Kumamoto J. Math.
Vol. 7, 61~66 March (1994)

THE BEST COMPONENT SELECTION PROCEDURES FOR A MULTIVARIATE EXPONENTIAL DISTRIBUTION

Hiroto HYAKUTAKE

(Received October 21, 1993)

1. Introduction

Marshall and Olkin [3] proposed a multivariate extension of exponential distributions which is much of interest in both theoretical development and applications. The k dimensional distribution has 2^k-1 parameters. Proschan and Sullo [4] considered a $k+1$ parameter version for likelihood estimation.We consider the selection problem for the $k+1$ parameter version in this paper.

Let $(X_{1r}, X_{2r}, \cdots, X_{kr})$, $r = 1, 2, \cdots, n$, be random samples from the multivariate exponential (MVE) distribution whose survival function is given by

$$\begin{aligned} & P(X_{1r} > x_1, X_{2r} > x_2, \cdots, X_{kr} > x_k) \\ = \ & \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \cdots - \lambda_k x_k - \lambda_0 \max(x_1, x_2, \cdots, x_k)] \end{aligned}$$

where $x_i > 0$, $\lambda_i > 0$ $(i = 1, 2, \cdots, k)$ and $\lambda_0 \geq 0$. This MVE distribution is derived by supposing that failure is caused by $k+1$ types of Poisson shocks on a system containing k components. The problem is to select the best component with respect to λ_i $(i = 1, 2, \cdots, k)$ which are viewed as the hazard of the components in lifetime analysis. We define that the j-th component is the best if $\lambda_j = \min(\lambda_1, \lambda_2, \cdots, \lambda_k)$. The best component has the largest mean because of $E(X_{ir}) = 1/(\lambda_i + \lambda_0)$. Throughout this paper, we assume that the 1st component is the best, that is,

$$\lambda_1 < \lambda_i \quad (i = 2, 3, \cdots, k) \tag{1.1}$$

without loss of generality and

$$\delta\lambda_1 = \lambda_2 = \cdots = \lambda_k \quad (\delta > 1). \tag{1.2}$$

Since $X_{1r}, X_{2r}, \cdots, X_{kr}$ are mutually independent if and only if $\lambda_0 = 0$, the problem is reduced to the problem of selecting the best of k univariate exponential populations under $\lambda_0 = 0$.

In Section 2, we propose two selection procedures, one of which is given by using multinomial distribution and the other is based on the marginal distribution. By using the central limit theorem, the probabilities of correct selection for proposed procedures are compared in Section 3. For $k = 2$, these are discussed in Hyakutake [2].

2. Selection procedure

2.1 Procedure R_1

Selecting the best component with respect to λ_i is equivalent to that with respect to $p_i = \lambda_i / \sum_{l=0}^{k} \lambda_l$. From Arnold [1],

$$P(X_{ir} < X_{i'r},\ i' = 1, \cdots, i-1, i+1, \cdots, k) = p_i$$

for $i = 1, 2, \cdots, k$ and

$$P(X_{1r} = X_{2r} = \cdots = X_{kr}) = p_0.$$

Let n_i and n_0 denote the number of observations in the region $\{x_i < x_{i'}\}$ and $\{x_1 = \cdots = x_k\}$, respectively, then $(n_1, n_2, \cdots, n_k, n_0)$ has a multinomial distribution with n and cell probability $(p_1, p_2, \cdots, p_k, p_0)$. Hence the simple procedure is based on n_j, that is, we select j-th component as the best when $n_j = \min(n_1, \cdots, n_k)$. Since we may express as

$$P(CS \mid R_1) = P(n_1 < n_2, \cdots, n_1 < n_k) \tag{2.1}$$

by the assumption (1.1), i.e. $p_1 = \min(p_1, \cdots p_k)$, it is easily seen that

$$\begin{aligned} P(CS \mid R_1) &= \sum_{n_1 < n_2, \cdots, n_1 < n_k} \frac{n!}{n_1! n_2! \cdots n_k! n_0!} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} p_0^{n_0} \\ &= \sum_{n_1 < n_2, \cdots, n_1 < n_k} \frac{n!}{n_1! n_2! \cdots n_k! n_0!} \delta^{n - n_1 - n_0} p_1^{n - n_0} p_0^{n_0} \end{aligned}$$

by (1.2), where $n = \sum_{l=0}^{k} n_l$. Arnold [1] gave estimates of λ_i, that is, $\tilde{\lambda}_i = n_i / \sum_{r=1}^{n} X_r^{(1)}$, where $X_r^{(1)} = \min(X_{1r}, X_{2r}, \cdots, X_{kr})$. If these estimates are used for the selection (procedure R_1'), then

$$P(CS \mid R_1') = P(\tilde{\lambda}_1 < \tilde{\lambda}_i,\ i = 2, \cdots, k),$$

which is equal to (2.1). Namely, the procedure R_1' is equivalent to the procedure R_1.

2.2 Procedure R_2

The next proposed procedure R_2 is based on the marginal distribution. The distribution of X_{ir} is the exponential with parameter $\lambda_i + \lambda_0$. The selection problem with respect to λ_i is equivalent to that of $\lambda_i + \lambda_0$. Let $\bar{X}_i = \sum_{r=1}^{n} X_{ir}/n$, then $\bar{X}_i$ is an unbiased estimate of $1/(\lambda_i + \lambda_0)$.

This $(\bar{X}_i)$ can be obtained as INT estimator by the functional relationship

$$m_i/\lambda_i + m_i^{(c)}/(\lambda_i + \lambda_0) = \sum_{r=1}^{n} X_{ir} \tag{2.2}$$

derived by Proschan and Sullo [4], where m_i is that the number of times i-th component is observed to fail strictly before the last failure and $m_i^{(c)}$ is that the number of times i-th component is observed to fail last, but not simultaneously with any other component. The left-side of (2.2) can be written by

$$\frac{m_i + m_i\lambda_0/\lambda_i + m_i^{(c)}}{\lambda_i + \lambda_0}.$$

Substituting $m_0(i)/m_i$ in λ_0/λ_i, we obtain the equation

$$n/(\lambda_i + \lambda_0) = \sum_{r=1}^{n} X_{ir} \tag{2.3}$$

by $m_i + m_0(i) + m_i^{(c)} = n$, where $m_0(i)$ is that the number of times i-th component is observed to have failed simultaneously with one or more of the other components. These are discussed in Proschan and Sullo [4]. It is easily seen that the INT estimator of $\lambda_i + \lambda_0$ is $1/\bar{X}_i$ by (2.3).

Hence we select j-th component as the best when $\bar{X}_j = \max(\bar{X}_1, \cdots, \bar{X}_k)$. By the assumption (1.1), we may express as

$$P(CS \mid R_2) = P(\bar{X}_1 > \bar{X}_2, \cdots, \bar{X}_1 > \bar{X}_k). \tag{2.4}$$

3. Asymptotic comparison

We compare $P(CS \mid R_1)$ and $P(CS \mid R_2)$ by normal approximation, because the exact comparison is quite difficult. By Marshall and Olkin [3],

$$V(X_{ir}) = 1/(\lambda_i + \lambda_0)^2, \quad Cov(X_{ir}, X_{jr}) = \lambda_0/(\lambda_i + \lambda_0)(\lambda_j + \lambda_0)(\lambda_i + \lambda_j + \lambda_0).$$

Under (1.2), we have

$$\begin{aligned}
\mu_1 &= E[(n_i - n_1)/n] = \delta(1 - p_1) \\
\sigma_1^2 &= V[(n_i - n_1)/n] = \{(\delta + 1)p_1 - (\delta - 1)^2 p_1^2\}/n \\
\rho_1\sigma_1^2 &= Cov[(n_i - n_1)/n, (n_j - n_1)/n] = \{p_1 - (\delta - 1)^2 p_1^2\}/n \\
\mu_2 &= E[\bar{X}_1 - \bar{X}_i] = \frac{(\delta - 1)\lambda_1}{(\lambda_1 + \lambda_0)(\delta\lambda_1 + \lambda_0)} \\
\sigma_2^2 &= V[\bar{X}_1 - \bar{X}_i] \\
&= \{\frac{1}{(\lambda_1 + \lambda_0)^2} + \frac{1}{(\delta\lambda_1 + \lambda_0)^2} - \frac{2\lambda_0}{(\lambda_1 + \lambda_0)(\delta\lambda_1 + \lambda_0)(\lambda_1 + \delta\lambda_1 + \lambda_0)}\}/n \\
\rho_2\sigma_2^2 &= Cov[\bar{X}_1 - \bar{X}_i, \bar{X}_1 - \bar{X}_j] \\
&= \{\frac{1}{(\lambda_1 + \lambda_0)^2} + \frac{\lambda_0}{(\delta\lambda_1 + \lambda_0)^2(2\delta\lambda_1 + \lambda_0)} - \frac{2\lambda_0}{(\lambda_1 + \lambda_0)(\delta\lambda_1 + \lambda_0)(\lambda_1 + \delta\lambda_1 + \lambda_0)}\}/n
\end{aligned}$$

for all i and j $(i \neq j, i \neq 1, j \neq 1)$, where $p_1 = \lambda_1/\{\lambda_1 + \delta(k-1)\lambda_1 + \lambda_0\}$. Let $Y_i^{(1)} = \{(n_i - n_1)/n - \mu_1\}/\sigma_1$ and $Y_i^{(2)} = (\bar{X}_1 - \bar{X}_i - \mu_2)/\sigma_2$ $(i = 2, \cdots, k)$, then (2.1) and (2.4) can be written by

$$P(CS \mid R_l) = P(Y_i^{(l)} > -c_l, \ i = 2, \cdots, k), \quad l = 1, 2$$

where

$$\begin{aligned}
c_1 = \frac{\mu_1}{\sigma_1} &= \{\frac{n(\delta - 1)^2}{(\delta + 1)(k\delta - \delta + 1 + d) - (\delta - 1)^2}\}^{1/2} \\
c_2 = \frac{\mu_2}{\sigma_2} &= \{\frac{n(\delta - 1)^2(\delta + 1 + d)}{2(\delta + 1)d^2 + (3\delta^2 + 2\delta + 3)d + (\delta^2 + 1)(\delta + 1)}\}^{1/2}
\end{aligned}$$

and $d = \lambda_0/\lambda_1$. By the central limit theorem, $(Y_2^{(l)}, \cdots, Y_k^{(l)})'$ has the $k-1$ variate normal distribution with mean vector $\mathbf{0}$ and covariance matrix $(1 - \rho_l)I_{k-1} + \rho_l \mathbf{1}_{k-1}\mathbf{1}_{k-1}'$, where I_{k-1}

is the $(k-1)\times(k-1)$ identity matrix and $1_{k-1}:(k-1)\times 1 = (1,\cdots,1)'$. The covariance matrix of $(Y_2^{(l)},\cdots,Y_k^{(l)})$ is the correlation matrix. In order to compare $P(CS \mid R_1)$ and $P(CS \mid R_2)$, we use the Slepian's inequality (see e.g. Tong [5]):

Let $(Z_1^{(l)},\cdots,Z_k^{(l)})'$ have the k variate normal distribution with mean $\mathbf{0}$ and correlation matrix $\Gamma^{(l)} = (\gamma_{ij}^{(l)})$ $(l=1,2)$. If $\gamma_{ij}^{(1)} \geq \gamma_{ij}^{(2)}$ holds for all i,j $(i \neq j)$, then

$$P(Z_1^{(1)} > a_1,\cdots,Z_k^{(1)} > a_k) \geq P(Z_1^{(2)} > a_1,\cdots,Z_k^{(2)} > a_k) \tag{3.1}$$

holds for all $(a_1,\cdots,a_k)$.

THEOREM. *Under the normal approximation, if $d < \sqrt{2}$ and $\delta \to 1$ or if $k \geq 3$ and $d \to 0$, then*

$$P(CS \mid R_1) \leq P(CS \mid R_2). \tag{3.2}$$

PROOF. We can obtain the follows:

$$\lim_{\delta\to 1}\rho_1 = 1/2, \quad \lim_{\delta\to 1}\rho_2 = \frac{1}{1+1/(d+1)} \geq \frac{1}{2}$$

by $d \geq 0$. If $d < \sqrt{2}$, then

$$\lim_{\delta\to 1}\frac{c_1^2}{c_2^2} = \frac{2(d+1)^2}{(d+2)(d+k)} < 1.$$

Next we have

$$\lim_{d\to 0}\rho_1 = \frac{1}{1+\delta(k\delta-\delta+1)/\{k\delta-\delta+1-(\delta-1)^2\}} < \frac{1}{2}, \quad \lim_{d\to 0}\rho_2 = \frac{1}{1+1/\delta^2} > \frac{1}{2}$$

by $\delta > 1$. If $k \geq 3$, then

$$\lim_{d\to 0}\frac{c_1^2}{c_2^2} = \frac{\delta^2+1}{(\delta+1)(k\delta-\delta+1)-(\delta-1)^2} < 1.$$

Hence (3.2) holds by applying (3.1).

This result means that R_2 is better procedure than R_1 under the conditions of Theorem. Hyakutake [2] proved the similar result for $k=2$. We recommend R_2 for selecting the best component of the MVE distribution.

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Department of Mathematics
Faculty of Engineering
Kumamoto University