Kumamoto J. Math.
Vol. 7, 51~59 March (1994)

IDEAL BOUNDARY OF A COMPLETE METRIC SPACE

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(Received Octorber 20, 1993)

Introduction

A map $f : X \to Y$ between metric spaces is called a rough isometry if the following two conditions hold:

(i) the δ-neighborhood of $f(X)$ in Y coincides with Y for some positive δ,

(ii) there exist constants $\alpha \geq 1$ and $\beta \geq 0$ such that

$$\frac{1}{\alpha}\, d(p,q) - \beta \leq d(f(p), f(q)) \leq \alpha\, d(p,q) + \beta$$

for every $p,\ q \in X$.

In his papers [9] and [10] Kanai showed that rough isometries preserve some asymptotic properties of Riemannian manifolds; the volume growth rate of geodesic balls, positivity of isoperimetric constants and so on. On the other hand, for some classes of complete Riemannian manifolds, ideal boundaries are considered for the sake of investigating their asymptotic behaviour at infinity; for Hadamard manifolds[6], manifolds of asymptotically nonnegative curvature[11], and open surfaces with finite total curvature[16]. Regarding their ideal boundaries Schröder[4] proved a rigidity theorem for Hadamard manifolds of rank ≥ 2, and Ohtsuka and the author[2] characterized the Euclidean factor of Hadamard manifolds. It is now interesting to study some relationship between properties of ideal boundaries and rough isometries. In this paper, as a first step of our problem, we consider Hausdorff approximations. Here a rough isometry is called a Hausdorff approximation if one can take $\alpha = 1$ in the second condition. In recent papers, Kubo[12] and Ohtsuka[14] showed that Hausdorff approximations between Riemannian manifolds in those three classes induce isometries of their ideal boundaries. But since Hausdorff approximations do not preserve curvature conditions, it is natural to consider a more wide class. We therefore introduce the ideal boundary of complete metric space as the set of equivalence classes of non-wandering quasi-geodesic rays, and define the Tits metric on this boundary. For the previous three cases of [6],[11] and [16], equivalence

Supported partially by The Sumitomo Foundation

classes of geodesic rays were taken into consideration respectively. What we do here is, in some sense, to unify these articles. Our definition is convenient because we need not take care of the base point: The quantity of geodesic rays depend on the emanating point (see for example [13]). As a benefit of our definition, we can easily get a generalization of the results of [12] and [14].

Theorem. *A Hausdorff approximation between two metric spaces includes an isometry on their ideal boundaries with respect to the Tits metrices.*

We devote section 1 to study an equivalence relation for quasi-geodesic rays. Since the set of all equivalence classes is too large, we only consider non-wandering quasi-geodesic rays which are "asymptotically straight". We define in section 2 the ideal boundary for complete metric spaces by using them.

1. Equivalence relation of quasi-geodesic rays

Let (X, d) be a complete metric space. A map $\gamma : [0, \infty) \to X$ is called a quasi-geodesic ray if it satisfies, for some $a \geq 1$ and $b \geq 0$, that

$$(*) \qquad \frac{1}{a}|t-s| - b \leq d(\gamma(s), \gamma(t)) \leq a\,|t-s| + b.$$

When we can choose $a = 1$ and $b = 0$, we call γ a geodesic ray. We should note that quasi-geodesic rays might not be continuous. We start with a lemma on some basic properties of quasi-geodesic rays which will be used later.

Lemma 1. *Let γ be a quasi-geodesic ray satisfying $(*)$. The set $\{d(\gamma(t), \gamma(0)) | t \geq 0\}$ is unbounded and*

$$[r, r+2b] \cap \{d(\gamma(t), \gamma(0)) | t \geq 0\} \neq \emptyset$$

for every positive r.

Proof. Put $T = \sup\,\{t \geq 0 | d(\gamma(s), \gamma(0)) < r \; for \; 0 \leq s \leq t\}$.
Suppose $[r, r+2b] \cap \{d(\gamma(t), \gamma(0)) | t \geq 0\} = \emptyset$. There exists T_ε ($T \leq T_\varepsilon \leq T + \varepsilon$) for every positive ε such that $d(\gamma(0), \gamma(T - \varepsilon)) < r < r + 2b < d(\gamma(0), \gamma(T_\varepsilon))$. This leads us to a contradiction:

$$2b < d(\gamma(0), \gamma(T_\varepsilon)) - d(\gamma(0), \gamma(T-\varepsilon)) < d(\gamma(T_\varepsilon), \gamma(T-\varepsilon)) < 2a\varepsilon + b. \qquad \square$$

Given two quasi-geodesic rays γ and σ, we set

$$d_\gamma(\sigma) = \limsup_{t\to\infty} \frac{d(\sigma(t), \gamma)}{d(\sigma(t), \gamma(0))},$$

which is nonnegative and is not greater than 1. We call they are equivalent if $d_\gamma(\sigma) = d_\sigma(\gamma) = 0$, and denote $\gamma \sim \sigma$. We show that this is an equivalence relation.

Lemma 2. *Let γ,σ and ρ be quasi-geodesic rays. We have*

$$d_\gamma(\sigma) \leq d_\gamma(\rho) + d_\rho(\sigma) + d_\gamma(\rho) \cdot d_\rho(\sigma).$$

Proof. If $d_\rho(\sigma) = 1$, the inequality trivially holds. We suppose $d_\rho(\sigma) < 1$. Given a positive number ε, We choose $s_t > 0$ for every t, such that $d(\sigma(t), \rho(s_t)) \leq d(\sigma(t), \rho) + \varepsilon$. Since it holds for every $p \in X$ that

$$\begin{aligned} d(\rho(s_t), p) &\geq d(\sigma(t), p) - d(\sigma(t), \rho(s_t)) \geq d(\sigma(t), p) - d(\sigma(t), \rho) - \varepsilon, \\ d(\rho(s_t), p) &\leq d(\sigma(t), p) + d(\sigma(t), \rho(s_t)) \leq d(\sigma(t), p) + d(\sigma(t), \rho) + \varepsilon, \end{aligned}$$

we have

$$1 - d_\rho(\sigma) \leq \liminf_{t\to\infty} \frac{d(\rho(s_t), p)}{d(\sigma(t), p)} \leq \limsup_{t\to\infty} \frac{d(\rho(s_t), p)}{d(\sigma(t), p)} \leq 1 + d_\gamma(\sigma).$$

Hence s_t goes infinity as t goes infinity. The following inequality

$$\frac{d(\sigma(t), \gamma)}{d(\sigma(t), \gamma(0))} \leq \frac{d(\sigma(t), \rho) + \varepsilon}{d(\sigma(t), \gamma(0))} + \frac{d(\rho(s_t), \gamma(0))}{d(\sigma(t), \gamma(0))} \frac{d(\rho(s_t), \gamma)}{d(\rho(s_t), \gamma(0))}$$

leads us to the conclusion.

This lemma guarantees that if γ_1 and γ_2 are equivalent quasi-geodesic rays then $d_{\gamma_1}(\sigma) = d_{\gamma_2}(\sigma)$ and $d_\sigma(\gamma_1) = d_\sigma(\gamma_2)$ for every quasi-geodesic ray σ. Hence we get that $\sim$ is an equivalence relation.

We here check that our equivalence relation coincides with the relation on (unit speed) geodesic rays given in the previous papers [4],[11] and [16].

Given two quasi-geodesic rays γ,σ, positive c, and a point $p \in X$, we define

$$\ell_c(\gamma, \sigma; p) = \limsup_{r\to\infty} \frac{1}{r} \sup \{d(\gamma(s), \sigma(t)) | s \in \mathcal{D}_c(r; \gamma, p),\ t \in \mathcal{D}_c(r; \sigma, p)\},$$

where $\mathcal{D}_c(r; \gamma, p)$ denotes the set $\{t \geq 0 | r \leq d(\gamma(t), p) \leq r + c\}$. By lemma 1 we can easily get that if γ and σ satisfy $(*)$ then $\ell_c(\gamma, \sigma; p) = \ell_{c'}(\gamma, \sigma; p)$ when c,$c' \geq 2b$. Hence we can define $\ell(\gamma, \sigma; p) = \sup_{c\geq 0} \ell_c(\gamma, \sigma; p)$, which coincides with $\lim_{t\to\infty} \frac{1}{t} d(\gamma(t), \sigma(t))$ when γ and σ are (unit speed) geodesics. Since $\ell(\gamma, \sigma; p)$ does not depend on the choice of p, we denote it by $\ell(\gamma, \sigma)$. It trivially holds that $d_\gamma(\sigma) \leq \ell(\gamma, \sigma)$. We shall give an opposite estimate. We shall call a quasi-geodesic ray γ non-wandering if it satisfies the following asymptotically straight conditon;

$$(**) \qquad \lim_{s\to\infty} \sup_{t\geq s} \frac{d(\gamma(0),\gamma(s)) + d(\gamma(s),\gamma(t)) - d(\gamma(0),\gamma(t))}{d(\gamma(0),\gamma(s))} = 0.$$

Geodesic rays are of course non-wandering.

Lemma 3. *We have for every non-wandering quasi-geodesic rays γ and σ that*

$$d_\gamma(\sigma) \leq \ell(\gamma,\sigma) \leq 2d_\gamma(\sigma).$$

In particular, $d_\gamma(\sigma) = 0$ if and only if $d_\sigma(\gamma) = 0$, and γ and σ are equivalent if and only if $\ell(\gamma,\sigma) = 0$.

Proof. We shall show the second inequality which is trivial when $d_\gamma(\sigma) = 1$. Suppose $d_\gamma(\sigma) < 1$. Given a positive ε we choose u_t for each t such that $d(\sigma(t),\gamma(u_t)) \leq d(\sigma(t),\gamma)+\varepsilon$. Since γ is a non-wandering quasi-geodesic ray and u_t goes infinity as t goes to infinity, we have for sufficient large s and t that

$$\begin{aligned} d(\sigma(t),\gamma(u_t)) &< (d_\gamma(\sigma)+\varepsilon)\cdot d(\sigma(t),\gamma(0)), \\ d(\gamma(u_t),\gamma(s)) &< |d(\gamma(s),\gamma(0)) - d(\gamma(u_t),\gamma(0))| \\ &\quad + \varepsilon \max\{\, d(\gamma(s),\gamma(0)),\, d(\gamma(u_t),\gamma(0))\,\}. \end{aligned}$$

When $s \in \mathcal{D}_c(r;\gamma,\gamma(0))$ and $t \in \mathcal{D}_c(r;\sigma,\gamma(0))$ we have for sufficient large r that

$$\begin{aligned} d(\gamma(s),\gamma(u_t)) &< |\,d(\sigma(t),\gamma(0)) - d(\gamma(u_t),\gamma(0))\,| + |\,d(\gamma(s),\gamma(0)) - d(\sigma(t),\gamma(0))\,| \\ &\quad + \varepsilon \max\{\, d(\gamma(s),\gamma(0)), d(\sigma(t),\gamma(0)) + d(\gamma(u_t),\sigma(t))\,\} \\ &\leq |\,d(\sigma(t),\gamma(0)) - d(\gamma(u_t),\gamma(0))\,| + c + \varepsilon(1+d_\gamma(\sigma)+\varepsilon)(r+c) \\ &\leq d(\sigma(t),\gamma(u_t)) + c + \varepsilon(1+d_\gamma(\sigma)+\varepsilon)(r+c) \\ &< (d_\gamma(\sigma)+\varepsilon)(r+c) + c + \varepsilon(1+d_\gamma(\sigma)+\varepsilon)(r+c). \end{aligned}$$

Hence we get

$$\begin{aligned} d(\sigma(t),\gamma(t)) &\leq d(\sigma(t),\gamma(u_t)) + d(\gamma(u_t),\gamma(s)) \\ &< 2(d_\gamma(\sigma)+\varepsilon)(r+c) + c + \varepsilon(1+d_\gamma(\sigma)+\varepsilon)(r+c). \end{aligned}$$

We can therefore conclude that $\ell(\gamma,\sigma) = \ell(\gamma,\sigma;\gamma(0)) \leq 2d_\gamma(\sigma)$.

2. Ideal boundary

If X is negatively curved space in the sense of Gromov, it is known that for every quasi-geodesic ray one can find a geodesic ray in a neighborhood of it ([3]). But in general the situation is not the same. Even if X is a Hadamard manifold there exists an equivalence class of quasi-geodesic rays containing no geodesic ray.

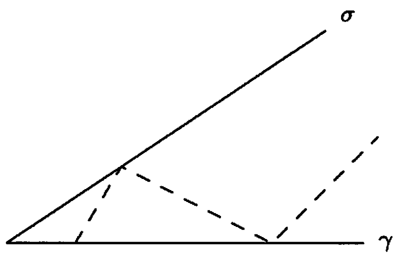
Example. Let γ and σ are half lines on $\mathbf{R}^2$ with $\gamma(0) = \sigma(0)$ and the angle θ of $\dot\gamma(0)$ and $\dot\sigma(0)$ is smaller than $\pi/2$. We inductively choose s_n and t_n so that

$$\begin{aligned} s_1 &= 1, \quad s_n = \max\Big(\frac{t_{n-1}}{\cos\theta}, \sum_{j=1}^{n-1} t_j\Big), \quad n \geq 2, \\ t_n &= \max\Big(\frac{s_n}{\cos\theta}, \sum_{j=1}^{n-1} t_j\Big), \quad n \geq 1. \end{aligned}$$

Then the broken line

$$\gamma(0) \to \gamma(s_1) \to \sigma(t_1) \to \gamma(s_2) \to \sigma(t_2) \to \cdots \to \gamma(s_n) \to \sigma(t_n) \to \cdots$$

is a quasi-geodesic ray, whose equivalence class does not contain geodesic rays.



In this reason we restrict ourselves to non-wandering quasi-geodesic rays. We define the ideal boundary $X(\infty)$ of X as the set of all equivalence classes of non-wandering quasi-geodesic rays. The fundamental neighborhood system for each $x \in X(\infty)$ can be given as follows. Let γ be a non-wandering quasi-geodesic ray which satisfies the inequality $(*)$ and $\gamma(\infty) = x$, where $\gamma(\infty)$ denotes the equivalence class of non-wandering quasi-geodesic rays containing γ. Take positive numbers R and ε, and put $C_{a,b}(\gamma, R, \varepsilon) = \bigcup_\sigma \sigma([0,\infty)) \cup \{\sigma(\infty)\}$, where σ runs over all non-wandering quasi-geodesic rays which satisfy $(*)$ and

$$\sup_{0 \leq t \leq R} \max\left(\frac{d(\sigma(t), \gamma)}{d(\sigma(t), \gamma(0))}, \frac{d(\gamma(t), \sigma)}{d(\gamma(t), \sigma(0))}\right) < \varepsilon.$$

This gives a topology on $\bar{X} = X \cup X(\infty)$, which is compatible with the topology on X, and is called the cone topology.

We now show that our definition of the ideal boundary coincides with the notion of the ideal boundary in case of Hadamard manifolds.

Proposition 1. *If X is a Hadamard manifold, then every equivalence class of non-wandering quasi-geodesic rays contains a (unit speed) geodesic ray, and is unique up to the emanating point.*

Proof. If there exist two equivalent geodesic rays γ_1 and γ_2, Lemma 3 assures that $\lim_{t\to\infty} \frac{1}{t} d(\gamma(t), \sigma(t)) = 0$. By the comparison theorem we get $\gamma_1 = \gamma_2$ if $\gamma_1(0) = \gamma_2(0)$.

We show the existence. Let σ be a non-wandering quasi-geodesic ray. We set $p = \sigma(0), r_t = d(\sigma(t), p)$ and choose unit tangent vectors $v_t \in U_pX$ so that $\sigma(t) = \exp_p(r_t\ v_t)$. Let $v_\infty \in U_pX$ be an accumulation point of $\{v_t\}_{t\geq 0}$. We denote by γ_t the unit speed geodesic ray joining $\sigma(0)$ and $\sigma(t)$, and by γ the geodesic ray satisfying $\dot{\gamma}(0) = v_\infty$. Since σ is non-wandering, for positive ε, if we take sufficient large s then we have

$$\sup_{t\geq s}\{r_s + d(\sigma(s), \sigma(t)) - r_t\} < \varepsilon r_s.$$

Let $p_{t,s} \in \gamma_t([0, \infty))$, $t > s$, be a point satisfying $d(\sigma(s), p_{t,s}) = d(\sigma(s), \gamma_t)$. If the geodesic triangle $\Delta(p, \sigma(s), \sigma(t))$ is Euclidean, we have

$$\begin{aligned} r_t &= \sqrt{r_s^2 - d(\sigma(s), p_{t,s})^2} + \sqrt{d(\sigma(s), \sigma(t))^2 - d(\sigma(s), p_{t,s})^2} \\ &\leq \sqrt{r_s^2 - d(\sigma(s), p_{t,s})^2} + d(\sigma(s), \sigma(t)), \end{aligned}$$

hence by use of the above inequality it is clear that $d(\sigma(s), p_{t,s}) < \sqrt{2\varepsilon} r_s$. Thus comparing associated Euclidean triangle we have for general case that $d(\sigma(s), p_{t,s}) < \sqrt{2\varepsilon} r_s$. Since $d(p, \gamma_t(r_s)) = r_s$ and $|\, d(p, p_{t,s}) - d(p, \sigma(s))\,| \leq d(\sigma(s), p_{t,s}) < \sqrt{2\varepsilon} r_s$, we get that $d(\sigma(s), \gamma_t(r_s)) \leq d(\sigma(s), p_{t,s}) + d(p_{t,s}, \gamma_t(r_s)) < 2\sqrt{2\varepsilon} r_s$. Therefore for sufficient large s we have

$$d(\sigma(s), \gamma(r_s)) = \lim_{j\to\infty} d(\sigma(s), \gamma_{t_j}(r_s)) < 2\sqrt{2\varepsilon} r_s,$$

and $d_\gamma(\sigma) < 2\sqrt{2\varepsilon}$, hence $d_\gamma(\sigma) = 0$. Therefore γ and σ are equivalent by Lemma 3, and we get the assertion.

This guarantees that our notion of the ideal boundary coincides with the previous definition given in [4]. We can apply the similar argument for complete Riemann surfaces with finite total curvature, since a non-wandering quasi-geodesic ray is contained in an end (see [12]). We therefore get the following.

Proposition 2. *Let X be a Riemann surface with finite total curvature.*

(1) Every equivalence class of non-wandering quasi-geodesic rays contains a (unit speed) geodesic ray.

(2) If γ and σ are (unit speed) geodesic rays in a same equivalence class, then they are equivalent in the sense of [16].

Next we introduce the Tits metric on $X(\infty)$. By Lemma 1 we have that $d_{\gamma_1}(\sigma_1) = d_{\gamma_2}(\sigma_2)$ if $\gamma_1 \sim \gamma_2$ and $\sigma_1 \sim \sigma_2$. We set $d_\infty(\gamma(\infty), \sigma(\infty)) = \max\ (d_\gamma(\sigma), d_\sigma(\gamma))$, which is well defined and gives a distance function on $X(\infty)$, though it does not give the same topology as the induced topology. The Tits metric Td is defined as the interior distance function of this distance. This metric measures somewhat expansiveness at infinity. We here point out that this metric is equivalent to the Tits metric due to Gromov[4] in the case that X is a Hadamard manifold. He define the Tits metric as the inner metric of the distance function ℓ on $X(\infty)$, which is defined by

$$\ell(x, y) = \lim_{t \to \infty} \frac{d(\gamma(t), \sigma(t))}{t}$$

where γ and σ are (unit speed) geodesic rays with $\gamma(\infty) = x$, $\sigma(\infty) = y$ and $\gamma(0) = \sigma(0)$. By the Lemma 3 we have

Proposition 3. *If X is a Hadamard manifold, the distance function d_∞ on $X(\infty)$ is equivalent to ℓ : $\frac{1}{2}\ell \le d_\infty \le \ell$.*

3. Hausdorff approximations

We now show that a Hausdorff approximation between complete metric spaces induces an isometry on their ideal boundaries. Let f : X $\to$ Y be a map between complete metric spaces satisfying the second condition of rough isometries;

$$\frac{1}{\alpha} d(p, q) - \beta \le d(f(p), f(q)) \le \alpha d(p, q) + \beta, \ p, q \in X.$$

It is clear that if γ and σ are quasi-geodesic rays then so are $f \circ \gamma$ and $f \circ \sigma$, and $\alpha^{-2} d_\gamma(\sigma) \le d_{f \circ \gamma}(f \circ \sigma) \le \alpha^2 d_\gamma(\sigma)$, which guarantees that $\gamma \sim \sigma$ if and only if $f \circ \gamma \sim f \circ \sigma$. When one can take $\alpha = 1$,it preserves the non-wandering property (**) of quasi-geodesic rays, hence induces an injective distance preserving map $\partial f : X(\infty) \to Y(\infty)$ with respect to the distance functions d_∞.

We shall show that ∂f is surjective when f is a Hausdorff approximation. Let λ be a non-wandering quasi-geodesic ray. Since $B_\delta(f(X)) = Y$ by the first condition, we can find

$\gamma(t) \in X$ with $d(\lambda(t), f\circ\gamma(t)) \leq \delta$ for each $t \geq 0$. Then

$$\begin{aligned} d(\gamma(s),\gamma(t)) &\leq d(f\circ\gamma(s), f\circ\gamma(t)) + \beta \\ &\leq d(\lambda(s),\lambda(t)) + \beta + 2\delta \leq a|t-s| + (b+\beta+2\delta) \end{aligned}$$

Similarly we find $d(\gamma(s),\gamma(t)) \geq \frac{1}{a}|t-s| - (b+\beta+2\delta)$ and

$$\frac{|\, d(\gamma(s),\gamma(t)) - d(\gamma(0),\gamma(t))\,|}{d(\gamma(0),\gamma(s))} \leq \frac{|\, d(\lambda(s),\gamma(t)) - d(\lambda(0),\lambda(t))\,| + 2\beta + 4\delta}{d(\lambda(0),\lambda(s)) - \beta - 2\delta}$$

Therefore γ is a non-wandering quasi-geodesic ray, and so is $f\circ\gamma$. We now get $\partial f(\gamma(\infty)) = (f\circ\gamma)(\infty) = \lambda(\infty)$ and ∂f is surjective.

Since ∂f preserves the topology on the ideal boundaries which are induced by the cone topologies, we get that $\partial f : X(\infty) \to Y(\infty)$ is an isometry with respect to the Tits metrices, and get our theorem.

As was shown in [14], Hausdorff approximations do not necessarily exist between manifolds with isometric ideal boundaries. Of course, a continuous extension of a rough isometry to the ideal boundary is usually impossible. But our consideration in this paper suggests us that ideal boundaries of two manifolds are equivalent in some sense if and only if these are of same dimension and rough isometric.

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