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ASYMPTOTIC DIAGONALIZATION OF A LINEAR ORDINARY DIFFERENTIAL SYSTEM

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1. Introduction

The process of diagonalization or block-diagonalization is important step in the study of a system of differential equations. The result of N. Levinson[L] plays an important role in the study of the asymptotic behavior of solutions of a linear system of differential equations

$$\frac{dy}{dt} = A(t)\,y, \tag{1.1}$$

as $t \to \infty$, where y is an n-dimensional vector and $A(t)$ is an $n \times n$ matrix continous on $\mathcal{I}_0 = [t_0, \infty)$ (t_0: finite). In order to state the Levinson theorem we need:

Assumption 1.1. The matrix $A(t)$ is in the form

$$A(t) = \Lambda(t) + R(t), \tag{1.2}$$

where $\Lambda(t)$ is an $n \times n$ diagonal matrix

$$\Lambda(t) = diag\{\lambda_1(t),\, \lambda_2(t),\, \cdots,\, \lambda_n(t)\,\}, \tag{1.3}$$

with $\lambda_j(t)\,(j = 1, 2, \cdots, n)$ continuous on $\mathcal{I}_0$.

Assumption 1.2. Let

$$D_{jk}(t) = \Re(\lambda_j(t) - \lambda_k(t)), \qquad (j, k = 1, 2, \cdots, n). \tag{1.4}$$

For each fixed j, the set of positive integers $\{1, 2, \cdots, n\}$ is the union of two disjoint subsets $\mathcal{P}_{j1}$ and $\mathcal{P}_{j2}$, where

(i) $k \in \mathcal{P}_{j1}$ if

$$\begin{cases} \lim_{t\to+\infty} \int_{t_0}^{t} D_{jk}(\tau)\, d\tau = -\infty, \\ \\ \int_{s}^{t} D_{jk}(\tau)\, d\tau < K \qquad \text{for} \qquad t_0 \le s \le t, \end{cases}$$

for some positive number K,

(ii) $k \in \mathcal{P}_{j2}$ if

$$\int_{s}^{t} D_{jk}(\tau)\, d\tau < K \qquad \text{for} \qquad s \ge t \ge t_0,$$

for some positive number K.

Assumption 1.3. The matrix $R(t)$ is an $n \times n$ matrix satisfying

$$R(t) \in L_1(\mathcal{I}_0). \tag{1.5}$$

A version of the Levinson theorem can be stated as the foilowing (e.g. see E. A. Coddington and N. Levinson[CL] or M. S. P. Eastham[E]).

Theorem 1.1. *Under the Assumptions 1.1, 1.2 and 1.3, there exists an $n \times n$ matrix $Q(t)$ such that*

(1) *the derivative $\frac{dQ(t)}{dt}$ exists and entries of $Q(t)$ and $\frac{dQ(t)}{dt}$ are continuous in t on the interval $\mathcal{I}_0$,*

(2) $\lim_{t\to+\infty} Q(t) = 0$,

(3) *the transformation:*

$$y = [I_n + Q(t)]\, z \tag{1.6}$$

changes system (1.1) to

$$\frac{dz}{dt} = \Lambda(t)\, z \tag{1.7}$$

on the interval $\mathcal{I}_0$, where I_n is the $n \times n$ identity matrix.

Remark 1.1. Assume that the functions $\lambda_1(t), \cdots, \lambda_n(t)$ are contiuous on the interval $\mathcal{I}_0$ and that

$$\lim_{t\to+\infty} \lambda_j(t) = \mu_j \qquad (j = 1, 2, \cdots, n)$$

exist. Then, if the real parts of $\mu_1, \cdots, \mu_n$ are mutually distinct, the functions $\lambda_1(t), \cdots, \lambda_n(t)$ satisfy Assumption 1.2.

Remark 1.2. Theorem 1.1 has been shown to be the basis of many important results for asymptotic integrations of diffrential equations. For instance, see W. A. Harris, Jr. and D. A. Lutz[HL3] and M. S. P. Eastham[E].

Remark 1.3. A result to Theorem 1.1 for the system (1.1) with coefficient $A(t, h(t))$ is obtained recently by W. A. Harris, Jr. and Y. Sibuya[HY]. Here $A(t, \varepsilon)$ is periodic in t for every fixed vector parameter ε and $h(t)$ is a vector function tending to zero vector as $t \to +\infty$,.

Remark 1.4. The conditions required for the existence of $Q(t)$ globally analytic on $\mathcal{I}_0$ and $\overline{\mathcal{I}}_0$ are studied recently by H. Gingold, P. F. Hsieh and Y. Sibuya[GHY].

Theorem 1.1 can be proved in the following manner. From (1.1), (1.2), (1.6) and (1.7), we see that $Q(t)$ satisfies a linear differential equation:

$$\frac{dQ(t)}{dt} = \Lambda(t)\, Q - Q\, \Lambda(t) + R(t)\, [I + Q]. \tag{1.8}$$

As (1.8) is a linear equation, if a solution $Q(t)$ is shown to satisfy condition (2) in an interval $\mathcal{I} = [t_1, \infty)$, for a large t_1, then, $Q(t)$ exists on $\mathcal{I}_0$ and satisfies Theorem 1.1. Assumption 1.2 and 1.3 are employed to show the existence of such $Q(t)$.

P. Hartman and A. Wintner[HW] studied the same problem for (1.1) under slightly different set of assumptions:

Assumption 1.4. There exists a positive constant δ such that for each pair of indices j and k, $(j \neq k)$,

$$|\, D_{jk}(t)\, | \geq \delta > 0 \qquad \text{for} \qquad t \in \mathcal{I}_0. \tag{1.9}$$

Assumption 1.5. There is a positive p, $(1 < p \leq 2)$, such that

$$|\, R(t)\, | \in L_p(\mathcal{I}_0), \tag{1.10}$$

where $|\cdot|$ denotes the maximum norm.

A version of the Hartman and Wintner Theorem can be given as follows:

Theorem 1.2. *Under the Assumptions 1.1, 1.4 and 1.5, there exists an $n \times n$ matrix $Q(t)$ such that*

(1) *the derivative* $\dfrac{dQ(t)}{dt}$ *exists and the entries of* $Q(t)$ *and* $\dfrac{dQ(t)}{dt}$ *are continuous in* t *on the interval* $\mathcal{I}_0$,

(2) $Q(t) \in L_p(\mathcal{I}_0)$, *diag* $Q(t) = 0$ *and* $\lim_{t\to+\infty} Q(t) = 0$,

(3) *the transformation:*

$$y = [I_n + Q(t)]\, z \tag{1.11}$$

changes system (1.1) to

$$\frac{dz}{dt} = [\Lambda(t) + diag\, R(t)]\, z \tag{1.12}$$

on the interval $\mathcal{I}_0$, *where* I_n *is the* $n \times n$ *identity matrix.*

For a system (1.1) with almost constant coefficient, consider the following:

Assumption 1.6. The matrix $A(t)$ is in the form

$$A(t) = A_1 + B(t) \tag{1.13}$$

satisfies the conditions:

(i) the constant matrix A_1 is in block-diagonal form

$$A_1 = diag\,\{A_{11}, A_{22}, \cdots, A_{mm}\},$$

where A_{jj} is an $n_j \times n_j$ matrix in the form

$$A_{jj} = \lambda_j I_{n_j} + E_j + N_j \quad (j = 1, 2, \cdots, m)$$

with λ_j a real number, E_j an $n_j \times n_j$ diagonal matrix of purely imaginary diagonal entries, N_j an $n_j \times n_j$ nilpotent matrix and $n_1 + n_2 + \cdots + n_m = n$;

(ii) $\lambda_m < \lambda_{m-1} < \cdots < \lambda_2 < \lambda_1$;

(iii) $N_j\, E_j = E_j\, N_j \quad (j = 1, 2, \cdots, m)$;

(iv) the matrix $B(t)$ is expressed in the form $(B_{jk}(t))_{j,k=1}^{m}$ with $B_{jk}(t)$ $n_j \times n_k$ matrices satisfying

$$\lim_{t\to\infty} B_{jk}(t) = 0 \qquad (j, k = 1, \cdots, m).$$

Let

$$y = [y_1\, y_2\, \cdots\, y_m]^T, \tag{1.14}$$

where y_j are n_j-column vectors. Y. Sibya[S] proved the following (see also P. Hartman[H, Ch. X, Lemm 12.1]):

Theorem 1.3. *Under the Assumption 1.6, there exists a linear transformation*

$$y_j = z_j + \sum_{j \neq k} T_{jk}(t) z_k \qquad (j = 1, 2, \cdots, m) \tag{1.15}$$

with $n_j \times n_k$ matrices $T_{jk}(t)$ such that

(1) *for every pair of indices j and k, $(j \neq k)$, the derivative $\dfrac{T_{jk}(t)}{dt}$ exists and the entries of $T_{jk}(t)$ and $\dfrac{T_{jk}(t)}{dt}$ are continuous on the interval $\mathcal{I}_0$;*

(2) $\displaystyle\lim_{t \to +\infty} T_{jk}(t) = 0, \quad (j \neq k)$;

(3) *the transformation (1.15) changes the system (1.1) with (1.13) to*

$$\frac{dz_j}{dt} = \left[A_{jj} + B_{jj}(t) + \sum_{h \neq j} B_{jh}(t) T_{hj}(t) \right] z_j \quad (j = 1, 2, \cdots, m). \tag{1.16}$$

For a more general setting, suppose that the system (1.1) is in the form

$$\frac{dy_i}{dt} = A_j(t) y_j + \sum_{k=1}^{m} B_{jk}(t) y_k, \tag{1.17}$$

where y_j are n_j-column vectors given in (1.14), $A_j(t)$ and $B_{jk}(t)$ are $n_j \times n_j$ and $n_j \times n_k$ matrices, respectively. Let $G_j(t, s)$ be the $n_j \times n_k$ matrix such that

$$\begin{cases} \dfrac{dG_j(t, s)}{dt} &= A_j(t)\, G_j(t, s), \\ G_j(s, s) &= I_{n_j} \qquad (j = 1, 2, \cdots, m) \end{cases} \tag{1.18}$$

for t, $s \in \mathcal{I}_0$. Assume the folloing:

Assumption 1.7. There exist two positive constants K and δ such that for any $n_j \times n_k$ matrix C_{jk}, we have

$$\| G_j(t, s)\, C_{jk}\, G_k(t, s)^{-1} \| \leq K e^{+\delta(t-s)} \| C_{jk} \|, \quad \text{for} \quad t \leq s,\ j > k \tag{1.19}$$

and

$$\| G_j(t, s)\, C_{jk}\, G_k(t, s)^{-1} \| \leq K e^{-\delta(t-s)} \| C_{jk} \|, \quad \text{for} \quad t \geq s,\ j < k \tag{1.20}$$

for $s, t \in \mathcal{I}_0$. Here $\| \cdot \|$ denotes the Euclidian norm.

Assumption 1.8. There exists a function $f(t)$ such that

$$\| B_{jk}(t) \| \leq f(t) \qquad (j,k = 1,2,\cdots,m) \tag{1.21}$$

and

$$\sup_{p \geq t} (1+p-t)^{-1} \int_t^p f(\tau)\, d\tau \to 0 \tag{1.22}$$

as $t \to +\infty$.

Y. Sibuya[S] proved the following

Theorem 1.4. *Under the Assumption 1.7 and 1.8, there exists a linear transformation*

$$y_j = z_j + \sum_{j \neq k} T_{jk}(t)\, z_k \qquad (j = 1,2,\cdots,m) \tag{1.23}$$

with $n_j \times n_k$ matrices $T_{jk}(t)$ such that

(1) *for every pair of indices j and k, $(j \neq k)$, the derivative $\dfrac{T_{jk}(t)}{dt}$ exists and the entries of $T_{jk}(t)$ and $\dfrac{T_{jk}(t)}{dt}$ are continuous on the interval $\mathcal{I}_0$;*

(2) $\lim_{t \to +\infty} T_{jk}(t) = 0, \qquad (j \neq k)$;

(3) *the transformation (1.23) reduces the system (1.17) to*

$$\frac{dz_j}{dt} = \left[A_j(t) + \sum_{h \neq j} B_{jh}(t)\, T_{hj}(t) \right] z_j \qquad (j = 1,2,\cdots,m). \tag{1.24}$$

Note that the reduced systems (1.16) and (1.24) are not necessarily diagonal and involve the entries of the respective transformation matrices, while the reduced systems (1.7) and (1.12) are diagonal and do not involve the entries of the transformation matrices. Further study of the asymptotic behavior of (1.1) are made to generalize these theorems and applied to topics such as adiabatic oscillators and deficiency index problems (e.g., see K. Chiba and T. Kimura[CK], A. Devinatz[D], M. S. P. Eastham[E] and W. A. Harris, Jr. and D. A. Lutz[HL1, HL2, HL3]). In these studies, the integrability of $R(t)$ and/or its derivatives are always assumed, except for Theorems 1.3 and 1.4. In this paper, we will generalize these results by assuming only the integrability of entries of $R(t)$ above (or below) the diagonal and replace the other half by zero limit as $t \to \infty$. Furthermore, the reduced system of our new result has a diagonal coefficient and does not involve the entries of the transformation matrix. In Section 2, the main theorem is to be stated and examples are to be given to illustrate the differences of the main theorem with Theorems 1.1 – 1.4. The proof of the main theorem is to be given in Section 3.

2. The Main Theorem and Examples

In order to formulate the main theorem, let

$$R(t) = \Big(r_{jk}(t)\Big)_{j,k=1}^{n} \tag{2.1}$$

and

$$r_1(t) = \max_{j>k} |\, r_{jk}(t)\,|, \qquad r_2(t) = \max_{j<k} |\, r_{jk}(t)\,|. \tag{2.2}$$

We assume the following

Assumption 2.1. There exist two positive constants δ and K and a constant α $(0 \leq \alpha < 1)$, independent of t, such that for each pair of indices j and k, $(j, k = 1, 2, \cdots, n; j \neq k)$,

$$\exp\left\{\int_t^s D_{jk}(\tau)d\tau\right\} \leq K \exp\bigl\{-\delta\,(s^{1-\alpha} - t^{1-\alpha})\bigr\}, \quad \text{whenever} \quad s \geq t \tag{2.3}$$

or

$$\exp\left\{\int_t^s D_{jk}(\tau)d\tau\right\} \leq K \exp\bigl\{-\delta\,(t^{1-\alpha} - s^{1-\alpha})\bigr\}, \quad \text{whenever} \quad s \leq t. \tag{2.4}$$

Assumption 2.2. The matrix $R(t)$ is continuous and satisfies:

$$|\, R(t)\,| = o(t^{-\alpha}) \qquad \text{as} \quad t \to +\infty \tag{2.5}$$

and either

$$\int_{t_0}^{+\infty} t^{-\alpha} \max_{s \geq t} [s^{\alpha} r_1(s)]\, dt < +\infty \tag{2.6}$$

or

$$\int_{t_0}^{+\infty} t^{-\alpha} \max_{s \geq t} [s^{\alpha} r_2(s)]\, dt < +\infty. \tag{2.7}$$

We will establish the following main theorem:

Theorem 2.1. *Under the Assumptions 2.1 and 2.2, there exists an $n \times n$ matrix $Q(t)$ such that*

(1) *the derivative $\dfrac{dQ(t)}{dt}$ exists and the entries of $Q(t)$ and $\dfrac{dQ(t)}{dt}$ are continuous in t on the interval $\mathcal{I}_0$,*

(2) $\lim\limits_{t \to +\infty} Q(t) = 0,$

(3) *the transformation:*

$$y = [I_n + Q(t)]\, z \tag{2.8}$$

changes system (1.1) to

$$\frac{dz}{dt} = [\Lambda(t) + diag R(t)]\, z \tag{2.9}$$

on the interval $\mathcal{I}_0$, where I_n is the $n \times n$ identity matrix.

Note that Assumption 2.1 is equivalent to Assumption 1.7 when $\alpha = 0$. On the other hand (2.5) implies Assumption 1.8. Thus Assumption 2.1 is weaker than Assumption 1.7 for $\alpha > 0$ and Assumption 2.2 is stronger than Assumption 1.8. However, the reduced system (2.9) does not involve the entries of the transformation matrix $Q(t)$ in (2.8). Moreover, the reduced system (2.9) has a diagonal coefficient.

The proof of this theorem will be given in Section 3. In order to utilize the Assumption 2.2, we will develop a "row-wise" (or "colum-wise") successive approximations method, similar to the Gauss-Seidel interaton (e.g., see [GVI] and [V]), in the proof. In this process, we need only the continuity of $R(t)$ along with either (2.6) or (2.7) and not the integrability of entire $R(t)$, or that of $R'(t)$ (e.g., see [CK], [E], [HL1] and [L]). Furthermore, as we do not require that $\lim_{t\to+\infty} \Lambda(t)$ has distinct eigenvalues, Theorem 2.1 can be applied to diagonalize some systems which are not doagonalizeable by previous results.

Remark 2.1. If $\{t^\alpha r_{jk}(t)\,|\,j > k\}$ (or $\{t^\alpha r_{jk}(t)\,|\,j < k\}$) are all monotonic decreasing, then $t^\alpha r_1(t)$ (or $t^\alpha r_2(t)$) is monotonic decreasing and, consequently, (2.6) (or (2.7)) can be replaced by

$$\int_{t_0}^{+\infty} r_1(t)\,dt < +\infty \qquad \left(or \quad \int_{t_0}^{+\infty} r_2(t)\,dt < +\infty \right). \tag{2.10}$$

The system (1.1) with $A(t)$ in each example below is compared for the applicability of Theorems 1.1, 1.2, 1.3, 1.4 and 2.1 on $[t_0, \infty)$.

Example 1. Consider

$$A(t) = \begin{bmatrix} 1 + \dfrac{1}{t^{\frac{2}{3}}} & \dfrac{|\sin t|}{t^2} \\ \dfrac{1}{\sqrt{t}\ln t} & 1 + \dfrac{1}{\sqrt{t}} \end{bmatrix}$$

for $t \in [2, +\infty)$. We consider $\lambda_1(t) = 1, \lambda_2(t) = 1 + \dfrac{1}{\sqrt{t}}$, $r_{11}(t) = \dfrac{1}{t^{\frac{2}{3}}}$ and $r_{22} = 0$. The system (1.1) with this $A(t)$ is not diagonalizeable by Theorem 1.1, because Assumption 1.3 is not satisfied. It is not diagonalizeable by Theorem 1.2 or 1.3 because $\lim_{t\to+\infty} \lambda_2(t) = \lim_{t\to+\infty} 1 + \dfrac{1}{\sqrt{t}} = 1 = \lambda_1(t)$. It is also not diagonalizeable by Theorem 1.4, because Assumption 1.7 is not satisfied. On the other hand, it is diagonalizeable by Theorem 2.1 with $\alpha = \dfrac{1}{2}$.

Example 2. Consider

$$A(t) = \begin{bmatrix} t & \dfrac{\cos t}{t^2} & \dfrac{1}{t^{\frac{3}{2}}} \\ \dfrac{1}{t} & 2 & \dfrac{1}{t^3} \\ \dfrac{\ln t}{t^{\frac{1}{2}}} & \dfrac{-2}{t^{\frac{1}{3}} \ln t} & 2 + \dfrac{1}{t^{\frac{1}{3}}} \end{bmatrix}$$

for $t \in [2, +\infty)$. The system (1.1) is not diagonalizeable by Theorem 1.1, 1.2, 1.3 or 1.4 because $\lim_{t \to +\infty} \lambda_3(t) = \lim_{t \to +\infty} \left[2 + \frac{1}{t^{\frac{1}{3}}} \right] = 2 = \lambda_2(t)$. On the other hand, it is diagonalizeable by Theorem 2.1 with $\alpha = \frac{1}{3}$. In this case, $\max r_2(t) = \frac{1}{t^{\frac{3}{2}}}$, thus, $[t^{\frac{1}{3}} \max r_2(t)] = \frac{1}{t^{\frac{7}{6}}}$ is monotonic decreasing for $t \in [2, +\infty)$ and (2.10) follows.

Example 3. Consider

$$A(t) = \begin{bmatrix} 1 & \dfrac{1}{t^{\frac{3}{2}}} \\ \dfrac{1}{\ln(t+2)} & e^{it} \end{bmatrix}$$

for $t \in [1, +\infty)$. The system (1.1) with this $A(t)$ is not diagonalizeable by Theorem 1.1 as Assumption 1.3 is not satisfied. As the diagonal elements coincide at $t = 2k\pi$, (k: positive integers), Theorem 1.2 and Theorem 1.3 are not applicable. On the other hand, the system (1.1) with this $A(t)$ is diagonalizeable by Theorem 2.1 with $\alpha = 0$. This system (1.1) is diagonalizeable by Theorem 1.4, however, the reduced system has different coeficients from that obtained by Theorem 2.1

Example 4. Consider

$$A(t) = \begin{bmatrix} 1 & \dfrac{1}{e^t} \\ \dfrac{2 + \sin t}{\sqrt{t}} & \sin t \end{bmatrix}$$

for $t \in [1, +\infty)$. The system (1.1) with this $A(t)$ is not diagonalizeable by Theorem 1.1 as Assumption 1.3 is not satisfied. As the diagonal elements coincide at $t = \left(2k + \frac{1}{2}\right) \pi$ (k: nonnegative integers), Theorems 1.2 and Theorems 1.3 are not applicable. The system (1.1) with this $A(t)$ is diagonalizeable by Theorem 2.1 with $\alpha = 0$. Note that in this case, $r_{21}(t) = \frac{2 + \sin t}{\sqrt{t}}$ is not monotonic decreasing. It is also diagonalizeable by Theorem 1.4 with different coefficients in the reduced system.

3. Proof of Theorem 2.1

We will prove Theorem 2.1 under the condition (2.7) in Assumption 2.2. The proof is to be given in seven steps. Similar proof is valid also under (2.6).

STEP 1: By differentiating both sides of (2.8) and by (1.1), we obtain

$$\frac{dQ}{dt}z + [I_n + Q]\frac{dz}{dt} = [\Lambda(t) + R(t)]\,[I_n + Q]z. \tag{3.1}$$

Hence, by (2.9), Q should satisfy the linear differential equation:

$$\frac{dQ}{dt} = [\Lambda(t) + R(t)]\,[I_n + Q] - [I_n + Q]\,[\Lambda(t) + diagR(t)] \tag{3.2}$$

or, equivalently,

$$\frac{dQ}{dt} = [\Lambda(t) + diagR(t)]\,Q - Q\,[\Lambda(t) + diagR(t)] + [R(t) - diagR(t)]\,[I_n + Q]. \tag{3.3}$$

A general solution $Q(t)$ of (3.2) can be written in the form

$$Q(t) = \Phi(t)\,C\Psi(t)^{-1} + \int^t \Phi(t)\Phi(s)^{-1}[R(s) - diagR(s)]\,\Psi(s)\Psi(t)^{-1}\,ds, \tag{3.4}$$

where C is an arbitrary constant matrix, $\Phi(t)$ is an $n \times n$ fundamental matrix of

$$\frac{d\Phi}{dt} = [\Lambda(t) + R(t)]\,\Phi \tag{3.5}$$

and $\Psi(t)$ is an $n \times n$ fundamental matrix of

$$\frac{d\Psi}{dt} = [\Lambda(t) + diagR(t)]\,\Psi. \tag{3.6}$$

Thus $Q(t)$ exists on $\mathcal{I}_0$ satisfying condition (1). We shall prove the existence of the solution of (3.3) satisfying condition (2) of Theorem 2.1, in an interval $\mathcal{I} = \{t \,:\, t_1 \leq t < +\infty\}$ for a large t_1, then, by (3.4), $Q(t)$ exists on $\mathcal{I}_0$ satisfying Theorem 2.1.

STEP 2: We shall construct $Q(t)$ by means of equation (3.3) and condition (2) of Theorem 2.1. To do this, let $\Phi(t,s)$ be the unique solution of the initial-value problem:

$$\frac{dY}{dt} = [\Lambda(t) + diagR(t)]\,Y, \qquad Y(s) = I_n. \tag{3.7}$$

Also, let

$$Q(t) = \Big(q_{jk}(t)\Big)_{j,k=1}^{n} \tag{3.8}$$

and

$$\begin{aligned} \tilde{\lambda}_j(t) &= \lambda_j(t) + r_{jj}(t), \quad j = 1, 2, \cdots, n; \\ \tilde{\lambda}_{jk}(t) &= \tilde{\lambda}_j(t) - \tilde{\lambda}_k(t), \quad j, k = 1, 2, \cdots, n, \ (j \neq k) \end{aligned} \tag{3.9}$$

Then, (3.3) is equivalent to the following linear integral equation:

$$Q(t) = \int^{t} \Phi(t,s)\,[R(s) - diagR(s)]\,[I_n + Q(s)]\,\Phi(t,s)^{-1}\,ds, \tag{3.10}$$

where

$$\begin{cases} \Phi(t,s) = \begin{bmatrix} F_1(t,s) & 0 & 0 & \cdots & 0 & 0 \\ 0 & F_2(t,s) & 0 & \cdots & 0 & 0 \\ 0 & 0 & F_3(t,s) & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & F_n(t,s) \end{bmatrix}, \\ \\ F_j(t,s) = \exp\left[\int_s^t \tilde{\lambda}_j(\tau)\,d\tau\right] \qquad (j = 1, 2, \cdots, n). \end{cases}$$

STEP 3: By Assumptions 2.1 and 2.2, there exists a positive constant ε, independent of t, such that for each pair of indices j and k, $(j, k = 1, 2, \cdots, n;\ j \neq k)$,

$$\exp\left\{\int_t^s \tilde{\lambda}_{jk}(\tau)\,d\tau\right\} \leq K \exp\bigl\{-\varepsilon\,(t^{1-\alpha} - s^{1-\alpha})\bigr\}, \quad \text{whenever} \quad t \geq s, \tag{3.11-i}$$

or

$$\exp\left\{\int_t^s \tilde{\lambda}_{jk}(\tau)\,d\tau\right\} \leq K \exp\bigl\{-\varepsilon\,(s^{1-\alpha} - t^{1-\alpha})\bigr\}, \quad \text{whenever} \quad t \leq s, \tag{3.11-ii}$$

for $t \in [t_1, \infty)$, where K and α are given in Assumption 2.1. Here (2.4) and (2.3) implies (3.11-i) and (3.11-ii), respectively. Let

$$\tilde{r}_1(t) = \max_{s \geq t}\,[s^\alpha r_1(s)] \qquad \text{and} \qquad \tilde{r}_2(t) = \max_{s \geq t}\,[s^\alpha r_2(s)]. \tag{3.12}$$

Then $\tilde{r}_1(t)$ and $\tilde{r}_2(t)$ are monotonic decreasing and tending to zero as $t \to +\infty$, by (2.5). Moreover, by (2.7), we have

$$\int_{t_0}^{\infty} t^{-\alpha}\tilde{r}_2(t)\,dt < +\infty. \tag{3.13}$$

By the notations (2.1), (3.8) and (3.9), we can write the integral equation (3.10) in the following form:

$$\begin{cases} q_{jj}(t) = \displaystyle\int_{\tau_{jj}}^{t} \left[\sum_{\substack{h=1 \\ h\neq j}}^{n} r_{jh}(s)\,q_{hj}(s)\right] ds, \\ q_{jk}(t) = \displaystyle\int_{\tau_{jk}}^{t} \exp\left[\int_s^t \tilde{\lambda}_{jk}(\tau)\,d\tau\right] \left[r_{jk}(s) + \sum_{\substack{h=1 \\ h\neq j}}^{n} r_{jh}(s)\,q_{hk}(s)\right] ds, \ (j \neq k), \end{cases} \tag{3.14}$$

where $j, k = 1, 2, \cdots, n$, and the initial points τ_{jk} are chosen as follows:

$$\tau_{jk} = \begin{cases} +\infty & \text{if} \quad j = k, \\ t_2 & \text{for the case (3.11-i)}, \quad (j \neq k), \\ +\infty & \text{for the case (3.11-ii)}, \quad (j \neq k), \end{cases} \tag{3.15}$$

where $t_2 \geq t_1$ is a suitable large number.

STEP 4: In order to utilize the condition (2.7), namely (3.13), define the "row-wise" successive approximations as follows:

$$\begin{cases} q_{0,jk}(t) \equiv 0, \\ q_{p,jj}(t) = \displaystyle\int_{\tau_{jj}}^{t} \left[\sum_{h=1}^{j-1} r_{jh}(s) q_{p,hj}(s) + \sum_{h=j+1}^{n} r_{jh}(s) q_{p-1,hj}(s) \right] ds, \\ q_{p,jk}(t) = \displaystyle\int_{\tau_{jk}}^{t} \exp\left[\int_{s}^{t} \tilde{\lambda}_{jk}(\tau)\, d\tau \right] \left[r_{jk}(s) + \sum_{h=1}^{j-1} r_{jh}(s)\, q_{p,hk}(s) \right. \\ \qquad \left. + \displaystyle\sum_{h=j+1}^{n} r_{jh}(s) q_{p-1,hk}(s) \right] ds, \qquad (j \neq k), \end{cases} \tag{3.16}$$

where $j, k = 1, 2, \cdots, n$, $p = 1, 2, \cdots$. Note that $q_{p,jk}(t)$ depends only on $q_{p-1,jk}(t)$, and $q_{p,jk}(t)$ depends only on $q_{p-1,hk}(t)$ for $h > j$ and $q_{p,hk}(t)$ for $h < j$. Therefore, $q_{p,jk}(t)$ is obtained in the increasing order of p and, for each p, in the increasing order of j, namely from the first row down, and thus defined successively.

STEP 5: We will see that $q_{p,jk}(t)$ defined by (3.16) is uniformly bounded for all p on the interval $[t_2, +\infty)$ for large enough t_2; namely,

$$|\, q_{p,jk}(t) \,| \leq G, \qquad t \in [t_2,\, +\infty), \tag{3.17}$$

for $j, k = 1, 2, \cdots, n; p = 1, 2, \cdots$. In order to do that, we will prove for each fixed p, ($p = 1, 2, \cdots$), the following

LEMMA 1. *Suppose that there exist two positive constants G and t_2, (t_2: large enough) such that:*

(a) $|\, q_{p-1,jk}(t) \,| \leq G$ *for* $t \in [t_2, +\infty)$, $j, k = 1, 2, \cdots, n$;

(b) $\displaystyle\int_{t_2}^{+\infty} s^{-\alpha} \tilde{r}_2 \left(\frac{s}{2^n} \right) ds < \frac{1}{2n}$;

(c) $$\tilde{r}_1(t_2)+\tilde{r}_2\left(\frac{t_2}{2^n}\right) \le \min\left\{\frac{G}{4(1+nG)K_1},\ \frac{1}{2nK_2}\right\},$$

where

$$K_1 = \frac{K}{\varepsilon(1-\alpha)} \quad \text{and} \quad K_2 = \frac{2^n}{\varepsilon(1-\alpha)^{n+1}}. \tag{3.18}$$

Then, we have:

$$|q_{p,jk}(t)| \le$$

$$\begin{cases} nGK_2[\tilde{r}_1(t_2)+\tilde{r}_2(t_2)]\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^n (t^{1-\alpha}-t_2^{1-\alpha})\right\} \\ \qquad +nG\displaystyle\int_t^{+\infty} s^{-\alpha}\tilde{r}_2\left(\frac{s}{2^n}\right)ds, & (j=k), \\ (1+nG)K_1\left\{[\tilde{r}_1(t_2)+\tilde{r}_2(t_2)]\exp\left[-\frac{\varepsilon(1-\alpha)(t^{1-\alpha}-t_2^{1-\alpha})}{2}\right]\right. \\ \qquad \left.+\tilde{r}_1\left(\frac{t+t_2}{2}\right)+\tilde{r}_2\left(\frac{t+t_2}{2}\right)\right\}, & (j>k), \\ 2(1+nG)K_1\left\{[\tilde{r}_1(t_2)+\tilde{r}_2(t_2)]\exp\left[-\varepsilon\left(\frac{1-\alpha}{2}\right)^j (t^{1-\alpha}-t_2^{1-\alpha})\right]\right. \\ \qquad \left.+\tilde{r}_2\left(\frac{t}{2^j}\right)\right\}, & (j<k), \end{cases} \tag{3.19}$$

for $t \in [t_2,\infty)$, *and* $j,k = 1,2,\cdots,n$.

As $\tilde{r}_2(t)$ is monotonic decreasing, it is clear that inequalities (b), (c) and (3.19) imply (3.17) for a fixed p. (This fact is to be used repeatedly in the proof of Lemma 1). Thus, by means of mathematical induction on p in (3.16) and utilize Lemma 1, (3.17) is true for every p for $t \in [t_2,\infty)$ and $j,k = 1,2,\cdots,n$. Therefore, we will concentrate on proving Lemma 1.

PROOF OF LEMMA 1: We will prove (3.19) in the increasing order of j.
Case I: $j=1$. For $k=1$, since

$$r_2(s) \le s^{-\alpha}\max_{t\ge s}[s^\alpha r_2(t)] \le s^{-\alpha}\max_{t\ge s}[t^\alpha r_2(t)] = s^{-\alpha}\tilde{r}_2(s) \tag{3.20}$$

by (3.16), we have

$$\begin{aligned} |q_{p,11}(t)| &\le \int_t^\infty \sum_{h=2}^n r_{1h}(s)q_{p-1,h1}(s)ds \\ &\le (n-1)G\int_t^\infty r_2(s)ds \le (n-1)G\int_t^\infty s^{-\alpha}\tilde{r}_2(s)\,ds. \end{aligned} \tag{3.21}$$

For $k = 2, 3, \cdots, n$, in case of (3.11-ii), for $t \in [t_2, \infty)$, by (3.16) and (3.15), we have

$$\begin{aligned} |q_{p,1k}(t)| &\leq \int_t^\infty K \exp\left\{-\varepsilon\left(s^{1-\alpha} - t^{1-\alpha}\right)\right\} \left| r_{1k}(s) + \sum_{h=2}^n r_{1h}(s) q_{p-1,hk}(s) \right| ds \\ &\leq [1 + (n-1)G]K \int_t^\infty \exp\left\{-\varepsilon\left(s^{1-\alpha} - t^{1-\alpha}\right)\right\} r_2(s)\, ds \\ &\leq [1 + (n-1)G]K_1 \left\{ \max_{s \geq t} [s^\alpha r_2(s)] \right\} = [1 + (n-1)G] K_1 \tilde{r}_2(t) \end{aligned} \tag{3.22}$$

by (3.18). In case of (3.11-i), for $t \in [t_2, \infty)$, by (3.16) and (3.15), we have

$$\begin{aligned} |q_{p,1k}(t)| &\leq \int_{t_2}^t K \exp\left\{-\varepsilon\left(t^{1-\alpha} - s^{1-\alpha}\right)\right\} \left| r_{1k}(s) + \sum_{h=2}^n r_{1h}(s) q_{p-1,hk}(s) \right| ds \\ &= \int_{t_2}^{\frac{(t+t_2)}{2}} K \exp\left\{-\varepsilon\left(t^{1-\alpha} - s^{1-\alpha}\right)\right\} \left| r_{1k}(s) + \sum_{h=2}^n r_{1h}(s) q_{p-1,hk}(s) \right| ds \\ &\quad + \int_{\frac{(t+t_2)}{2}}^t K \exp\left\{-\varepsilon\left(t^{1-\alpha} - s^{1-\alpha}\right)\right\} \left| r_{1k}(s) + \sum_{h=2}^n r_{1h}(s) q_{p-1,hk}(s) \right| ds \\ &\leq [1 + (n-1)G] \left[\max_{s \geq t_2} \{s^\alpha r_2(s)\} \right] \int_{t_2}^{\frac{(t+t_2)}{2}} K s^{-\alpha} \exp\left\{-\varepsilon(t^{1-\alpha} - s^{1-\alpha})\right\} ds \\ &\quad + [1 + (n-1)G] \left[\max_{s \geq \frac{t+t_2}{2}} \{s^\alpha r_2(s)\} \right] \int_{\frac{(t+t_2)}{2}}^t K s^{-\alpha} \exp\left\{-\varepsilon(t^{1-\alpha} - s^{1-\alpha})\right\} ds. \end{aligned} \tag{3.23}$$

Since

$$\begin{aligned} &\int_{t_2}^{\frac{(t+t_2)}{2}} K s^{-\alpha} \exp\left\{-\varepsilon(t^{1-\alpha} - s^{1-\alpha})\right\} ds \\ &\quad = K \exp\left\{-\varepsilon t^{1-\alpha}\right\} \frac{1}{1-\alpha} \int_{t_2^{1-\alpha}}^{(\frac{t+t_2}{2})^{1-\alpha}} e^{\varepsilon u}\, du \\ &\quad < K_1 \exp\left\{-\varepsilon\, t^{1-\alpha}\right\} \cdot \exp\left\{\varepsilon \left(\frac{t+t_2}{2}\right)^{1-\alpha}\right\} \\ &\quad = K_1 \exp\left\{-\varepsilon\left[t^{1-\alpha} - \left(\frac{t+t_2}{2}\right)^{1-\alpha}\right]\right\} \\ &\quad = K_1 \exp\left\{-\varepsilon(1-\alpha) u^{-\alpha}\left[t - \frac{t+t_2}{2}\right]\right\}, \quad \text{with} \quad u \in \left(\frac{t+t_2}{2}, t\right), \\ &\quad \leq K_1 \exp\left\{-\varepsilon(1-\alpha) \cdot t^{-\alpha} \cdot \frac{t-t_2}{2}\right\} \\ &\quad \leq K_1 \exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)\left(t^{1-\alpha} - t_2^{1-\alpha}\right)\right\} \end{aligned} \tag{3.24}$$

as $t \geq t_2$, and

$$\begin{aligned}\int_{\frac{(t+t_2)}{2}}^{t} & K s^{-\alpha} \exp\left\{-\varepsilon\left(t^{1-\alpha}-s^{1-\alpha}\right)\right\} ds \\ &= K_1 \exp\left\{-\varepsilon t^{1-\alpha}\right\}\left[\exp\{\varepsilon t^{1-\alpha}\} - \exp\left\{\varepsilon\left(\frac{t+t_2}{2}\right)^{1-\alpha}\right\}\right] < K_1,\end{aligned} \tag{3.25}$$

we have

$$\begin{aligned}|\, q_{p,1k}(t)\,| \leq K_1[1+(n-1)G]\tilde{r}_2(t_2)\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)\left(t^{1-\alpha}-t_2^{1-\alpha}\right)\right\} \\ +K_1[1+(n-1)\,G]\tilde{r}_2\left(\frac{t+t_2}{2}\right).\end{aligned} \tag{3.26}$$

Hence,

$$|\, q_{p,1k}(t)\,| \leq \begin{cases} (n-1)\,G\displaystyle\int_t^\infty s^{-\alpha}\tilde{r}_2(s)\,ds, & k=1, \\ [1+(n-1)\,G]\,K_1\left[\tilde{r}_2(t_2)\exp\left\{-\varepsilon\left(\dfrac{1-\alpha}{2}\right)\left(t^{1-\alpha}-t_2^{1-\alpha}\right)\right\}\right. & \\ \qquad\qquad\left. +\,\tilde{r}_2\left(\dfrac{t+t_2}{2}\right)\right], & k\neq 1.\end{cases} \tag{3.27}$$

Thus, (3.19) is true for $j=1$ and $k=1,2,\cdots,n$.
Case II: Assume that (3.19) is true for $j<m$, $(m>1)$, we want to show that (3.19) is true for $j=m$.

Case IIa. For $k<m$, in case of (3.11-ii), from (3.16) and the assumption (3.19) for this case, (i.e. (3.17) is ture for $j<m$), we have

$$\begin{aligned}|\, q_{p,mk}(t)\,| &\leq \int_t^\infty K\exp\left\{-\varepsilon(s^{1-\alpha}-t^{1-\alpha})\right\}\left| r_{mk}(s)\right. \\ &\quad \left. +\sum_{h=1}^{m-1} r_{mh}(s)q_{p,hk}(s) + \sum_{h=m+1}^{n} r_{mh}(s)q_{p-1,hk}(s)\right| ds \\ &\leq [1+(m-1)\,G]\left[\max_{s\geq t}\{s^{-\alpha}r_1(s)\}\right]\int_t^\infty K s^{-\alpha}\exp\left\{-\varepsilon\left(s^{1-\alpha}-t^{1-\alpha}\right)\right\} ds \\ &\quad +(n-m)\,G\left[\max_{s\geq t}\{s^{-\alpha}r_2(s)\}\right]\int_t^\infty K s^{-\alpha}\exp\left\{-\varepsilon\left(s^{1-\alpha}-t^{1-\alpha}\right)\right\} ds \\ &= K_1[1+(m-1)\,G]\tilde{r}_1(t)+K_1(n-m)\,G\tilde{r}_2(t).\end{aligned} \tag{3.28}$$

In case of (3.11-i), from (3.16) and by (3.24) and (3.25), with similar reasons for (3.23), we have

$$
\begin{aligned}
|\,q_{p,mk}(t)\,| \le & \int_{t_2}^{t} K\exp\left\{-\varepsilon\left(t^{1-\alpha}-s^{1-\alpha}\right)\right\}\left| r_{mk}(s)+\sum_{h=1}^{m-1} r_{mh}(s)q_{p,hk}(s)\right.\\
&\left. +\sum_{h=m+1}^{n} r_{mh}(s)q_{p-1,hk}(s)\right| ds\\
\le & \left\{[1+(m-1)\,G]\left[\max_{s\ge t_2}\{s^{\alpha}r_1(s)\}\right]+(n-m)\,G\left[\max_{s\ge t_2}\{s^{\alpha}r_2(s)\}\right]\right\}\\
&\cdot\int_{t_2}^{\frac{(t+t_2)}{2}} Ks^{-\alpha}\exp\left\{-\varepsilon\left(t^{1-\alpha}-s^{1-\alpha}\right)\right\} ds\\
&+\left\{[1+(m-1)\,G]\left[\max_{s\ge\frac{t+t_2}{2}}\{s^{\alpha}r_1(s)\}\right]+(n-m)\,G\left[\max_{s\ge\frac{t+t_2}{2}}\{s^{-\alpha}r_2(s)\}\right]\right\}\\
&\cdot\int_{\frac{(t+t_2)}{2}}^{t} Ks^{-\alpha}\exp\left\{-\varepsilon\left(t^{1-\alpha}-s^{1-\alpha}\right)\right\} ds\\
\le & K_1\Big[\{1+(m-1)\,G\}\tilde r_1(t_2)+(n-m)\,G\tilde r_2(t_2)\Big]\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)\left(t^{1-\alpha}-t_2^{1-\alpha}\right)\right\}\\
&+K_1\left[\{1+(m-1)\,G\}\tilde r_1\left(\frac{t+t_2}{2}\right)+(n-m)\,G\tilde r_2\left(\frac{t+t_2}{2}\right)\right]\\
\le & K_1(1+nG)\Big[[\tilde r_1(t_2)+\tilde r_2(t_2)]\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)\left(t^{1-\alpha}-t_2^{1-\alpha}\right)\right\}\\
&+\tilde r_1\left(\frac{t+t_2}{2}\right)+\tilde r_2\left(\frac{t+t_2}{2}\right)\Big].
\end{aligned}
\tag{3.29}
$$

Thus, (3.19) is true for $k<j=m$.

<u>Case IIb</u>. For $k=m$, from (3.16), we have

$$
\begin{aligned}
|\,q_{p,mm}(t)\,| \le & \int_{t}^{\infty}\left\{\sum_{h=1}^{m-1}|r_{mh}(s)|\,|q_{p,hm}(s)|+\sum_{h=m+1}^{n}|r_{mh}(s)|\,|q_{p-1,hm}(s)|\right\} ds\\
\le & \left\{\sum_{h=1}^{m-1}\int_{t}^{\infty} s^{-\alpha}|\,q_{p,hm}(s)\,|\,ds\right\}\left[\max_{s\ge t}\{s^{\alpha}r_1(s)\}\right]+(n-m)G\int_{t}^{\infty} r_2(s)\,ds.
\end{aligned}
\tag{3.30}
$$

Since $h<m$ in the first summation, by the assumption (3.19) of this case, we have

$$
\begin{aligned}
|\,q_{p,hm}(t)\,| \le 2(1+nG)K_1&\left[[\tilde r_1(t_2)+\tilde r_2(t_2)]\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^{h}\left(t^{1-\alpha}-t_2^{1-\alpha}\right)\right\}\right.\\
&\left.+\tilde r_2\left(\frac{t}{2^{h}}\right)\right].
\end{aligned}
\tag{3.31}
$$

Hence,

$$\int_t^\infty s^{-\alpha}|\,q_{p,hm}(s)\,|\,ds$$
$$\leq 2(1+nG)K_1\left[[\tilde r_1(t_2)+\tilde r_2(t_2)]\int_t^\infty s^{-\alpha}\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^h(s^{1-\alpha}-t_2^{1-\alpha})\right\}ds\right.$$
$$\left.+\int_t^\infty s^{-\alpha}\tilde r_2\left(\frac{s}{2^h}\right)ds\right]. \tag{3.32}$$

Since

$$\begin{aligned}
&\int_t^\infty s^{-\alpha}\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^h(s^{1-\alpha}-t_2^{1-\alpha})\right\}ds\\
&=\exp\left\{\varepsilon\left(\frac{1-\alpha}{2}\right)^h t_2^{1-\alpha}\right\}\int_t^\infty s^{-\alpha}\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^h s^{1-\alpha}\right\}ds\\
&=\exp\left\{\varepsilon\left(\frac{1-\alpha}{2}\right)^h t_2^{1-\alpha}\right\}\int_{t^{1-\alpha}}^\infty \frac{1}{1-\alpha}\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^h u\right\}du\\
&\leq K_2\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^h(t^{1-\alpha}-t_2^{1-\alpha})\right\},
\end{aligned} \tag{3.33}$$

where K_2 is given in (3.18), we have

$$\int_t^\infty s^{-\alpha}|\,q_{p,hm}(s)\,|\,ds$$
$$\leq 2K_1(1+nG)\left[[\tilde r_1(t_2)+\tilde r_2(t_2)]K_2\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^h(t^{1-\alpha}-t_2^{1-\alpha})\right\}\right. \tag{3.34}$$
$$\left.+\int_t^\infty s^{-\alpha}\tilde r_2\left(\frac{s}{2^h}\right)ds\right]$$

for all $h<m$. Note here that $\tilde r_1(t)$ and $\tilde r_2(t)$ are decreasing functions. Substituting (3.34) into (3.30), we obtain

$$\begin{aligned}
&|\,q_{p,mm}(t)\,|\\
&\leq 2K_1K_2(m-1)(1+nG)\tilde r_1(t)[\tilde r_1(t_2)+\tilde r_2(t_2)]\exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^h(t^{1-\alpha}-t_2^{1-\alpha})\right\}\\
&\quad+2K_1(m-1)(1+nG)\tilde r_1(t)\int_t^\infty s^{-\alpha}\tilde r_2\left(\frac{s}{2^m}\right)ds+(n-m)G\int_t^\infty r_2(s)\,ds.
\end{aligned} \tag{3.35}$$

By condition (c), we have

$$2K_1(1+nG)\tilde r_1(t)\leq\frac{G}{2}<G \tag{3.36}$$

for $t \in [t_2, \infty)$. Hence,

$$
\begin{aligned}
&|\, q_{p,mm}(t) \,| \\
&\quad \le K_2(m-1)G[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^m (t^{1-\alpha} - t_2^{1-\alpha})\right\} \\
&\qquad + (m-1)G\int_t^\infty s^{-\alpha}\tilde{r}_2\left(\frac{s}{2^m}\right) ds + (n-m)G\int_t^\infty r_2(s)\, ds \\
&\quad \le K_2(m-1)G[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^m (t^{1-\alpha} - t_2^{1-\alpha})\right\} \\
&\qquad + nG\int_t^\infty s^{-\alpha}\tilde{r}_2\left(\frac{s}{2^m}\right) ds.
\end{aligned}
\tag{3.37}
$$

Thus, (3.19) holds for $j = k = m$.

Case IIc. For $k > m$ in case of (3.11-ii), from (3.16), we have

$$
\begin{aligned}
&|\, q_{p,mk}(t) \,| \\
&\le \int_t^\infty \exp\left\{-\varepsilon(s^{1-\alpha} - t^{1-\alpha})\right\} \Bigg| r_{mk}(s) + \sum_{h=1}^{m-1} r_{mh}(s) q_{p,hk}(s) \\
&\qquad + \sum_{h=m+1}^{n} r_{mh}(s) q_{p-1,hk}(s) \Bigg| ds \\
&\le \left\{ \left[\max_{s\ge t}\{s^\alpha r_2(s)\}\right] + \sum_{h=1}^{m-1}\left[\max_{s\ge t} |s^\alpha r_{mh}(s) q_{p,hk}(s)|\right] \right. \\
&\qquad \left. + (n-m)G\left[\max_{s\ge t}\{s^\alpha r_2(s)\}\right]\right\} \int_t^\infty K s^{-\alpha} \exp\left\{-\varepsilon(s^{1-\alpha} - t^{1-\alpha})\right\} ds \\
&\le K_1\left\{ \tilde{r}_2(t) + (m-1)\tilde{r}_1(t)\left[\max_{\substack{s\ge t\\ 1\le h\le m-1}} |q_{p,hk}(s)|\right] + (n-m)G\tilde{r}_2(t)\right\}.
\end{aligned}
\tag{3.38}
$$

Since $h < m$, by the assumption (3.19) of this case, we have

$$
\begin{aligned}
&|\, q_{p,hk}(t) \,| \\
&\quad \le 2(1+nG)K_1\left[[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^{m-1} (t^{1-\alpha} - t_2^{1-\alpha})\right\}\right. \\
&\qquad \left. + \tilde{r}_2\left(\frac{t}{2^{m-1}}\right)\right].
\end{aligned}
\tag{3.39}
$$

Substituting these into (3.38), we obtain

$$\begin{aligned}|q_{p,mk}(t)| \le \Bigg[&\tilde{r}_2(t)\\
&+ 2K_1(m-1)(1+nG)\tilde{r}_1(t)\Bigg([\tilde{r}_1(t)+\tilde{r}_2(t)]\exp\left\{-\varepsilon\Big(\frac{1-\alpha}{2}\Big)^{m-1}(t^{1-\alpha}-t_2^{1-\alpha})\right\}\\
&+\tilde{r}_2\Big(\frac{t}{2^{m-1}}\Big)\Bigg) + (n-m)G\tilde{r}_2(t)\Bigg].\end{aligned}\tag{3.40}$$

By (3.36), we have

$$\begin{aligned}|q_{p,mk}(t)| &\le K_1\left[\tilde{r}_2(t) + (m-1)G[\tilde{r}_1(t)+\tilde{r}_2(t)]\exp\left\{-\varepsilon\Big(\frac{1-\alpha}{2}\Big)^{m-1}(t^{1-\alpha}-t_2^{1-\alpha})\right\}\right.\\
&\qquad\left. + (m-1)G\tilde{r}_2\Big(\frac{t}{2^{m-1}}\Big) + (n-m)G\tilde{r}_2(t)\right]\\
&\le K_1(m-1)G[\tilde{r}_1(t)+\tilde{r}_2(t)]\exp\left\{-\varepsilon\Big(\frac{1-\alpha}{2}\Big)^{m-1}(t^{1-\alpha}-t_2^{1-\alpha})\right\}\\
&\qquad + K_1(1+nG)\tilde{r}_2\Big(\frac{t}{2^{m-1}}\Big)\\
&\le 2K_1(1+nG)\left\{[\tilde{r}_1(t)+\tilde{r}_2(t)]\exp\left\{-\varepsilon\Big(\frac{1-\alpha}{2}\Big)^{m-1}(t^{1-\alpha}-t_2^{1-\alpha})\right\}+\tilde{r}_2\Big(\frac{t}{2^m}\Big)\right\}.\end{aligned}\tag{3.41}$$

Thus, (3.19) holds for this case.

In case of (3.11-i), from (3.16) and the assumption (3.19) of this case, since $h < m < k$, we have

$$\begin{aligned}|q_{p,mk}(t)| \le \int_{t_2}^{t} &K\exp\left\{-\varepsilon(t^{1-\alpha}-s^{1-\alpha})\right\}\left|r_{mk}(s)+\sum_{h=1}^{m-1} r_{mh}(s)q_{p,hk}(s)\right.\\
&\left.+\sum_{h=m+1}^{n} r_{mh}(s)q_{p-1,hk}(s)\right| ds\\
\le \Bigg\{&\left[\max_{s\ge t_2}\{s^\alpha r_2(s)\}\right] + (m-1)G\left[\max_{s\ge t_2}\{s^\alpha r_1(s)\}\right]\\
&+(n-m)G\left[\max_{s\ge t_2}\{s^\alpha r_2(s)\}\right]\Bigg\}\int_{t_2}^{\frac{(t+t_2)}{2}} Ks^{-\alpha}\exp\left\{-\varepsilon(t^{1-\alpha}-s^{1-\alpha})\right\} ds\\
&+\Bigg\{\left[\max_{s\ge\frac{t+t_2}{2}}\{s^\alpha r_2(s)\}\right] + (m-1)\left[\max_{\substack{s\ge\frac{t+t_2}{2}\\ 1\le h\le m-1}} |q_{p,hk}(s)|\right]\left[\max_{s\ge\frac{t+t_2}{2}}\{s^\alpha r_1(s)\}\right]\end{aligned}\tag{3.42}$$

$$
\begin{aligned}
&+ (n-m)G\left[\max_{s\geq\frac{t+t_2}{2}}\{s^\alpha r_2(s)\}\right]\Bigg\}\int_{\frac{(t+t_2)}{2}}^{t} Ks^{-\alpha}\exp\Big\{-\varepsilon(t^{1-\alpha}-s^{1-\alpha})\Big\}\,ds\\
&\leq K_1\Big\{\tilde r_2(t_2)+(m-1)G\tilde r_1(t_2)+(n-m)G\tilde r_2(t_2)\Big\}\exp\left\{-\frac{\varepsilon(1-\alpha)}{2}(t^{1-\alpha}-t_2^{1-\alpha})\right\}\\
&\quad + K_1\Bigg[\tilde r_2\Big(\frac{t+t_2}{2}\Big)+2K_1(m-1)(1+nG)\tilde r_1\Big(\frac{t+t_2}{2}\Big)[\tilde r_1(t_2)+\tilde r_2(t_2)]\\
&\qquad\cdot\exp\left\{-\varepsilon\Big(\frac{1-\alpha}{2}\Big)^{m-1}\left[\Big(\frac{t+t_2}{2}\Big)^{1-\alpha}-t_2^{1-\alpha}\right]\right\}\\
&\quad + 2K_1(m-1)(1+nG)\tilde r_1\Big(\frac{t+t_2}{2}\Big)\tilde r_2\Big(\frac{t}{2^m}\Big)+(n-m)G\tilde r_2\Big(\frac{t+t_2}{2}\Big)\Bigg].
\end{aligned}
\tag{3.42}
$$

By (3.36) and the fact that $\tilde r_2(t)$ is monotonic decreasing, we have

$$
\begin{aligned}
&|\,q_{p,mk}(t)\,|\\
&\leq K_1\Big[\tilde r_2(t_2)+(m-1)G\tilde r_1(t_2)+(n-m)G\tilde r_2(t_2)\Big]\exp\left\{-\frac{\varepsilon(1-\alpha)}{2}(t^{1-\alpha}-t_2^{1-\alpha})\right\}\\
&\quad + K_1\Bigg[\tilde r_2\Big(\frac{1+t_2}{2}\Big)+(m-1)G[\tilde r_1(t_2)+\tilde r_2(t_2)]\\
&\qquad\cdot\exp\left\{-\varepsilon\Big(\frac{1-\alpha}{2}\Big)^{m}(t^{1-\alpha}-t_2^{1-\alpha})\right\}\\
&\quad + (m-1)G\tilde r_2\Big(\frac{t}{2^m}\Big)+(n-m)G\tilde r_2\Big(\frac{t}{2}\Big)\Bigg]\\
&\leq K_1\Big[\tilde r_2(t_2)+nG\tilde r_1(t_2)+nG\tilde r_2(t_2)\\
&\quad + nG[\tilde r_1(t_2)+\tilde r_2(t_2)]\Big]\exp\left\{-\varepsilon\Big(\frac{1-\alpha}{2}\Big)^{m}(t^{1-\alpha}-t_2^{1-\alpha})\right\}\\
&\quad + K_1\left[\tilde r_2\Big(\frac{t}{2}\Big)+(m-1)G\tilde r_2\Big(\frac{t}{2^m}\Big)+(n-m)G\tilde r_2\Big(\frac{t}{2}\Big)\right]\\
&\leq 2K_1(1+nG)[\tilde r_1(t_2)+\tilde r_2(t_2)]\exp\left\{-\varepsilon\Big(\frac{1-\alpha}{2}\Big)^{m}(t^{1-\alpha}-t_2^{1-\alpha})\right\}\\
&\quad + K_1(1+nG)\tilde r_2\Big(\frac{t}{2^m}\Big)\\
&\leq 2K_1(1+nG)\left[[\tilde r_1(t_2)+\tilde r_2(t_2)]\exp\left\{-\varepsilon\Big(\frac{1-\alpha}{2}\Big)^{m}(t^{1-\alpha}-t_2^{1-\alpha})\right\}+\tilde r_2\Big(\frac{t}{2^m}\Big)\right].
\end{aligned}
\tag{3.43}
$$

Thus, by (3.29), (3.37) and (3.43), we have for $j=m$,

$$| q_{p,mk}(t) | \leq \begin{cases} K_2 nG[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^m (t^{1-\alpha} - t_2^{1-\alpha})\right\} \\ \qquad + nG \int_t^{+\infty} s^{-\alpha}\tilde{r}_2\left(\frac{s}{2^m}\right) ds, \qquad (k = m), \\ K_1(1+nG)\left[[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)(t^{1-\alpha} - t_2^{1-\alpha})\right\}\right. \\ \qquad \left. + \tilde{r}_1\left(\frac{t+t_2}{2}\right) + \tilde{r}_2\left(\frac{t+t_2}{2}\right)\right], \qquad (k < m), \\ 2K_1(1+nG)\left[[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^m (t^{1-\alpha} - t_2^{1-\alpha})\right\}\right. \\ \qquad \left. + \tilde{r}_2\left(\frac{t}{2^m}\right)\right], \qquad (k > m). \end{cases} \tag{3.44}$$

Consequently, (3.19) is true for $j, k = 1, 2, \cdots, n$, and Lemma 1 is proved.

STEP 6: In this step, we will show that, for large enough t_2, the sequence $\{q_{p,jk}(t) \mid p = 1, 2, \cdots\}$ converges uniformly to functions $q_{jk}(t)$ on $[t_2, \infty)$ satisfying the expressions (3.14), for $j, k = 1, 2, \cdots, n$. In order to do that, let

$$\| q_p - q_{p-1} \|_t = \max_{\substack{s \geq t \\ 1 \leq j,k \leq n}} | q_{p,jk}(s) - q_{p-1,jk}(s) |. \tag{3.45}$$

Note that, from (3.16),

$$q_{p+1,jk}(t) - q_{p,jk}(t) = \begin{cases} \int_\infty^t \left\{ \sum_{h=1}^{j-1} r_{jh}(s)[q_{p+1,hj}(s) - q_{p,hj}(s)] + \sum_{h=j+1}^{n} r_{jh}(s)[q_{p,hj}(s) - q_{p-1,hj}(s)] \right\} ds, \\ \qquad (j = k), \\ \int_{\tau_{jk}}^t \exp\left[\int_s^t \tilde{\lambda}_{jk}(\tau)\, d\tau\right] \left\{ \sum_{h=1}^{j-1} r_{jh}(s)[q_{p+1,hj}(s) - q_{p,hj}(s)] \right. \\ \qquad \left. + \sum_{h=j+1}^{n} r_{jh}(s)[q_{p,hj}(s) - q_{p-1,hj}(s)] \right\} ds, \qquad (j \neq k). \end{cases} \tag{3.46}$$

We will establish

LEMMA 2. *If t_2 is large enough such that*

(a) $$\int_t^\infty s^{-\alpha}\tilde{r}_2\left(\frac{t}{2^n}\right) ds < \frac{1}{4n}$$

and

(b) $$\tilde{r}_1(t_2) + \tilde{r}_2\left(\frac{t_2}{2^n}\right) < \min\left\{\frac{1}{8nK_1}, \frac{1}{4nK_2}\right\},$$

then we have

$$| q_{p+1,jk}(t) - q_{p,jk}(t) | \leq$$

$$\begin{cases} \left[nK_2[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^n (t^{1-\alpha} - t_2^{1-\alpha})\right\}\right. \\ \qquad \left. + n\int_t^{+\infty} s^{-\alpha}\tilde{r}_2\left(\frac{s}{2^n}\right) ds\right] \| q_p - q_{p-1} \|_{t_2}, \quad (j = k), \\ nK_1\left[[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)(t^{1-\alpha} - t_2^{1-\alpha})\right\} + \tilde{r}_1\left(\frac{t+t_2}{2}\right)\right. \\ \qquad \left. + \tilde{r}_2\left(\frac{t+t_2}{2}\right)\right] \| q_p - q_{p-1} \|_{t_2}, \quad (j > k), \\ 2nK_1\left[[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\frac{1-\alpha}{2}\right)^j (t^{1-\alpha} - t_2^{1-\alpha})\right\}\right. \\ \qquad \left. + \tilde{r}_2\left(\frac{t}{2^j}\right)\right] \| q_p - q_{p-1} \|_{t_2}, \quad (j < k), \end{cases} \tag{3.47}$$

for $t \geq t_2$ and $j, k = 1, 2, \cdots, n$; $p = 1, 2, \cdots$.

In fact, for t_2 satisfying Lemma 2, we have

$$| q_{p+1,jk}(t) - q_{p,jk}(t) | \leq \frac{1}{2} \| q_p - q_{p-1} \|_{t_2} \tag{3.48}$$

for $t \geq t_2$ and $j, k = 1, 2, \cdots, n; p = 1, 2, \cdots$. Hence,

$$\| q_{p+1} - q_p \|_{t_2} \leq \frac{1}{2} \| q_p - q_{p-1} \|_{t_2} \tag{3.49}$$

and

$$\| q_{p+1} - q_p \|_{t_2} \leq \frac{1}{2^p} \| q_1 - q_0 \|_{t_2} = \frac{1}{2^p} G \tag{3.50}$$

for $p = 1, 2, \cdots$. Therefore, the sequence $\{q_{p,jk}(t) \,|\, p = 1, 2, \cdots\}$ converges uniformly to function $q_{jk}(t)$ on $[t_2, \infty)$, for $j, k = 1, 2, \cdots, n$, and furthermore, they satisfy the expressions (3.14).

Lemma 2 can be proved in a fashion similar to that for Lemma 1 with slight modifications.

STEP 7: In order to see that

$$\lim_{t \to \infty} q_{jk}(t) = 0, \qquad j, k = 1, 2, \cdots, n, \tag{3.51}$$

note from Lemma 1, we have

$$|q_{jk}(t)| = \lim_{p\to\infty} |q_{p,jk}(t)| \leq$$

$$\begin{cases} K_2 nG[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\dfrac{1-\alpha}{2}\right)^n (t^{1-\alpha} - t_2^{1-\alpha})\right\} \\ \qquad + nG \displaystyle\int_t^{+\infty} s^{-\alpha} \tilde{r}_2\left(\frac{s}{2^n}\right) ds, & (j = k), \\ K_1(1+nG)\left[[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\dfrac{1-\alpha}{2}\right)(t^{1-\alpha} - t_2^{1-\alpha})\right\}\right. \\ \qquad \left. + \tilde{r}_1\left(\dfrac{t+t_2}{2}\right) + \tilde{r}_2\left(\dfrac{t+t_2}{2}\right)\right], & (j > k), \\ 2K_1(1+nG)\left[[\tilde{r}_1(t_2) + \tilde{r}_2(t_2)] \exp\left\{-\varepsilon\left(\dfrac{1-\alpha}{2}\right)^j (t^{1-\alpha} - t_2^{1-\alpha})\right\}\right. \\ \qquad \left. + \tilde{r}_2\left(\dfrac{t}{2^j}\right)\right], & (j < k), \end{cases} \tag{3.52}$$

for $t \in [t_2, \infty)$, and $j, k = 1, 2, \cdots, n$. By (2.5), we have

$$\lim_{t\to\infty} \tilde{r}_j(t) = 0, \qquad (j = 1, 2). \tag{3.53}$$

Thus, by (2.7) and (3.53), (3.51) follows.

This completes the proof of Theorem 2.1.

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