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SOME MATHEMATICAL PROBLEMS ON THE UNIFICATION OF FORCES

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1. Introduction

The unification of forces seems to be the main theme of elementary particle physics. The same idea can happen to reach mathematically interesting conclusions. The objective of the present paper is to give some such examples, and especially to show that the canonical connections ω on the universal bundles over the Grassmann mainfolds of dimension n satisfy the Yang-Mills equation

$$D*F=d*F+\omega\wedge*F-(-1)^{n}*F\wedge\omega=0. \tag{0. 1}$$

where F is the curvature form of ω. It should be noted that Yang-Mills fields in higher dimensions hold their own value even physically [14].

The electroweak theory or the Glashow-Salam-Weinberg theory [1], [12], unites the electromagnetic force and the weak force. Let M be a Riemannian manifold. An electromagnetic gauge field is given by a $u(1)$-connection on a principal $U(1)$-bundle over M, while the intermediate vector bosons are given by the components of a Yang-Mills $su(2)$-connection form on a principal $SU(2)$-bundle over M. They are united in an $su(2)\times u(1)$-valued form on a principal $SU(2)\times U(1)$-bundle over the same manifold. We sometimes encounter similar situations in mathematics. Take the example of a flag manifold

$$M=\frac{U_{K}(m)}{U_{K}(m_{1})\times\cdots\times U_{K}(m_{k})},$$

over which there are k canonical bundles: a $U_{K}(m_{1})$-bundle, a $U_{K}(m_{2})$-bundle, $\cdots$, a $U_{K}(m_{k})$-bundle, where $m=m_{1}+\cdots+m_{k}$ (see **Sec.** 4 for the notations). Each of these bundles has its own canonical connection. These connections are united in a gravitational field. It follows from this fact that each canonical connection satisfies the Yang-Mills equation, though we follow a little bit different line, namely the Kaluza-Klein frame work in the proof.

In **Sec. 2** we review some aspects of the Kaluza-Klein metric from a mathematical point of view.

In **Sec. 3** we explain some mathematics of the unification of forces.

In **Sec. 4** we prove the following fact. *Let* $G_n(\mathbf{K}^N)$ *be the Grassmann manifold of* n *-planes in vector* N*-space over* **K** *where* **K** *is the field of real numbers, complex numbers or quaternions. Then the universal connection on* $G_n(\mathbf{K}^N)$ (*more correctly the canonical connection on the Stiefel bundle* [11]) *satisfies the Yang-Mills equation* (0. 1).

After a brief exposition of the Glashow-Salam-Weinberg theory, **Sec. 3** is devoted to the calculation of the Maxwell equation in the electroweak field. Suppose there are an electromagnetic field and a Yang-Mills Gauge $SU(2)$-field. The electromagnetic field, interacting with intermediate vector bosons, generates more current than usually known. The extra term, mathematically unavoidable, is related to the size of the internal space, i. e. the fibre of the principal bundle on which the unified connection form is defined. We calculate the exact form of this current.

In **Sec. 5** we retale our story on the special case of quaternionic projective space and the canonical connection. The curvature form F, in this case, satisfies a self-dual type equation

$$*F=\pm F\wedge\cdots\wedge F.$$

The origin of the present paper was a conversation with Professor S. Kobayashi when the author stayed in University of California at Berkeley a couple of years ago. He would like to express his gratitude to Professor Kobayashi.

2. Kaluza-Klein Metric [7], [13], [14]

Let G be a compact connected r-dimensional Lie group with Lie algebra **G**. Let P be a principal G-bundle over a manifold M of dimension m. $A\in\mathbf{G}$ generates a 1-parameter subgroup $a_t\in G$ $(-\infty<t<\infty)$ and defines a vector field on P, denoted A^* and called a fundamental vector field [8], such that

$$A_p^*=\frac{d}{dt}(pa_t)_{t=0}$$

for $p\in P$. A connection form ω on P is a **G**-valued 1-form on P with the following two properties

$$\omega(Xg)=Ad(g^{-1})\omega(X) \qquad \text{for } X\in T(P) \text{ and } g\in G$$

$$\omega(A_p^*)=A \qquad \text{for } A\in G$$

where $T(P)$ denotes the tangent bundle of P[1]. We write $G_m(P)_p$ for the Grassmann manifold of m-planes in $T(P)_p$, the tangent space at p of P. Then

$$G_m(P)=\bigcup_{p\in P} G_m(P)_p$$

with the natural projection is a bundle over P with fibre $G_m(\mathbf{R}^{m+r})$, the Grassman manifold of m-planes in $\mathbf{R}^{m+r}$. G acts on $G_m(P)$ naturally. We denote by $\Gamma(P, G_m(P))^G$ the set of G-invariant sections of $G_m(P)$. The sections $\sigma\in\Gamma(P, G_m(P))^G$ such that $\sigma(p)$ is complementary to $T(pG)_p$ for any $p\in P$ (where pG is the fibre though p of P) will be denoted by $\Gamma_h(P, G_m(P))^G$. Given $\sigma\in\Gamma_h(P, G_m(P))^G$, $X\in\Gamma(P, T(P))$ decomposes into the sum

$$X_p=(X_h)_p+(X_v)_p \qquad (p\in P)$$

where $(X_h)_p\in\sigma(p)$ *and* $(X_v)_p\in T(pG)_p$.

From now on assume that G *is equipped with a bi-invariant Riemannian metric* $\mathbf{g}_G$. Let *Rie* (M), *Rie* (P), ... be the totality of Riemannian metrics on M, P, $\cdots$. Then a pair $(\mathbf{g}, \sigma)\in$ *Rie* $(M)\times\Gamma_h(P, G_m(P))^G$ gives rise to a metric $\tilde{\mathbf{g}}\in$ *Rie* (P) if we put

$$\tilde{\mathbf{g}}(X)=\mathbf{g}(\pi_*(X))+\mathbf{g}_G(X_v)$$

where we denote the projection: $P\to M$ by π. We write *Rie* $(P)^G$ for the subset of G-invariant metrics in *Rie* (P). Since the metric $\tilde{\mathbf{g}}$ defined above is clearly G-invariant, we have a natural map:

$$Rie\ (M)\times\Gamma_h(P, G_m(P))^G\to Rie\ (P)^G \tag{2. 1}$$

Fix a metric $\mathbf{g}$ on M and we have a map:
$\Gamma_h(P, G_m(P))^G\to$ *Rie* $(P)^G$. Conversely given a G-invariant metric on P we can assign the orthogonal complement of $T(pG)_p$ to each $p\in P$ and get an element of $\Gamma_h(P, G_m(P))$.

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[1] We denote tangent bundles by $T(P)$, $T(M)$, $\cdots$ in this paper.

Hence if we denote by $Rie\ (P)^G_{\mathbf{g}}$ the set

$$\{\tilde{\mathbf{g}} \in Rie\ (P)^G \mid \tilde{\mathbf{g}}(X_v) = \mathbf{g}(\pi_*(X_v)) \text{ for any } X \in T(P)\},$$

then we have

$$\Gamma_h(P,\ G_m(P)) \simeq Rie\ (P)^G_{\mathbf{g}}.$$

The left-hand side is in 1-1 correspondence with the set of connections on P, which is denoted by $\mathcal{A}(P)$. We, therefore, have

$$\mathcal{A}(P) \simeq Rie\ (P)^G_{\mathbf{g}} \tag{2. 2}$$

The map (2. 1), therefore, can be considered as the one:

$$Rie\ (M) \times \mathcal{A}(P) \to Rie\ (P)^G. \tag{2. 3}$$

The image $\tilde{\mathbf{g}}$ of $(\mathbf{g},\ \omega) \in Rie\ (M) \times \mathcal{A}(P)$ is called the Kaluza-Klein metric corresponding to $(\mathbf{g},\ \omega)$. We note that the Kaluza-Klein metric corresponding to $(\mathbf{g},\ \omega)$ is defined as follows.

$$\tilde{\mathbf{g}}(X,\ Y) = \mathbf{g}(\pi_*(X),\ \pi_*(Y)) + \mathbf{b}(\omega(X),\ \omega(Y)) \text{ for } X,\ Y \in T(P)_p \quad (p \in P)$$

where $\mathbf{b}$ is the inner product in $\mathbf{G}$ corresponding to $\mathbf{g}_G$.

Now let us seek for the explicit form of the Kaluza-Klein metric. Take a section $\sigma \in \Gamma(U,\ P)$ of the bundle P where U is an open set of M. Then we get a diffeomorphism:

$$U \times G \ni (x,\ g) \mapsto \tilde{x} = \sigma(x) g \in \pi^{-1}(U).$$

We identify $T(P)_{\tilde{x}}$ with $T(M)_x \times T(G)_g$ by this isomorphism. Let $A \in \mathbf{G}$ and A^* be the fundamental vector field corresponding to A. Then we have $\mathbf{G} \simeq T(G)_g$ by $A \mapsto A^*$. Hence we can identify $T(P)_{\tilde{x}}$ with $T(M)_x \times \mathbf{G}$. Take a frame $e_1,\ \ldots,\ e_m$ for $T(M)_x$ and another one $A_1,\ \ldots,\ A_r$ for $\mathbf{G}$. Then the connection form ω is expressed with respect to the chosen frames by an $r \times (m+r)$ matrix

$$(\Omega \ 1_r).$$

Define $\omega_U(X)=\omega(\sigma_*(X))$ where

$$X=\sum_{i=1}^{m} e_i X_i \in T(M)$$

Then we find

$$\omega_U(X)=(A_1 \ldots A_r)\,\Omega \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}$$

Further, as easily calculated, the Kaluza-Klein metric has the matrix form

$$\begin{pmatrix} (g_{ij})+{}^t\Omega(b_{\alpha\beta})\Omega & {}^t\Omega(b_{\alpha\beta}) \\ (b_{\alpha\beta})\Omega & (b_{\alpha\beta}) \end{pmatrix}$$

with respect to these frames (c. f. [7]), where (g_{ij}) (resp. $(b_{\alpha\beta})$) is the matrix representing $\mathbf{g}$ (resp. $\mathbf{b}$) with respect to the frames.

We assume that G is compact in what follows. Let $\tilde{R}$ (resp. R) be the scalar curvature of the Kaluza-Klein metric $\tilde{\mathbf{g}}$ (resp. the one $\mathbf{g}$ of M). Then the following celebrated equality holds. Suppose M is compact for convenience.

$$\int_P (\tilde{R}+\lambda)*1=\text{const}\left\{\int_M (R+\lambda)*1 + \int_M (T_r(F\wedge *F)\right\}$$

where $*1$ denotes the volume element of each manifold, λ is a constant and F is the curvature form $d\omega+\frac{1}{2}[\omega, \omega]$.

The following lemma follows directly from the above equality.

LEMMA 2. 1. *A Gauge field of M is Yang-Mills if the corresponding Kaluza-Klein metric on P is Einstein.*

3. Unification of Forces

Suppose there are two Gauge field ω, ω' on a manifold M. Let G, G' be the corresponding groups of local symmetry and P, P' the corresponding principal bundles with projections π, π' respectively. Let us denote by $\mathbf{G}$, $\mathbf{G}'$ the Lie algebras of G, G'. Then ω (resp. ω') is a $\mathbf{G}$-valued (resp. $\mathbf{G}'$-valued) 1-form on P (resp. P'). The bundle induced on P from P' by π is a principal G'-bundle, whose total space $\tilde{P}$ is set-theoretically.

$$\{(p, p')\in P\times P' \mid \pi(p)=\pi'(p')\}$$

and whose projection π'' is given by $(p, p')\mapsto p$. Further G hence $G\times G'$ can act on $\tilde{P}$. Actually the action is given by $(p, p')\mapsto (pg, p'g')$ for any $(g, g')\in G\times G'$. Then $(\tilde{P}, \pi\circ\pi'', M)$ turns out to be a $G\times G'$-bundle over M. We write $\tilde{\pi}$ for $\pi\circ\pi''$. We denote by the symbol $\tilde{P}$ the bundle itself in the sequel and write $\tilde{\omega}$ for the pull-back of ω by π''.

Switching P and P', we can get a $G'\times G$ bundle over M whose total space is

$$\{(p', p)\in P\times P' \mid \pi'(p')=\pi(p)\}.$$

We also obtain a $\mathbf{G}$-valued form $\tilde{\omega}'$ as the pull-back of ω'.

By the diffeomorphism: $(p', p)\mapsto (p, p')$ we identify the latter bundle with the former. We use the same symbol $\tilde{P}$ for the identified bundle. Then we have a $\mathbf{G}\times\mathbf{G}'$-valued form $\tilde{\omega}\times\tilde{\omega}'$ on $\tilde{P}$

PROPOSITION 3. 1. *$\tilde{\omega}\times\tilde{\omega}'$ is a connection form on $\tilde{P}$, i. e. a Gauge field on M.*

Proof. The proof follows from a direct check of the two properties in **Sec. 2** of the connection form.

We write π''' for the projection: $\tilde{P}\to P'$ which is defined by $(p, p')\mapsto p'$. Introduce a Riemannian metric $\mathbf{g}$ in M and denote by $\mathbf{k}$ (resp. $\mathbf{k}'$) the Kaluza-Klein metric in P (resp. P') corresponding to ω (resp. ω'). Define a quadratic form $\tilde{\mathbf{g}}$ by

$$\tilde{\mathbf{g}}(X)=\mathbf{k}(\pi''_*(X))+\mathbf{k}'(\pi'''_*(X))-\mathbf{g}(\tilde{\pi}_*(X)).$$

Then $\tilde{\mathbf{g}}$ is a $G\times G'$-invariant positive definite metric, as is easily seen. Moreover $(\tilde{\omega}\times\tilde{\omega}')^{-1}(0)$ at $\tilde{p}\in\tilde{P}$ is the orthogonal complement of $T(\tilde{p}(G\times G'))_{\tilde{p}}$ with respect to $\tilde{\mathbf{g}}$. We, therefore, see that $\tilde{\omega}\times\tilde{\omega}'$ corresponds to $\tilde{\mathbf{g}}$ by the map (2. 2), provided that P there is replaced by $\tilde{P}$ and G

by $G\times G'$. $\tilde{\omega}\times\tilde{\omega}'$ *is called the unified field of* ω *and* ω'.

4. Canonical Connections on Grassmann Manifolds

Let **K** be one of the filed **R**, **C** and **H**. We write $U_K(l)$ for $O(l)$ if $\mathbf{K}=\mathbf{R}$, for $U(l)$ if $\mathbf{K}=\mathbf{C}$, for $S_p(l)$ if $\mathbf{K}=\mathbf{H}$. A flag in $\mathbf{K}^N$ of type $\nu(=(n_1, \cdots, n_s)$ where $0<n_1<\cdots<n_s<N)$ is, by definition, a chain $V_1\subset\cdots\subset V_s$ of planes in $\mathbf{K}^N$ such that dim $V_i=n_i(i=1, \cdots, s)$. A flag manifold F_ν of type ν is the manifold all flags of type ν, which is viewed as a homogeneous space

$$\frac{U_K(N)}{U_K(n_1)\times U_K(n_2-n_1)\times\cdots\times U_K(N-n_s)}$$

Consider the principal bundles P_i over F_ν:

$$\frac{U_K(N)}{U_K(n_1)\times U_K(n_2-n_1)\times\cdots\times U_K(N-n_s)}$$

i−th factor omitted

for $i=1, \cdots, s+1$. Take an orthonormal frame $\zeta_1, \cdots, \zeta_{h(i)}$ for the orthogonal complement of V_{i-1} in V_i where $h(i)=n_i-n_{i-1}$. Then

$$w_i=d(\xi_1\cdots\xi_{h(i)})^*\ (\xi_1\cdots\xi_{h(i)})$$

are connection forms on P_i where $\xi_1, \cdots, \xi_{h(i)}$ are considered as column vectors and $*$ stands for the transposed conjugate. They are called canonical connection forms. We claim that the canonical connections are Yang-Mills. To avoid mere cumbersomeness we only prove the case of Grassmann manifolds, i. e., that of $s=1$. Let P_n, P_k be the universal principal bundles over Grassmann manifold $G_n(\mathbf{K}^{n+k})(\simeq G_k(\mathbf{K}^{n+k}))$, the manifold of n-planes in $\mathbf{K}^{n+k}(\simeq$ that of k-planes in $\mathbf{K}^{n+k}$). These are $U_K(n)-$, $U_K(k)$-bundles respectively. The projections are denoted by π_n, π_k, though somewhat confusing in the case $n=k$. Further the projection of $U_K(n)\times U_K(k)$-bundle $\tilde{P}$; $U_K(n+k)\rightarrow\dfrac{U_K(n+k)}{U_K(n)\times U_K(k)}\simeq G_n(\mathbf{K}^{n+k})$ is denoted by $\tilde{\pi}$. Since $P_n\simeq\dfrac{U_K(n+k)}{U_K(k)}$, we have a cononical projection $U_K(n+k)\rightarrow P_n$, which is denoted by π'_n. Similarly π'_k.

LEMMA 4. 1. *The induced bundle from* P_n *by* π_k: $P_k\rightarrow G_n(\mathbf{K}^{n+k})$ *is equivalent to* $(U_K(n+k)$,

$\pi'_k, P_k)$.

Proof. Consider a sequence of product spaces:

$$U_K(n+k)\times U_K(n+k) \xrightarrow{\pi'_n\times\pi'_k} P_n\times P_k \xrightarrow{\pi_n\times\pi_k} G_n(\mathbf{K}^{n+k})\times G_n(\mathbf{K}^{n+k}).$$

Set $\triangle=\{(x, x\} \mid x\in G(\mathbf{K}^{n+k})\}$. The total space of the induced bundle is $(\pi_n\times\pi_k)^{-1}(\triangle)$, which we denote by P below. Imbed $U_K(n)$ in $U_K(n+k)\times U_K(n+k)$ by ι; $g\mapsto(g, g)(g\in U_K(n+k))$. Then what we have to show is that (1):

$$\pi'_n\times\pi'_k\circ\iota;\ U_K(n+k)\to P_n\times P_k$$

gives rise to a bijection between $U_K(n+k)$ and P and that (2): the right actions of $g\in U_K(n+k)$ on $U_K(n+k)$ and P correspond with each other.

First note that $(\pi_n\times\pi_k)\circ(\pi'_n\times\pi'_k)\ (g, g')\in\triangle$ means to have the same coset

$$g(U_K(n)\times U_K(k))=g'(U_K(n)\times U_K(k)),$$

in other words

$$g'=gh \text{ for some } h\in U_K(n)\times U_K(k).$$

(1) Set $h=h_1\times h_2$ $(h_1\in U_K(n),\ h_2\in U_K(k))$. Then

$$\pi'_n\times\pi'_k(g, gh)=\pi'_n\times\pi'_k(g(h_1\times 0), gh(0\times h_2^{-1}))=\pi'_n\times\pi'_k(g(h_1\times 0), g(h_1\times 0)).$$

Hence $\pi'_n\times\pi'_k\circ\iota\,(U_K(n+k))=P$. On the other hand $\pi'_n\times\pi'_k\circ\iota$ clearly is injective.

(2) Let $g\in U_K(n+k)$ and $h\in U_K(n)$. Then

$$\pi'_n\times\pi'_k\circ\iota\,(gh)=(\pi'_n(gh), \pi'_k(g))$$
$$=(\pi'_n(g), \pi'_k(g))=(\pi'_n\times\pi'_k\circ\iota\,(g))h.$$

This completes the proof.

From now on we introduce on $U_K(l)$ a specific inner product by

$$\langle A, B\rangle = \text{the real part of } T_r(A^*B) \ (A, B \in u_K(l)) \tag{4. 1}$$

where $u_K(l)$ is the Lie algebra of $U_K(l)$. Then there remains no ambiguity in the correspondence (2. 3), in what follows.

LEMMA 4. 2. *P_k is equipped with the metric induced from that of $U_K(n+k)$. The Kaluza-Klein metric corresponding to the connection induced on $U_K(n+k)$ by π_k from ω_n coincides with the metric* (4. 1) *for $U_K(n+k)$.*

Proof. The horizontal space of the induced connection is just the orthogonal complement of the vertical space as is easily seen. The lemma follows from it immediately.

LEMMA 4. 3. *The connection form $(\pi'_n)^*(\omega_n)$ on the principal bundle $U_K(n+k) \to P_k$ is a Yang-Mills potential.*

Proof. $U_K(n+k)$ is a symmetric space since it has a bi-invariant metric [10]. It is (locally) irreducible (as a Riemannian space). Hence it is a Einstein space [3] (see [4] for the semi-simple case). The metric of $U_K(n+k)$ is the Kaluza-Klein metric corresponding to $(\pi'_n)^*(\omega_n)$ according to **LEMMA 4. 2.** Therefore we can see from **LEMMA 2. 1.** that $(\pi'_n)^*(\omega_n)$ satisfies the Yang-Mills equation.

THEOREM 4. 4. *The canonical connection on the canonical bundle over the Grassmann manifold $G_n(\mathbf{K}^{n+k})$ is a Yang-Mills Gauge potential where* $\mathbf{K}=\mathbf{R}, \mathbf{C}, \mathbf{H}$.

Proof. Let ω be any connection on P_n. For a section $\sigma: U \to P_n$ (where $U \subset G_n(\mathbf{K}^{n+k})$ is open) we set $\omega_U = \sigma^*\omega$. The curvature form

$$F(\omega_U) = d\omega_U + \frac{1}{2}[\omega_U, \omega_U]$$

has the trace not dependent on the choice of section σ [8]. We write

$$Ym(\omega) = \int_{G_n(K^{n+k})} T_r(F_U \wedge * F_U)$$

Recall that P is the bundle induced from P_n by $\pi_k: P_k \to G_n(\mathbf{K}^{n+k})$. Define $\tilde{\sigma}(y) = (y, \sigma \circ \pi_k(y))$ for $y \in \pi_k^{-1}(U)$. Then $\tilde{\sigma}$ is a (local) section of P.

Let ω_n, ω_k be the canonical connection forms on P_n, P_k respectively. Let $\tilde{\omega}_1, \cdots, \tilde{\omega}_I$ ($I = nk$) be an orthogonal coframe on $G_n(\mathrm{K}^{n+k})$. To avoid cumbersome notations we use $\tilde{\omega}_i$ themselves for $\pi_k^*(\tilde{\omega}_i)$.

Let us introduce a bi-invariant metric on $U_{\mathrm{K}}(k)$. Take an orthonormal base for $u_{\mathrm{K}}(k)$ and denote by $W_1, \cdots, W_J$ ($J=\dim_{\mathrm{R}} U_{\mathrm{K}}(k)$) the components of ω_k with respect to this base. Then

$$ds^2 = \tilde{\omega}_1^2 + \cdots + \tilde{\omega}_I^2 + W_1^2 + \cdots + W_J^2 \tag{4. 2}$$

turns out to be the Kaluza-Klein metric on P_k corresponding to ω_k. We write ω^* instead of $(\pi_n')^*\omega$ for notational simplicity. Then we have

$$\tilde{\sigma}^*(F(\omega^*)) = \pi_k^*(F_U)$$

where $F(\omega^*)$ is the curvature form of ω^*. Note that $\pi_k^*(F_U)$ is a linear combination of $\tilde{\omega}_i \wedge \tilde{\omega}_j$ ($1 \leq i \leq j \leq I$) with constant coefficients along the fibres. It follow that

$$* \, \tilde{\sigma}^*(F(\omega^*)) = \pi_k^*(F_U) \wedge W_1 \wedge \cdots \wedge W_J.$$

Hence we have

$$\begin{aligned} &\tilde{\sigma}^*(F(\omega^*)) \wedge * \, \tilde{\sigma}^*(F(\omega^*)) \\ &= \pi_k^*(F_U \wedge * F_U) \wedge W_1 \wedge \cdots \wedge W_J, \end{aligned}$$

whence we can find

$$\begin{aligned} &\int_P T_r(\tilde{\sigma}^*(F(\omega^*)) \wedge * \, \tilde{\sigma}^*(F(\omega^*))) = \\ &\int_{G_n(K^{n+k})} T_r(F_U \wedge * F_U) \int_{U_{\mathrm{K}}(k)} W_1 \wedge \cdots \wedge W_J \\ &= \mathrm{const} \times Ym(\omega). \end{aligned} \tag{4. 3}$$

Now replace ω in the above by $\omega_n + tA$ where A is a $u_{\mathrm{K}}(n+k)$-valued 1-form on P_n and $t \in \mathbf{R}$ is a parameter. By differentiating both sides of (4. 3) with respect to t at $t=0$, we obtain

$$\int T_r(A \wedge D^* F_U(\omega_n)) = \text{const} \times$$
$$\int T_r(A \wedge D^* F_U(\omega_n^*))$$

where D^* is the adjoint of D. The right hand side vanishes according to **LEMMA 4. 3.** Since A is arbitrary, we can conclude

$$D^* F_U(\omega_n) = 0.$$

This completes the proof of **THEOREM 4. 4.**

5. The Electroweak Theory and its Currents

In this section we show that there appears more electric current than we expect in text books (e. g. [12]). The exact form of the added term is determined, which is connected with the size of the internal space i. e. the typical fiber of the principal bundle with a Kaluza-Klein metric, on which the Gauge potentials are defined. We first review the electroweak theory from a mathematical standpoint. An electromagnetic field on a Riemannian manifold M is given by a triple (Q, r, ω) where Q is a principal $U(1)$-bundle over M, r a linear representation of $U(1)$, ω a connection form on P. Similarly a weak force field is given by (S, s, W) when S is a principal $SU(2)$-bundle and so on. Then these two Gauge fields are unified in a field $\omega \times W$ on a principal $U(1) \times SU(2)$-bundle P as we saw in **Sec. 3.**

As bases, take

$$\frac{i}{2} g' Y \quad \text{for } u(1)$$

and

$$igT_1 = \frac{i}{2} g\sigma_1, \quad igT_2 = \frac{i}{2} g\sigma_2, \quad igT_3 = \frac{i}{2} g\sigma_3 \quad \text{for } su(2)$$

where Y is the hypercharge, σ_1, σ_2, σ_3 the Pauli matrices and g, g' the coupling constants in the theory. (For example $Y = -1$ for the neutrino-electron multiplet $\binom{\nu_e}{e}$ and $Y = \frac{1}{3}$ for the quark multiplet $\binom{d}{u}$). Then the following structure equations hold:

$$[igT_3,\ igT_1]=g(igT_2),\ [igT_2,\ igT_3]=g(igT_1),\ [igT_1,\ igT_2]=g(igT_3).$$

Let us write W componentwise as

$$W=W_1(igT_1)+W_2(igT_2)+W_3(igT_3)$$

and set

$$W_i=\sum_{\mu=1}^{m} W_{i\mu}dx^{\mu} \qquad (i=1,\ 2,\ 3)$$

where $m=\dim M$. Then the covectors

$$W_{\pm}=\frac{1}{2}(W_1 \mp iW_2)$$

give rise to charged vector bosons. We write ω instead of $B(\frac{i}{2}g'Y)$. We are required to make a rotation of the base

$$(ei\mathbf{Q},\ iZ_0)=(\frac{i}{2}g'Y\ \ igT_3)\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

so that the Gell-mann-Nishijima relation should hold where $\mathbf{Q}$ is the electric charge, iZ_0 the other vector in the new base and where e means that there are e units of the electric charge. We, therefore, have

$$\cos\theta=\frac{g}{\sqrt{g^2+g'^2}},\quad \sin\theta=\frac{g'}{\sqrt{g^2+g'^2}},\quad e=\frac{gg'}{\sqrt{g^2+g'^2}}.$$

and

$$\omega \times W = \cdots + (ie\mathbf{Q}iZ_0) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} B \\ W_3 \end{pmatrix}$$

in the componentweise way. Hence $\cos\theta\, B + \sin\theta\, W_3$ is the potential of our electromagnetic field (while $(-\sin\theta\, B + \cos\theta\, W_3)$ is that of the neutral vector boson).

Let us calculate the Maxwell equation for the above potential. It should be noted that this potential $A = \cos\theta\, B + \sin\theta\, W_3$ can not be considered as any connection form in the strict sense.

In what follows, we write $\hat{T}_i$ for $igT_i (i=1, 2, 3)$ in brief and F_W stands for the curvature form of W. Then

$$F_W = (dW^1 + gW^2 \wedge W^3)\hat{T}_1 + (dW^2 + gW^3 \wedge W_1)\hat{T}_2 + (dW^3 + gW_1 \wedge W_2)\hat{T}_3.$$

We denote by π the projection: $P \to M$. Take an orthonormal coframe $\tilde{\omega}_1, \cdots, \tilde{\omega}_m$ on M. As before (see **Sec. 4**), we use $\tilde{\omega}_1, \cdots, \tilde{\omega}_m$ instead of $\pi^*(\tilde{\omega}_1), \cdots, \pi^*(\tilde{\omega}_m)$. Note that $\hat{T}_1, \hat{T}_2, \hat{T}_3$ are orthogonal with respect to the inner product (4. 1). Then a Kaluza-Klein metric can be introduced on P by

$$ds^2 = \tilde{\omega}_1^2 + \cdots + \tilde{\omega}_m^2 + a^2((W^1)^2 + (W^2)^2 + (W^3)^2) + b^2B^2, \tag{5. 1}$$

where a, b are constants which should be determined physically [2]. Since F_W can be spanned by $\tilde{\omega}_i \wedge \tilde{\omega}_j$ $(1 \le i \le j \le m)$, we have

$$*F_W = \text{something} \wedge W^1 \wedge W^2 \wedge W^3 \wedge B \tag{5. 2}$$

where, of course, the $*$-operator is taken with respect to the metric (5. 1). On the other hand

$$*F_W = \sum_{i=1}^{3} (*dW^i)\hat{T}_i + g\Sigma' (*W^i \wedge W^j)\hat{T}_k$$

where Σ' expresses the sum taken cyclically with respect to i, j, k $(1 \le i, j, k \le 3)$. Writing $F_W = \sum_{i=1}^{3} F_W^i \hat{T}_i$, we have

$$D*F_W = \Sigma d(*F_W^i)\hat{T}_i + g\Sigma'(W^i \wedge *F_W^j \pm *F_W^i \wedge W^j)\hat{T}_k.$$

It follows from (5. 2) that the second term in the right hand side vanishes (it is interesting to note that this always happens if we take any Kaluza-Klein metric). Setting $D*F_W=\sum_{i=1}^{3}(D*F_W)i\hat{T}_i$,

$$(D*F_W)^3=d*F_W^3.$$

Hence

$$d*dW^3=-gd*(W^1\wedge W^2)+(D*F_W)^3$$

dB, $d\tilde{\omega}_i$ are spanned by $\tilde{\omega}_j\wedge\tilde{\omega}_i$ $(1\leq j\leq k\leq m)$ as F_W^3, because dB is the curvature form and $\tilde{\omega}_i$ are forms on M. Consequently we can see

$$d*(W^1\wedge W^2)=\frac{b}{a}d(W^3\wedge B\wedge\tilde{\omega}^1\wedge\cdots\wedge\tilde{\omega}^m)$$

$$=-\frac{b}{a}(W^1\wedge W^2\wedge B\wedge\tilde{\omega}^1\wedge\cdots\wedge\tilde{\omega}^m).$$

We, therefore, have

$$*d*dW^3=\frac{g}{a^2}W^3+(*D*F_W)^3.$$

We set as usual

$$*d*dB=\frac{g'}{2}j_Y$$

$$*D*F_W^3=gJ^3$$

Then we have

$$*d*dA=\frac{e}{a^2}W^3+(\cos\theta)*d*dB$$

$$+(\sin\theta)(*D*F_W)^3$$

$$=e\left(\frac{1}{a^2}W^3+\frac{g'}{2}j_Y+gJ^3\right).$$

Hence we can obtain the following theorem.

THEOREM 5. 1. *Let V be the volume of the $SU(2)$-part of the internal space with respect to the Kaluza-Klein metric. Then the Maxwell equation in the electroweak field can be written as follows:*

$$*d*dA=e\left\{\left(\frac{\int_{SU(2)} W^1\wedge W^2\wedge W^3}{V}\right)^{\frac{2}{3}} W^3+j_{\mathrm{em}}\right\}$$

where $j_{\mathrm{em}}=\frac{1}{2}j_Y+J^3$.

6. Quaternionic Projective Space

Consider the $(n+1)$-dimensional quaternionic right-vector space $\mathbf{H}^{n+1}$. We give it an inner product

$$(y, y')=\bar{y}_0y'_0+\cdots+\bar{y}_ny'_n \tag{6. 1}$$

where $y=(y_0, \cdots, y_n)$, $y'=(y'_0, \cdots, y'_n)\in\mathbf{H}^{n+1}$. The underlying set of quaternionic projective n-space $\mathbf{HP}(n)$ is the set of quaternionic lines through the origin, $\{y\mathbf{H} \mid y\in\mathbf{H}^{n+1}-0\}$. $\mathbf{HP}(n)$ is also considered as $S_P(n+1)/S_P(n)\times S_P(1)$. Let $x\in\mathbf{HP}(n)$. Let $x_0, \cdots, x_2$ be the homogeneous coordinates of x. Set $y=(x_0, \cdots, x_n)$ and

$$\tilde{x}=y/\sqrt{(y, y)}.$$

Then $\tilde{x}\in S^{4n+3}$ (the round unit sphere with the origin as center, in $\mathbf{H}^{n+1}$). Considering $\tilde{x}\mathbf{H}$ as fibre over x, we get the canonical line bundle over $\mathbf{HP}(n)$. We denote it by L. We write dx for the identity isomorphism of the tangent bundle $T(S^{4n+3})$. Let $e_0(=\tilde{x})$, $e_1, \cdots, e_n$ be an orthonormal frame for $\mathbf{H}^{n+1}$. Then $e_1, \cdots, e_n$ span the tangent space $T(S^{4n+3})_{\tilde{x}}$ and can be viewed as an orthonormal frame for $T(S^{4n+3})$. We write

$$dx=e_0\tilde{\omega}_0+e_1\tilde{\omega}_1+\cdots+e_n\tilde{\omega}_n$$

Then $\tilde{\omega}_0$ is purely quaternionic. Define $\pi: S^{4n+3}\rightarrow\mathbf{HP}(n)$ by $\pi(\tilde{x})=x$. Then we have a principal $SU(2)$-bundle $(S^{4n+3}, \pi, \mathbf{HP}(n))$, which is the associated bundle to L. The canonical connection form is given by

$$\tilde{\omega}_0 = x^* dx$$

where $*$ means the transposed conjugate. $\bar{\tilde{\omega}}_0 \tilde{\omega}_0$ gives rise to the natural metric, i. e., the metric induced from (6. 1), on each fibre. We denote by $\tilde{\omega}_1, \cdots, \tilde{\omega}_n$ the pull-backs $\pi^*(\tilde{\omega}_1), \cdots, \pi^*(\tilde{\omega}_m)$ for simplicity. As in **Sec. 4**,

$$ds^2 = \bar{\tilde{\omega}}_0 \tilde{\omega}_0 + \bar{\tilde{\omega}}_1 \tilde{\omega}_1 + \cdots + \bar{\tilde{\omega}}_n \tilde{\omega}_n.$$

is the Kaluza-Klein metric on S^{4n+3}, which is nothing but the ordinary metric of the round sphere. Hence it is an Einstein metric. It follows from this that $\tilde{\omega}_0$ is a Yang-Mills potential

We can rediscover this fact more directly. Namely an easy calculation shows us

$$\begin{aligned} F(\tilde{\omega}_0) \, (= d\tilde{\omega}_0 + \tilde{\omega}_0 \wedge \tilde{\omega}_0) &= (dx, dx,) - (dx, x) \wedge (x, dx) \\ &= \bar{\tilde{\omega}}_1 \wedge \tilde{\omega}_1 + \cdots + \bar{\tilde{\omega}}_n \wedge \tilde{\omega}_n. \end{aligned}$$

This expression can be considered as one on $\mathbf{HP}(n)$. Through a calculation we find

$$* F(\tilde{\omega}_0) = \pm \underbrace{F(\tilde{\omega}_0) \wedge \cdots \wedge F(\tilde{\omega}_0)}_{(2n-1) \text{ copies}}$$

on $\mathbf{HP}(n)$. We might call this type of F(anti-) selfdual. From this it follows rather directly that $\tilde{\omega}_0$ is Yang-Mills.

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