CENTERS OF TWISTED CHEVALLEY GROUPS
OVER COMMUTATIVE RINGS

Kazuo SUZUKI

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Introduction.

In [3], E. Abe and J. F. Hurley have studied the structure of centers of Chevalley groups over commutative rings. In this paper, applying the same method as used in [3], we study the structure of centers of twisted Chevalley groups over commutative rings. The main theorem is stated in § 1. In § 2, we deal with some properties related to maximal tori, and in § 3, we prove the main theorem.

We shall freely use the definitions, the notations and the relations between elements of the group given in [2] on twisted Chevalley groups over commutative rings.

§ 1. Statement of the main theorem.

1. 1. Let \( G \) be an almost simple Chevalley-Demazure group scheme of type \( \Phi = A_n(n \geq 2), D_m(m \geq 4) \) or \( E_6 \), and \( \Gamma \) the lattice of weights of the representation which defines \( G \) (as for definition, see E. Abe [1]).

We set \( \Gamma^c \) or \( \Gamma^{ad} \) instead of \( \Gamma \) respectively, if \( G \) is of universal or adjoint type. Then \( \Gamma^{ad} \subseteq \Gamma \leq \Gamma^c \). Let \( \sigma \) be the canonical involutive automorphism of \( \Phi \) and denote by the same symbol as \( \sigma \) the involutive automorphism of \( \Gamma^c \) or \( \Gamma^{ad} \) induced by \( \sigma \). Then \( \Gamma \) is \( \sigma \)-stable, that is, \( \sigma \Gamma \subseteq \Gamma \), except for \( \Phi = D_n, n \) is even, \( \geq 4 \) and \( \Gamma^{ad} \not\subseteq \Gamma \).

Let \( A \) be a commutative ring with \( 1 \) and with an involutive automorphism \( \sigma \) and assume that \( \Gamma \) is \( \sigma \)-stable. Let \( G(\Phi, A) \) be the group of \( A \)-valued points of \( G \). We can set \( G(\Phi, A) = \text{Hom}(\mathbb{Z}[G], A) \) where \( \mathbb{Z}[G] \) is a Hopf-algebra over \( \mathbb{Z} \). Then we see that \( \sigma \) induces an involutive automorphism \( \sigma \) of Hopf-algebra \( \mathbb{Z}[G] \). Let \( T(\Phi, A) = \text{Hom}(\Gamma, A^*) \) be the standard maximal torus of \( G(\Phi, A) \) where \( A^* \) is the group of units of \( A \), and denote by \( h(\chi) \) the element of \( T(\Phi, A) \) corresponding to \( \chi \in \text{Hom}(\Gamma, A^*) \). We set \( \sigma(\chi)(\gamma) = \sigma(\chi(\sigma(\gamma))) \) for any \( \gamma \in \Gamma \). Then we have the automorphism \( \sigma \) of \( G(\Phi, A) = \text{Hom}(\mathbb{Z}[G], A) \) induced by \( \sigma \) which satisfies the following (see E. Abe [2]). For an unipotent element \( x_\Phi(a) \) of \( G(\Phi, A) \) where \( a \in \Phi, a \in A \),
\[
s(\tau(a)) = x_{\tilde{a}}(c_{\tilde{a}} \tilde{a}) \quad \text{where} \quad \tilde{a} = \sigma(a) \text{ and } \tilde{a} = \sigma(a), \ c_{\tilde{a}} = \pm 1
\]

and for any \( h(\chi) \in T(\Phi, A) \),
\[
\sigma(h(\chi)) = h(\tilde{\chi}) \quad \text{where} \quad \tilde{\chi} = \sigma(\chi).
\]

We put
\[
G_\sigma(\Phi, A) = \{ x \in G(\Phi, A) \mid \sigma(x) = x \}.
\]

It is called the twisted Chevalley group over \( A \) of type \( \Phi_\sigma \) associated with \( G \).

Denote \( A_\sigma = \{ u \in A \mid u = \tilde{u} \} \) and \( \mathcal{A} = \{ (a, b) \in A \times A \mid a\tilde{a} = b + \tilde{b} \} \). For any \( R \in \Phi_\sigma \), we define
\[
\begin{align*}
(G\ I) & \quad x_\sigma(u) = x_\tau(u) \quad \text{for} \ u \in A_\sigma, \ R = \{ r \} \\
(G\ II) & \quad x_\sigma(a) = x_\tau(a)x_\tau(\tilde{a}) \quad \text{for} \ a \in A, \ R = \{ r, \tilde{r} \} \\
(G\ III) & \quad x_\sigma(\xi) = x_\tau(a)x_\tau(\tilde{a})x_\tau(\tilde{a})x_\tau(N_r, \tilde{r}, b) \quad \text{for} \ \xi = (a, b) \in \mathcal{A}, \\
& \quad \quad \quad R = \{ r, \tilde{r}, \ r + \tilde{r} \} \text{ and } N_r, \tilde{r} = \pm 1.
\end{align*}
\]

Then these elements are contained in \( G_\sigma(\Phi, A) \). We denote by \( E(\Phi_\sigma, A) \) the group generated by these elements, and call it the elementary subgroup of \( G_\sigma(\Phi, A) \).

For the product of \( x_\sigma(\xi), x_\sigma(\eta) \), we use the following notation
\[
x_\sigma(\xi)x_\sigma(\eta) = x_\sigma(\xi + \eta) \quad \text{for} \ \xi, \eta \in \mathcal{A} \text{ and } \mathcal{S} = \{ r, \tilde{r}, \ r + \tilde{r} \}
\]

Then for any \( \xi = (a, b), \ \eta = (c, d) \in \mathcal{A} \), we have \( \xi + \eta = (a + c, b + d + a \tilde{c}) \). Thus the set \( \mathcal{A} \) has a structure of group with the composition \( + \) and the inverse of \( \xi = (a, b) \) is \( \xi^* = (-a, \tilde{b}) \).

Further we have the following operation \( \neg \) of \( A \) on \( \mathcal{A} \), that is, \( c \neg (a, b) = (ca, c\tilde{c}b) \) for any \( c \in A \) and \( \xi = (a, b) \in \mathcal{A} \).

1. 2. Let \( C_{G_\sigma}(A) \) and \( C_{E_\sigma}(A) \) be the centers of \( G_\sigma(\Phi, A) \) and \( E(\Phi_\sigma, A) \) respectively, and \( C_{G_\sigma}(A) \) the centralizer of \( E(\Phi_\sigma, A) \) in \( G_\sigma(\Phi, A) \). Then by definition, \( C_{G_\sigma}(A) \subseteq C_{E_\sigma}(A) \) and \( C_{E_\sigma}(A) = C_{G_\sigma}(A) \cap E(\Phi_\sigma, A) \). We put \( \mathcal{A}^* = \{ (a, b) \in \mathcal{A} \mid b \in A^* \} \) and
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\[ \text{Hom}(\Gamma', A^*) = \{ \chi \in \text{Hom}(\Gamma', A^*) \mid \chi = \overline{\chi} \}. \]

Here we state the main theorem in the following.

**THEOREM.** Suppose that \( A \) has trivial Jacobson radical or that \( \Phi_\sigma \) has rank at least 2. Furthermore assume that \( \mathfrak{a} \) has an element \((a, b) \in \mathfrak{a}, a \in A^* \) and \( \mathfrak{a}^* \neq \phi, \) if \( \Phi_\sigma \) is of type \( 2^{2n} \) and the radical of \( A \) is not trivial. Then we have

\[ C_{G, \sigma}(A) = C_{G, \sigma}(A) = \text{Hom}(\Gamma'_{ad}, A^*). \]

If \( G \) is of universal or adjoint type, then

\[ C_{G, \sigma}(A) = C_{G, \sigma}(A) = C_{G, \sigma}(A). \]

If \( G \) is of adjoint type, then \( C_{G, \sigma}(A) \) is trivial.

For any elements \( x, y \) and \( z \) of a group, we write \( ^x y = x y x^{-1} \) and \( [x, y] = x y x^{-1} y^{-1} \). We shall use the commutator relation

\[ [x, y z] = [x, y] \cdot ^x [x, z]. \]

§ 2. Properties related to maximal tori.

2. 1. Let \( A \) be a commutative ring with \( 1 \) and with an involutive automorphism \( \sigma \), and \( G_\sigma(\Phi, A) \) the twisted Chevalley group over \( A \). We put

\[ w_b(u) = \text{id} x_b(u)(-u^{-1})x_b(u) \quad \text{and} \quad h_b(u) = w_b(u)w_b(-1) \]

for \( u \in A_b^* \) or \( u \in A^* \) if \( R = \{ r \} \) or \( R = \{ r, \overline{r} \} \) respectively,

and

\[ w_b(\xi) = \text{id} x_b(\xi)(-b^{-1} \rightarrow \xi)x_b(b \overline{b}^{-1} \rightarrow \xi) \]

\[ h_b(\xi, \eta) = w_b(\xi)w_b(\eta) \quad \text{for} \quad \xi = (a, b), \eta \in A^* \quad \text{if} \quad R = \{ r, \overline{r}, r + \overline{r} \}. \]
Then we have

\[ h_s(u) = h_r(u) \quad \text{if} \quad R = \{ r \} \]

\[ h_s(u) = h_r(u) h_r(u) \quad \text{if} \quad R = \{ r, \bar{r} \} \]

\[ h_s(\xi, \eta) = h_r(c) h_r(\bar{c}) \quad \text{for} \quad c = b(\xi)b(\eta)^{-1} \quad \text{if} \quad R = \{ r, \bar{r}, r + \bar{r} \} \]

where we denote by \( b(\xi) \) the second component of \( \xi \).

Furthermore we put

\[ T_1(A) = \{ h(\chi) \mid \chi \in Hom_1(\Gamma, A^*) \}, \]

\[ T_0(A) = \{ h(\chi) \in Hom_0(\Gamma, A)^* \mid h(\chi) \text{ is centralized by } E(\Phi, A) \}. \]

Then

\[ [h(\chi), x_s(a)] = x_s((\chi(r) - 1)a) \quad \text{for} \quad a \in A_0 \text{ or } a \in A \text{ respectively} \]

\[ \text{if} \quad R = \{ r \} \text{ or } R = \{ r, \bar{r} \}, \]

\[ [h(\chi), x_s(\xi)] = x_s((\chi(r) - \xi) + \xi^*) \quad \text{if} \quad R = \{ r, \bar{r}, r + \bar{r} \}. \]

Therefore, we have \( T_0(A) \cong Hom_0(\Gamma/\Gamma_{\text{ad}}, A^*) \).

2. 2. PROPOSITION 1. Let \( C'_{c,e,\sigma}(A) = C_{c,e,\sigma}(A) \cap T_1(A) \) and \( C'_{c,\sigma}(A) = C_{c,\sigma}(A) \cap T_1(A) \). Then we have

\[ C'_{c,\sigma}(A) = C'_{c,e,\sigma}(A) = T_0(A) \]

PROOF. By definition \( C'_{c,\sigma}(A) \subseteq C'_{c,e,\sigma}(A) \). On the other hand we have shown that \( C'_{c,e,\sigma}(A) = T_0(A) \cong Hom_0(\Gamma/\Gamma_{\text{ad}}, A^*) \). In [3] we see that \( Hom(\Gamma/\Gamma_{\text{ad}}, A^*) \) is isomorphic to the center of \( G(\Phi, A) \), then we have \( T_0(A) \subseteq C'_{c,\sigma}(A) \).

To prove \( C_{c,\sigma}(A) = C_{c,e,\sigma}(A) \), it is sufficient to prove that
\[ C_{c,e,s}(A) \subseteq T_0(A). \]

In next section, we shall show this fact under the hypotheses of the main theorem (see section 1).

\section*{§ 3. Proof of the main theorem.}

3. 1. Let \( \Phi \) be a root system and \( \Pi \) a fixed fundamental basis of \( \Phi \). Denote by \( \Phi^+ \) or \( \Phi^- \) the set of positive or negative roots associated with \( \Pi \) respectively. Here we shall fix a regular order in \( \Phi \). By definition, if \( \alpha < \beta \) for \( \alpha, \beta \in \Phi \), then \( ht(\alpha) \leq ht(\beta) \) where \( ht(\alpha) = \Sigma m_i \) for \( \alpha = \Sigma m_i \alpha_i, m_i \in \mathbb{Z} \) and \( \Pi = \{\alpha_i, \ldots, \alpha_l\} \). Then the regular order in \( \Phi \) induces the following order in \( \Phi \).

\textbf{Definition.} 1) For \( R, S \in \Phi^\prime \),
\[ R \triangledown S \text{ if and only if } \text{Min}(R) < \text{Min}(S). \]
2) For \( \alpha \in R, \beta \in S \)
\[ \alpha \triangledown \beta \text{ if and only if } \begin{cases} a) & R = S, \alpha < \beta \\ or & b) & R \neq S, R \triangledown S. \end{cases} \]

We set \( h(R) = \text{Min}(ht(r) \mid r \in R) \) and call it the height of \( R \) (see N. Iwahori[5]).

We set
\[ U^\prime(A) = U^+(A) \cap G_0(\Phi, A) \text{ and } U^\prime(A) = U^-(A) \cap G_0(\Phi, A) \]
where \( U^+(A) = \prod_{\alpha \in \Phi^+} x_\alpha(A) \) and \( U^-(A) = \prod_{\alpha \in \Phi^-} x_\alpha(A) \).

Then for any \( x \in U^\prime(A) \) and \( y \in U^\prime(A) \), we have
\[ x = \prod_{\alpha \in \Phi^+} x_\alpha(a) \]
\[ y = \prod_{\alpha \in \Phi^-} x_\alpha(a) \text{ where } a \in A_0, A \text{ or } \mathfrak{a}, \]
and this expression is unique where the product is taken in the order \( \triangledown \).

3. 2. Let \( W \) be the Weyl group associated with root system \( \Phi \) and \( \sigma \) the canonical automorphism of \( \Phi \). Put \( W = \{w \in W \mid w = \sigma w \sigma^{-1}\} \). Let \( \mathfrak{B} \) be the subgroup of \( G_0(\Phi, A) \)
generated by the elements of \( \{ w_R(u), w_R(\xi) \mid R \in \Phi_\sigma, u \in A_0^* \text{ or } A^* \text{ and } \xi \in \mathfrak{W} \} \) and \( T_1(A) \), then we see that \( \mathfrak{W}_1/T_1(A) \cong W_1 \). Let \( w \) be a representative of \( \mathfrak{W}_1 \) modulo \( T_1(A) \), we identify \( w \) with the corresponding element of the group \( W_1 \).

For a field \( K \), we have the Bruhat decomposition of \( G_0(\Phi, K) \), that is, for any element \( g \) of \( G_0(\Phi, K) \) we can write

\[
g = utwu' \quad \text{for some } u \in U_t(K), \, t \in T_1(K), \, w \in W_1 \text{ and } u' \in U_{v,1}^*. \]

where \( U_{v,1}^* \) is the subgroup of \( U_t(K) \) generated by the elements of \( \{ x(a) \mid R \in \Phi_\sigma^*, w(R) \leq 0, \, a \in K_0, \, K \text{ or } \mathfrak{W} \} \). This expression is unique.

3.3. Lemma 1. Let \( K \) be a field, Then we have \( C_{G,\Phi,\sigma}(K) \subseteq T_1(K) \).

Proof. If \( w \in W_1, \, w \neq 1 \), then we have a root \( R \geq 0 \) such that \( w(R) \leq 0 \). Then for \( z \in C_{G,\Phi,\sigma}(K) \), we set

\[
x_{w}(a)z = x_{w}(a)utwu' = zx_{w}(a) = utwu'x_{w}(a). \]

By uniqueness of expression, we must have \( x_{w}(a)u = u \) for any \( a \in A_0, \, A \) or \( \mathfrak{W} \). This is a contradiction, and \( w = 1 \), therefore \( C_{G,\Phi,\sigma}(K) \subseteq U_t(K)T_1(K) \). Since there exists an element \( w_0 \) of \( W_1 \) such that \( w_0(\Phi_\sigma^*) \subseteq \Phi_\sigma \), we have

\[
C_{G,\Phi,\sigma}(K) \subseteq U_t(K)T_1(K) \cap U_1(K)T_1(K) = T_1(K). \]

3.4. We set \( \Omega(A) = U^*(A)T(A)U^-(A) \). Then there exists an element \( d \) in \( Z[G] \) such that \( \Omega(A) = (x \in G(\Phi, A) \mid d(x) \in A^*) \) ([4], Section 4). Since for \( g \in \Omega(A) \), \( g = utv \) where \( u \in U^+(A), \, t \in T(A), \, v \in U^-(A) \) is unique expression, we have

\[
\Omega_1(A) = \Omega(A) \cap G_0(\Phi, A) = U^+(A)T_1(A)U_1(A). \]

Lemma 2. \( C_{G,\Phi,\sigma}(A) \subseteq U_1(J)T_1(A)U_1(J) \)

where \( J \) is the Jacobson radical of \( A \). In particular if \( J = 0 \), then
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\[ C_{G,F,A} \subseteq T_i(A) \]

PROOF. Let \( M \) be any maximal ideal of \( A \) and \( \pi: G_0(\Phi, A) \to G_0(\Phi, A/M) \) the homomorphism induced by the canonical map \( A \to A/M \). Then for any element \( z \) of \( C_{G,F,A}(A) \), \( \pi(z) \) belongs to \( C_{G,F,A}(A/M) \). By lemma 1 \( \pi(z) \) belongs to \( T_i(A/M) \), and then there exists an element \( d \) of \( Z[G] \) such that \( d(z) \notin M \). Thus \( d(z) \notin A^* \), that is, \( z \in \Omega_i(A) \). We can set \( z = xhy \) for some \( x \in U_i(A) \), \( h \in T_i(A) \) and \( y \in U_i(A) \) and \( \pi(x) = \pi(x) \pi(h) \pi(y) \in T_i(A/M) \), then \( \pi(x) = \pi(y) = 1 \), that is, \( x \in U_i(M) \) and \( y \in U_i(M) \) for any maximal ideal \( M \) of \( A \). Therefore \( z \in U_i(J) T_i(A) U_i(J) \).

3.4. PROPOSITION 3. If rank \( \Phi_0 \geq 1 \), then \( C_{G,F,A}(A) \subseteq T_i(A) \).

PROOF. For \( z \in C_{G,F,A}(A) \), we set

\[ z = xhy \quad \text{where} \quad x \in U_i(J), \ h \in T_i(A) \text{ and } y \in U_i(J) \]

and

\[ x = \prod_{R \leq 0} x_R(a_R) \quad \text{and} \quad y = \prod_{R \leq 0} x_{-R}(b_R) \]

where \( a_R \) and \( b_R \) belong to \( J \) or \( \mathcal{Y} = \{(a, b) \in \mathcal{Y} | a, b \in J\} \). The product of this expression is taken in the order fixed above. We shall use induction on heights to show that for every positive root \( R \), \( a_R \) and \( b_R \) are zero element in \( A \) or \( \mathcal{Y} \).

(1) First, suppose that \( a_R \) is not zero element with \( ht(R) = 1 \). Since rank \( \Phi_0 \geq 1 \), there is a root \( S \) such that \( R + S \in \Phi_0 \) and \( ht(S) = 1 \). Then

\[ [x, a_R, z] = [x, a_R, xh] \cdot x^k[x, a_R, y] \]

this is conjugate to

\[ x^{ht(R)} [x, a_R, xh][x, a_R, y] \]

and
\([x_{\delta}(\pm), x_{-\delta}(\pm)] \in x_{\delta}(J) T_i(A)x_{-\delta}(J)\) for \(u \in J\) or \(u \in \mathbb{N}_j\).

then we have

\([x_{\delta}(\pm), y] \in x_{\delta}(v) T_i(A) U_i(J)\) for some \(v \in J\) or \(\mathbb{N}_j\).

On the other hand, we have

\([x_{\delta}(\pm), xh] \in U_i(J) T_i(A)\)

If \(a_\delta \in A^*\) or \(a_\delta = (a, b) \in \mathbb{N}, a \in A^*\), then \([x_{\delta}(\pm), xh]\) has a factor \(x_{\delta + \delta}(u) \neq 1\). Thus

\[x^{-1}\sigma^{-1}[x_{\delta}(\pm), z] = x^{-1}\sigma^{-1}[x_{\delta}(\pm), xh][x_{\delta}(\pm), y] = x_i h_i y_i,\]

where \(x_i \in U_i(A), h_i \in T_i(A)\) and \(y_i \in U_i(A)\). By \(z \in C_{C_{c_{e}}} A\) and uniqueness of the expression of \(z\), \(x_i = h_i = y_i = 1\), this is contradictory.

Let \(\omega_0\) be the element of \(W_i\) such that \(\omega_0(\Phi_\delta) = \Phi_\sigma\). Then the elements

\[z = \omega_0 z\quad \text{and} \quad x_i = \omega_0 x_i \quad \text{and} \quad x_i = \omega_0 x_i \quad \text{and} \quad h_i = \omega_0 h_i\]

are conjugate, where \(x_i = \omega_0 y = u_0 w_0 y \in U_i(J)\), \(h_i = \omega_0 h \in T_i(A)\) and \(y_i = \omega_0 h^{-1} x_i \omega_0 h \in U_i(J)\).

By uniqueness of the expression, \(y\) contains no factors \(x_{-\delta}(a_\delta)\) with \(ht(R) = 1\).

(2) We assume that \(x\) and \(y\) contain no factors \(x_{\delta}(a_\delta)\) or \(x_{-\delta}(a_{-\delta})\) for \(R \in \Phi_\delta\) with \(ht(R) \leq m\). Then if \(a_\delta \neq 0\) for some \(R \in \Phi_\delta\) of height \(m + 1\), there exists a root \(S\) such that \(ht(S) = 1\) and \(ht(w_0(R)) \leq m\) where \(w_0\) is the reflection of \(\Phi_\delta\) corresponding to \(S\). Thus

\[z = \omega_0 u_0 z = \omega_0 u_0 x_i \quad \text{and} \quad \omega_0 u_0 h i = x_i h_i y_i,\]

where \(u_0 \in A^*\) or \(u_0 = (a, b) \in \mathbb{N}^*\), and \(x_i \in U_i, h_i \in T_i(A)\) and \(y_i \in U_i\). In \(x_i\) there exists the non-trivial factor \(x_{\omega_0(R)}(a)\). This would contradict the inductive hypothesis. Hence \(a_\delta = 0\) and as above also, \(a_{-\delta} = 0\) for all \(R \in \Phi_\delta\) with \(ht(R) = m + 1\), therefore \(z = h_i\) belong to \(T_i(A)\).
References.


Department of Mathematics
Faculty of Education
Kumamoto University