-1} \cdot {}^s f$ is uniquely determinde by $f'^{-1} \cdot {}^s f'$. Indeed if $f(X) \in C \otimes_k \bar{k}$ for every $X \in M_i(k)$ then $f([X_{k,l}]) = [f'(X_{k,l})]_{k,l}$ for each $[X_{k,l}] \in M_r(M_i(k))$. We have $f^{-1} \cdot {}^s f([X_{k,l}]) = ([f'^{-1} \cdot {}^s f'])$. On the other hand $f'^{-1} \cdot {}^s f'$ is an automorphism of $M_i(k)$. Then it is an inner automorphism. Therefore there exists an $q_s \in GL(i)$ such that $f'^{-1} \cdot {}^s f'(X) = q_s X q_s$. $f^{-1} \cdot {}^s f([X_{k,l}]) = ([f'^{-1} \cdot {}^s f' X_{k,l}]) = [q_s X_{k,l} q_s^{-1}] =$

$\begin{pmatrix} q_s & & \\ & \ddots & \\ & & q_s \end{pmatrix} [X_{k,l}] \begin{pmatrix} q_s & & \\ & \ddots & \\ & & q_s \end{pmatrix}^{-1}$. We define $p_s = \begin{pmatrix} q_s & & \\ & \ddots & \\ & & q_s \end{pmatrix}$. The (p_s) is a 1-cocycle of H^1 (k, PGL (n)) corresponding to A. Then $A \subset IM_{i, i, \cdots, i} \subset IM_{i_1, i_2, \cdots, i_l}$.

q.e.d.

The necessity does not require the condition of the basic field. So we conclude the following proposition.

PROPOSITION 7. *Let k be a perfect field. If there exists an* $C\in A\ (d, k)$, *where d is the greatest common measure of* $(i_1, i_2, \cdots, i_l)$, *such that* $A=M_{\frac{n}{d}}(C)$, *then* $A\in A\ (n, k)$ *is in* $IM_{i_1, i_2, \cdots, i_l}$.

(2.8)If k=**R**, then Br $(\mathbf{R})=\mathbf{Z}/2\mathbf{Z}$.
If k=**C**, then Br $(\mathbf{C})=0$.

PROPOSITION 8. *If* $k=\mathbf{C}$, *then* $IM_{i_1, i_2, \cdots, i_l}=0$.

If $k=\mathbf{R}$, *then* $IM_{i_1, i_2, \cdots, i_l}=\begin{cases} 0 & (2\nmid d) \\ \mathbf{Z}/2\mathbf{Z} & (2\mid d) \end{cases}$

PROOF. When $k=\mathbf{C}$, $IM_{i_1, i_2, \cdots, i_l}\subset A\ (n, \mathbf{C})\subset Br\ (\mathbf{C})=0$. When $k=\mathbf{R}$ and $2\mid d$, $IM_{i_1, i_2, \cdots, i_l}\subset A\ (n, \mathbf{R})\subset Br\ (\mathbf{R})$. By proposition 7 the order of $IM_{i_1, i_2, \cdots, i_l}$ is two. Then $IM_{i_1, i_2, \cdots, i_l}=A\ (n, \mathbf{R})$. When k=**R** and $2\nmid d$, there is an j such that $u=i_1+\cdots+i_j$ is odd. We will say that $IM_{u,v}$ is trivial for odd u. Let $\begin{pmatrix} p_{1s} & * \\ & p_{1s} \end{pmatrix}$ be the corresponding 1-cocycle to $A\in IM_{u,v}$. Then we can choose (p_{1s}mod G_m) trivial since u is odd. Then $IM_{u,v}\subset IM_{1,\dots,1,v}=0$.

q.e.d.

We can rewrite these two propositions as follows.

PROPOSITION 9. *Let k be a completion of an algebraic number field relative to a place. For* $A\in A\ (n, k)$ *to be in* $IM_{i_1, i_2, \cdots, i_l}$ *it is necessary and sufficient that the order of A divides the greatest common measure of* $(i_1, i_2, \cdots, i_l)$.

Let us review the fundamental theorem of central simple algebras over an algebraic number field.

(2.9) Let k be an algebraic number field, v be a place, k_v be the completion relative to v.

(1) $1\rightarrow Br\ (k)\rightarrow \Pi_v Br\ (k_v)\rightarrow \mathbf{Q}/\mathbf{Z}\rightarrow 1$ (exact)

(2) For $A\in Br\ (k)$ (exactly speaking, the the class of A) to have the order of divisor of r, it is necessary and sufficient that there is an $C\in A\ (r, k)$ such that $A=M_{\frac{n}{r}}(C)$ where $r\mid n$.

Now we will have two main propositions.

PROPOSITION 10. *Let k be an algebraic number field. The statement that* $IM_{i_1, i_2, \cdots, i_l}$ *is trivial is equivalent to that* $d=1$ *where d is the greatest common measure of* $(i_1, i_2, \cdots, i_l)$.

PROOF. Assume that d=1. If $A \in IM_{i_1, i_2, \cdots, i_l}$, then $A \otimes_k k_v$ is in $IM_{i_1, i_2, \cdots, i_l}$. For each local field $IM_{i_1, i_2, \cdots, i_l}$ is trivial by Proposition 9. The statement (1) of the previous Review, especially its injectivity, say that A is trivial.

Conversely, if $d \neq 1$, let us choose two different finite places, v_1 and v_2. Let A_{v_1} and A_{v_2} be in $IM_{i_1, i_2, \cdots, i_l}$ for k_{v_1} and k_{v_2} such that $inv_{v_1} A_{v_1} = \frac{1}{d}$ and $inv_{v_2} A_{v_2} = -\frac{1}{d}$ respectively. For every place v except for v_1 and v_2 we define A_v trivial algebra. The statement (1) of the previous (2.9), especially its surjectivity, says that there is an nontrivial algebra $C \in A$ (d, K). Therefore $IM_{i_1, i_2, \cdots, i_l}$ is not trivial.

q.e.d.

PROPOSITION 11. *Let k be an algebraic number field and* $A \in A$ *(n, k). If* $A \otimes_k k_v$ *in in* $IM_{i_1, i_2, \cdots, i_l}$ *for all places, then A is in* $IM_{i_1, i_2, \cdots, i_l}$ *relative to the field k.*

PROOF. By Proposition 9, the assumption suggests that the order of $A \otimes_k k_v$ in Br (k_v) divides d. By the exact sequence of the (2.9), so does the order of A in Br (k). Then there is an $C \in A$ (d, k) such that $A = M_{\frac{n}{d}}$ (C). Furthermore, by Proposition 7, A is in $IM_{i_1, i_2, \cdots, i_l}$ relative to the filed k.

q.e.d.

§3. Conclusion

We translate proposition 10. and 11. back into the original language in section 1. We can get the following two theorems.

THEOREM 2. *We assume that* $(i_1, i_2, \cdots, i_l) \neq (i_l, i_{l-1}, \cdots, i_1)$.

Let k be an algebraic number field or a $\mathfrak{z}$*-adic field. A* k-form of $H_{i_1, i_2, \cdots, i_l}$ have the Châtelet property if and only if the greatest common measure of $(i_1, i_2, \cdots, i_l)$ *is equal to* 1.

Let k be **R**. *A* k-form of $H_{i_1, i_2, \cdots, i_l}$ have the Châtelet property if and only if the greatest common measure of $(2, i_1, i_2, \cdots, i_l)$ *is equal to* 1.

THEOREM 3. *Let k be an algebraic number field.*
If $(i_1, i_2, \cdots, i_l) \neq (i_l, i_{l-1}, \cdots, i_1)$, *then the each k-form V of* $H_{i_1, i_2, \cdots, i_l}$ *holds the Hasse principle ;*

If $V \otimes_k k_v$ *has a* k_v*-rational point for every place v, then the variety V* has a k-rational point.

Now, we have answered our two problems for certain homogeneous space of type PGL (n)/ $P_{i_1, i_2, \cdots, i_l}$. That is, whenever we could translate the problems into those of central simple algebras, we could solve them.

However we can extend Theorem 3 even in the case where $(i_1, i_2, \cdots, i_l) = (i_l, i_{l-1}, \cdots, i_1)$ and the greatest common measure of $(i_1, i_2, \cdots, i_l)$ is 1. Furthermore, in Theorem 2 we can omit the assumption: $(i_1, i_2, \cdots, i_l) \neq (i_l, i_{l-1}, \cdots, i_1)$. But we have to use the language of algebras with involution and Hermitian forms over algebraic number fields. We shall treat these subjects in the next paper.

References

[1] E. Artin and J. Tate, Class field theory, Benjamin, Massachusetts, 1967.
[2] S. Cassels and A. Frolich, Algebraic number theory, Academic press, London, 1967.
[3] M. Demazure, Automorphismes et déformations des varietes de Borel, Invent. Math., **39** (1977), 179-186.
[4] J. Dieudonné, On the automorphisms of classical groups, AMS, 1951.
[5] J. P. Serre, Cohomologie Galoisienne (Lecture notes in Mathematics No. 5), Springer-Verlag, Berlin•Heidelberg•New York, 1962.
[6] J. P. Serre, Corps Locaux, Hermann, Paris, 1962.
[7] A. Weil, Basic Number Theory (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen Band 144), Springer-Verlag, Berlin•Heidelberg•New York, 1967.

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