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CENTERS OF TWISTED CHEVALLEY GROUPS OVER COMMUTATIVE RINGS

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Introduction.

In [3], E. Abe and J. F. Hurley have studied the structure of centers of Chevalley groups over commutative rings. In this paper, applying the same method as used in [3], we study the structure of centers of twisted Chevalley groups over commutative rings. The main theorem is stated in § 1. In § 2, we deal with some properties related to maximal tori, and in § 3, we prove the main theorem.

We shall freely use the definitions, the notations and the relations between elements of the group given in [2] on twisted Chevalley groups over commutative rings.

§ 1. Statement of the main theorem.

1. 1. Let G be an almost simple Chevalley-Demazure group scheme of type $\Phi=A_n (n\geqq 2)$, $D_m (m\geqq 4)$ or E_6, and Γ the lattice of weights of the representation which defines G (as for definition, see E. Abe [1]).

We set Γ_{sc} or Γ_{ad} instead of Γ respectively, if G is of universal or adjoint type. Then $\Gamma_{ad} \subseteq \Gamma \subseteq \Gamma_{sc}$. Let σ be the canonical involutive automorphism of Φ and denote by the same symbol as σ the involutive automorphism of Γ_{sc} or Γ_{ad} induced by σ. Then Γ is σ-stable, that is, $\sigma\Gamma \subseteq \Gamma$, except for $\Phi=D_n$, n is even, $\geqq 4$ and $\Gamma_{ad} \subsetneqq \Gamma \subsetneqq \Gamma_{sc}$.

Let A be a commutative ring with 1 and with an involutive automorphism σ and assume that Γ is σ-stable. Let $G(\Phi, A)$ be the group of A-valued points of G. We can set $G(\Phi, A) = Hom(Z[G], A)$ where $Z[G]$ is a Hopf-algebra over Z. Then we see that σ induces an involutive automorphism σ of Hopf-algebra $Z[G]$. Let $T(\Phi, A) \cong Hom(\Gamma, A^*)$ be the standard maximal torus of $G(\Phi, A)$ where A^* is the group of units of A, and denote by $h(\chi)$ the element of $T(\Phi, A)$ corresponding to $\chi \in Hom(\Gamma, A^*)$. We set $\sigma(\chi)(\gamma)=\sigma(\chi(\sigma(\gamma)))$ for any $\gamma \in \Gamma$. Then we have the automorphism σ of $G(\Phi, A)=Hom(Z[G], A)$ induced by σ which satisfies the following (see E. Abe [2]). For an unipotent element $x_\alpha(a)$ of $G(\Phi, A)$ where $\alpha \in \Phi$, $a \in A$,

$$\sigma(x_a(a))=x_{\bar{a}}(c_a\bar{a}) \quad \text{where } \bar{a}=\sigma(a) \text{ and } \bar{a}=\sigma(a),\ c_a=\pm 1$$

and for any $h(\chi)\in T(\Phi, A)$,

$$\sigma(h(\chi))=h(\bar{\chi}) \qquad \text{where } \bar{\chi}=\sigma(\chi).$$

We put

$$G_\sigma(\Phi, A)=\{x\in G(\Phi, A)\mid \sigma x=x\}.$$

It is called the twisted Chevalley group over A of type Φ_σ associated with G.

Denote $A_0=\{u\in A\mid u=\bar{u}\}$ and $\mathfrak{A}=\{(a, b)\in A\times A\mid a\bar{a}=b+\bar{b}\}$. For any $R\in\Phi_\sigma$, we define

$$(G\,\mathrm{I}) \qquad x_R(u)=x_r(u) \qquad \text{for } u\in A_0,\ R=\{r\}$$

$$(G\,\mathrm{II}) \qquad x_R(a)=x_r(a)x_{\bar{r}}(\bar{a}) \qquad \text{for } a\in A,\ R=\{r, \bar{r}\}$$

$$(G\,\mathrm{III}) \qquad x_R(\xi)=x_r(a)x_{\bar{r}}(\bar{a})x_{r+\bar{r}}(N_{r,\bar{r}}b) \qquad \text{for } \xi=(a, b)\in\mathfrak{A},\ R=\{r, \bar{r}, r+\bar{r}\} \text{ and } N_{r,\bar{r}}=\pm 1.$$

Then these elements are contained in $G_\sigma(\Phi, A)$. We denote by $E(\Phi_\sigma, A)$ the group generated by these elements, and call it the elementary subgroup of $G_\sigma(\Phi, A)$.
For the product of $x_s(\xi)$, $x_s(\eta)$, we use the following notation

$$x_s(\xi)x_s(\eta)=x_s(\xi\dotplus\eta) \qquad \text{for } \xi,\ \eta\in\mathfrak{A} \text{ and } S=\{r, \bar{r}, r+\bar{r}\}$$

Then for any $\xi=(a, b)$, $\eta=(c, d)\in\mathfrak{A}$, we have $\xi\dotplus\eta=(a+c, b+d+a\bar{c})$. Thus the set $\mathfrak{A}$ has a structure of group with the composition $\dotplus$ and the inverse of $\xi=(a, b)$ is $\xi^*=(-a, \bar{b})$. Further we have the following operation $\rightharpoonup$ of A on $\mathfrak{A}$, that is, $c\rightharpoonup(a, b)=(ca, c\bar{c}b)$ for any $c\in A$ and $\xi=(a, b)\in\mathfrak{A}$.

1. 2. Let $C_{G,\sigma}(A)$ and $C_{E,\sigma}(A)$ be the centers of $G_\sigma(\Phi, A)$ and $E(\Phi_\sigma, A)$ respectively, and $C_{G,E,\sigma}(A)$ the centralizer of $E(\Phi_\sigma, A)$ in $G_\sigma(\Phi, A)$. Then by definition, $C_{G,\sigma}(A)\subseteq C_{G,E,\sigma}(A)$ and $C_{E,\sigma}(A)=C_{G,E,\sigma}(A)\cap E(\Phi_\sigma, A)$. We put $\mathfrak{A}^*=\{(a, b)\in\mathfrak{A}\mid b\in A^*\}$ and

$$Hom_1(\Gamma, A^*)=\{\chi\in Hom(\Gamma, A^*) \mid \chi=\bar{\chi}\}.$$

Here we state the main theorem in the following.

THEOREM. *Suppose that A has trivial Jacobson radical or that Φ_σ has rank at least 2. Furthermore assume that $\mathfrak{A}$ has an element $(a, b)\in\mathfrak{A}$, $a\in A^*$ and $\mathfrak{A}^*\neq\phi$, if Φ_σ is of type ${}^2A_{2n}$ and the radical of A is not trivial. Then we have*

$$C_{G,E,\sigma}(A)=C_{G,\sigma}(A)=Hom_1(\Gamma/\Gamma_{ad}, A^*).$$

If G is of universal or adjoint type, then

$$C_{G,\sigma}(A)=C_{E,\sigma}(A)=C_{G,E,\sigma}(A).$$

If G is of adjoint type, then $C_{G,\sigma}(A)$ is trivial.

For any elements x, y and z of a group, we write ${}^xy=xyx^{-1}$ and $[x, y]=xyx^{-1}y^{-1}$. We shall use the commutator relation

$$[x, yz]=[x, y]\cdot {}^y[x, z].$$

§ 2. Properties related to maximal tori.

2. 1. Let A be a commutative ring with 1 and with an involutive automorphism σ, and $G_\sigma(\Phi, A)$ the twisted Chevalley group over A. We put

$$w_R(u)=x_R(u)x_{-R}(-u^{-1})x_R(u) \text{ and } h_R(u)=w_R(u)w_R(-1)$$
$$\text{for } u\in A_0^* \text{ or } u\in A^* \text{ if } R=\{r\} \text{ or } R=\{r, \bar{r}\} \text{ respectively,}$$

and

$$w_R(\xi)=x_R(\xi)x_{-R}(-b^{-1}\rightharpoonup\xi)x_R(b\bar{b}^{-1}\rightharpoonup\xi)$$
$$h_R(\xi, \eta)=w_R(\xi)w_R(\eta) \qquad \text{for } \xi=(a, b),\ \eta\in\mathfrak{A}^* \text{ if } R=\{r, \bar{r}, r+\bar{r}\}.$$

Then we have

$$h_R(u)=h_r(u) \qquad \text{if } R=\{r\}$$
$$h_R(u)=h_r(u)h_{\bar r}(\bar u) \qquad \text{if } R=\{r, \bar r\}$$
$$h_R(\xi, \eta)=h_r(c)h_{\bar r}(\bar c) \qquad \text{for } c=b(\xi)b(\eta)^{-1} \text{ if } R=\{r, \bar r, r+\bar r\}$$

where we denote by $b(\xi)$ the second component of ξ.

Furthermore we put

$$T_1(A)=\{h(\chi) \mid \chi\in Hom_1(\Gamma, A^*)\},$$
$$T_0(A)=\{h(\chi)\in Hom_1(\Gamma, A)^* \mid h(\chi) \text{ is centralized by } E(\Phi_\sigma, A)\}.$$

Then

$$[h(\chi), x_R(a)]=x_R((\chi(r)-1)a) \qquad \text{for } a\in A_0 \text{ or } a\in A \text{ respectively if } R=\{r\} \text{ or } R=\{r, \bar r\},$$
$$[h(\chi), x_R(\xi)]=x_R((\chi(r)\rightharpoonup\xi)\dotplus\xi^*) \quad \text{if } R=\{r, \bar r, r+\bar r\}.$$

Therefore, we have $T_0(A)\cong Hom_1(\Gamma/\Gamma_{ad}, A^*)$.

2. 2. PROPOSITION 1. *Let* $C'_{G,E,\sigma}(A)=C_{G,E,\sigma}(A)\cap T_1(A)$ *and* $C'_{G,\sigma}(A)=C_{G,\sigma}(A)\cap T_1(A)$. *Then we have*

$$C'_{G,\sigma}(A)=C'_{G,E,\sigma}(A)=T_0(A)$$

PROOF. By definition $C'_{G,\sigma}(A)\subseteq C'_{G,E,\sigma}(A)$. On the other hand we have shown that $C'_{G,E,\sigma}(A)=T_0(A)\cong Hom_1(\Gamma/\Gamma_{ad}, A^*)$. In [3] we see that $Hom(\Gamma/\Gamma_{ad}, A^*)$ is isomorphic to the center of $G(\Phi, A)$, then we have $T_0(A)\subseteq C'_{G,\sigma}(A)$.

To prove $C_{G,\sigma}(A)=C_{G,E,\sigma}(A)$, it is sufficient to prove that

$$C_{G,E,\sigma}(A) \subseteq T_0(A).$$

In next section, we shall show this fact under the hypotheses of the main theorem (see section 1).

§ 3. Proof of the main theorem.

3. 1. Let Φ be a root system and Π a fixed fundamental basis of Φ. Denote by Φ^+ or Φ^- the set of positive or negative roots associated with Π respectively. Here we shall fix a regular order in Φ. By definition, if $\alpha < \beta$ for $\alpha, \beta \in \Phi$, then $ht(\alpha) \leq ht(\beta)$ where $ht(\alpha) = \Sigma m_i$, for $\alpha = \Sigma m_i \alpha_i$, $m_i \in Z$ and $\Pi = \{\alpha_1, \cdots, \alpha_l\}$. Then the regular order in Φ induces the following order in Φ.

DEFINITION. *1) For $R, S \in \Phi_1^+$,*
$R \angle S$ if and only if $Min(R) < Min(S)$.
2) For $\alpha \in R$, $\beta \in S$
$\alpha \angle \beta$ if and only if a) $R = S$, $\alpha < \beta$
or b) $R \neq S$, $R \angle S$.

We set $ht(R) = Min\{ht(r) \mid r \in R\}$ and call it the height of R (see N. Iwahori[5]).
We set

$$U_1^+(A) = U^+(A) \cap G_\sigma(\Phi, A) \text{ and } U_1^-(A) = U^-(A) \cap G_\sigma(\Phi, A)$$

where $U^+(A) = \prod_{\alpha \in \Phi^+} x_\alpha(A)$ and $U^-(A) = \prod_{\beta \in \Phi^-} x_\beta(A)$.

Then for any $x \in U_1^+(A)$ and $y \in U_1^-(A)$, we have

$$x = \prod_{R \in \Phi_\sigma^+} x_R(a)$$

$$y = \prod_{R \in \Phi_\sigma^-} x_R(a) \qquad \text{where } a \in A_0,\ A \text{ or } \mathfrak{A},$$

and this expression is unique where the product is taken in the order $\angle$.

3. 2. Let W be the Weyl group associated with root system Φ and σ the canonical automorphism of Φ. Put $W_1 = \{w \in W \mid w = \sigma w \sigma^{-1}\}$. Let $\mathfrak{W}_1$ be the subgroup of $G_\sigma(\Phi, A)$

generated by the elements of $\{w_R(u), w_R(\xi) \mid R\in\Phi_\sigma, u\in A_0^* \text{ or } A^* \text{ and } \xi\in\mathfrak{A}^*\}$ and $T_1(A)$, then we see that $\mathfrak{W}_1/T_1(A)\cong W_1$. Let w be a representative of $\mathfrak{W}_1$ modulo $T_1(A)$, we identify w with the corresponding element of the group W_1.

For a field K, we have the Bruhat decomposition of $G_\sigma(\Phi, K)$, that is, for any element g of $G_\sigma(\Phi, K)$ we can write

$$g=utwu' \text{ for some } u\in U_1^+(K),\ t\in T_1(K),\ w\in W_1 \text{ and } u'\in U_{w,1}^+,$$

where $U_{w,1}^+$ is the subgroup of $U_1^+(K)$ generated by the elements of $\{x_R(a) \mid R\in\Phi_\sigma^+, w(R)\angle 0, a\in K_0, K \text{ or } \mathfrak{A}\}$. This expression is unique.

3. 3. LEMMA 1. *Let K be a field, Then we have $C_{G,E,\sigma}(K)\subseteq T_1(K)$.*

PROOF. If $w\in W_1$, $w\neq 1$, then we have a root $R\succeq 0$ such that $w(R)\angle 0$. Then for $z\in C_{G,E,\sigma}(K)$, we set

$$x_R(a)z=x_R(a)utwu'=zx_R(a)=utwu'x_R(a).$$

By uniqueness of expression, we must have $x_R(a)u=u$ for any $a\in A_0$, A or $\mathfrak{A}$. This is a contradiction, and $w=1$, therefore $C_{G,E,\sigma}(K)\subseteq U_1^+(K)T_1(K)$. Since there exists an element w_0 of W_1 such that $w_0(\Phi_\sigma^+)\subseteq\Phi_\sigma^-$, we have

$$C_{G,E,\sigma}(K)\subseteq U_1^+(K)T_1(K)\cap U_1^-(K)T_1(K)=T_1(K).$$

3. 4. We set $\Omega(A)=U^+(A)T(A)U^-(A)$. Then there exists an element d in $Z[G]$ such that $\Omega(A)=\{x\in G(\Phi, A) \mid d(x)\in A^*\}$ ([4], Section 4). Since for $g\in\Omega(A)$, $g=utv$ where $u\in U^+(A)$, $t\in T(A)$, $v\in U^-(A)$ is unique expression, we have

$$\Omega_1(A)=\Omega(A)\cap G_\sigma(\Phi, A)=U_1^+(A)T_1(A)U_1^-(A).$$

LEMMA 2. $C_{G,E,\sigma}(A)\subseteq U_1^+(J)T_1(A)U_1^-(J)$
where J is the Jacobson radical of A. In particular if $J=0$, then

$$C_{G,E,\sigma}(A)\subseteq T_1(A)$$

PROOF. Let M be any maximal ideal of A and $\pi: G_\sigma(\Phi, A)\rightarrow G_\sigma(\Phi, A/M)$ the homomorphism induced by the canonical map: $A\rightarrow A/M$. Then for any element z of $C_{G,E,\sigma}(A)$, $\pi(z)$ belongs to $C_{G,E,\sigma}(A/M)$. By lemma 1 $\pi(z)$ belongs to $T_1(A/M)$, and then there exists an element d of $Z[G]$ such that $d(z)\notin M$. Thus $d(z)\in A^*$, that is, $z\in\Omega_1(A)$. We can set $z=xhy$ for some $x\in U_1^+(A)$, $h\in T_1(A)$ and $y\in U_1^-(A)$ and $\pi(z)=\pi(x)\pi(h)\pi(y)\in T_1(A/M)$, then $\pi(x)=\pi(y)=1$, that is, $x\in U_1^+(M)$ and $y\in U_1^-(M)$ for any maximal ideal M of A. Therefore $z\in U_1^+(J)T_1(A)U_1^-(J)$.

3. 4. PROPOSITION 3. If *rank* $\Phi_\sigma>1$, *then* $C_{G,E,\sigma}(A)\subseteq T_1(A)$.

PROOF. For $z\in C_{G,E,\sigma}(A)$, we set

$$z=xhy \qquad \text{where } x\in U_1^+(J),\ h\in T_1(A) \text{ and } y\in U_1^-(J)$$

and

$$x=\prod_{R>0}x_R(a_R) \text{ and } y=\prod_{R>0}x_{-R}(b_R)$$

where a_R and b_R belong to J or $\mathfrak{A}_J=\{(a, b)\in\mathfrak{A}\mid a, b\in J\}$. The product of this expression is taken in the order fixed above. We shall use induction on heights to show that for every positive root R, a_R and b_R are zero element in A or $\mathfrak{A}$.

(1) First, suppose that a_R is not zero element with $ht(R)=1$. since *rank* $\Phi_\sigma>1$, there is a root S such that $R+S\in\Phi_\sigma$ and $ht(S)=1$. Then

$$[x_S(a_S), z]=[x_S(a_S), xh]\cdot {}^{xh}[x_S(a_S), y],$$

this is conjugate to

$$^{h^{-1}x^{-1}}[x_S(a_S), xh][x_S(a_S), y],$$

and

$$[x_s(a_s),\ x_{-s}(u)] \in x_s(J)\,T_1(A)x_{-s}(J) \text{ for } u\in J \text{ or } u\in \mathfrak{A}_J,$$

then we have

$$[x_s(a_s),\ y] \in x_s(v)\,T_1(A)\,U_1^-(J) \text{ for some } v\in J \text{ or } \mathfrak{A}_J$$

On the other hand, we have

$$[x_s(a_s),\ xh] \in U_1^+(J)\,T_1(A)$$

If $a_s \in A^*$ or $a_s=(a,\ b)\in \mathfrak{A}$, $a\in A^*$, then $[x_s(a_s),\ xh]$ has a factor $x_{s+R}(u)\neq 1$. Thus

$$^{h^{-1}x^{-1}}[x_s(a_s),\ z] = {}^{h^{-1}x^{-1}}[x_s(a_s),\ xh][x_s(a_s),\ y] = x_1h_1y_1,$$

where $x_1 \in U_1^+(A)$, $h_1\in T_1(A)$ and $y_1\in U_1^-(A)$. By $z\in C_{G,E,\sigma}(A)$ and uniqueness of the expression of z, $x_1=h_1=y_1=1$, this is contradictory.
Let w_0 be the element of W_1 such that $\omega_0(\Phi_\sigma^+)=\Phi_\sigma^-$. Then the elements

$$z={}^{w_0}z \text{ and } {}^{w_0}y\cdot{}^{w_0}x\cdot{}^{w_0}h=x_1'h_1'y_1'$$

are conjugate, where $x_1'={}^{w_0}y=w_0yw_0^{-1}\in U_1^+(J)$, $h_1'={}^{w_0}h\in T_1(A)$ and $y_1'={}^{w_0}h^{-1}\cdot{}^{w_0}x\cdot{}^{w_0}h\in U_1^-(J)$. By uniqueness of the expression, y contains no fuctors $x_{-R}(a_{-R})$ with $ht(R)=1$.

(2) We assume that x and y contain no factors $x_R(a_R)$ or $x_{-R}(a_{-R})$ for $R\in\Phi_\sigma^+$ with $ht(R)\leqq m$. Then if $a_R\neq 0$ for some $R\in\Phi_\sigma^+$ of height $m+1$, there exists a root S such that $ht(S)=1$ and $ht(w_s(R))\leqq m$ where w_s is the refrection of Φ_σ corresponding to S. Thus

$$z={}^{W_s(u_S)}z={}^{W_s(u_S)}x\cdot{}^{W_s(u_S)}h\cdot{}^{W_s(u_S)}y=x_2h_2y_2$$

where $u_s\in A^*$ or $u_s=(a_s,\ b_s)\in\mathfrak{A}^*$, and $x_2\in U_1^+$, $h_2\in T_1(A)$ and $y_2\in U_1^-$. In x_2 there exists the non-trivial factor $x_{w_s(R)}(a)$. This would contradict the inductive hypothesis. Hence $a_R=0$ and as above also, $a_{-R}=0$ for all $R\in\Phi_\sigma^+$ with $ht(R)=m+1$, therefore $z=h_1$ belong to $T_1(A)$.

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