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A REMARK ON AN ABELIANNESS OF INVARIANT STATES ON C*-DYNAMICS

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1. Introduction.

In [2], we considered a subspace $\mathcal{H}'_\varphi$ to get a characterization of the ergodicity of an invariant state φ on a C*-dynamics (A, G, α), which makes the same role as $L^2(\varphi)$ in the case the C*-algebra A is abelian. And, by the use of the subspace $\mathcal{H}'_\varphi$, we gave a characterization of an abelianness of invariant states on C*-dynamics.

In this note we will give another characterization of an abelianness of invariant states on C*-dynamics to claim that it is important to consider the subspace $\mathcal{H}'_\varphi$.

Let A be a C*-algebra with unit element and α an action of a group G on A. We say that (A, G, α) is a *C*-dynamics*. A state φ on A is said to be *invariant* if $\varphi(\alpha_g(x)) = \varphi(x)$ for $x \in A$ and $g \in G$. By $\mathcal{S}_G$, we denote the set of all invariant states on the C*-dynamics (A, G, α). Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the cyclic representation of A induced by a state φ on A. If φ is an invariant state on (A, G, α), then it induces a unitary representation u^φ (or simply u) of G on the Hilbert space $\mathcal{H}_\varphi$ such that $u_g \pi_\varphi(x) u_g^* = \pi_\varphi \circ \alpha(x)$ for $x \in A$ and $g \in G$ and $u_g \xi_\varphi = \xi_\varphi$ for $g \in G$. In fact, it is defined by $u_g^\varphi \pi_\varphi(x)\xi_\varphi = \pi_\varphi(\alpha_g(x))\xi_\varphi$. By $\mathcal{H}'_\varphi$, we denote the closed subspace $[\pi_\varphi(A)'\xi_\varphi]$ of $\mathcal{H}_\varphi$ which is the closed linear span of the subset $\pi_\varphi(A)'\xi_\varphi$ and by e_φ, the projection onto $\mathcal{H}'_\varphi$. For $\varphi \in \mathcal{S}_G$, we denote p_φ and q_φ be the projections onto the subspaces $\{\xi \in \mathcal{H}_\varphi : u_g\xi = \xi \ (g \in G)\}$ and $\{\eta \in \mathcal{H}'_\varphi : u_g\eta = \eta \ (g \in G)\}$, respectively. The pair (A, φ) is defined to be *G-abelian* if

$$\inf\{|(\pi_\varphi([a', b])\xi|\xi)| : a' \in Co(\alpha_G(a))\} = 0$$

for all $a, b \in A$ and all $\xi \in p_\varphi \mathcal{H}_\varphi$, where $Co(\alpha_G(a))$ is the convex hull of $\{\alpha_g(a) : g \in G\}$ ([1]). Moreover, we say that (A, φ) is *wG-abelian* (*weakly G-abelian*) if

$$\inf\{|(\langle \pi_\varphi(a'), \pi_\varphi(b)\rangle \eta|\eta)| : a' \in Co(\alpha_G(a))\} = 0$$

for all $a, b \in A$ and all $\eta \in q_\varphi \mathcal{H}_\varphi$, where $\langle x, y\rangle = xe_\varphi y - ye_\varphi x$. Then we will show that *the following conditions are equivalent* : (1) *the pair* (A, φ) *is wG-abelian*; (2) $q_\varphi \pi_\varphi(A) q_\varphi$ *is* *abelian*; (3) $\{\pi_\varphi(A), u_G\}'$ *is abelian* (Theorem 2).

2. wG-abelian systems.

By a slight modification of the proof of the equivalence of the conditions (1) and (2) in Proposition 4.3.7.([1]), we can prove the following

PROPOSITION 1. *Let φ be an invariant state on a C^*-dynamics (A, G, α) . Then (A, φ) is wG-abelian if and only if $q_\varphi \pi_\varphi(A) q_\varphi$ is abelian, in the sense that the operators in $q_\varphi \pi_\varphi(A) q_\varphi$ commute mutually.*

PROOF. Suppose that (A, φ) is wG-abelian. Given $\epsilon > 0, a = a^*$ and $\eta \in q_\varphi \mathcal{H}_\varphi$, there exists a convex combination $\sum_{i=1}^n \lambda_i u_{g_i}$ of u^φ such that

$$\|(\sum_{i=1}^n \lambda_i u_{g_i} - q_\varphi) e_\varphi \pi_\varphi(a)\eta\| < \epsilon$$

([1] Proposition 4.3.4.). For any other convex combination $\sum_{j=1}^m \mu_j u_{h_j}$ of u^φ, we have

$$\|q_\varphi \pi_\varphi(a)\eta - e_\varphi \pi_\varphi(\sum_{j=1}^m \mu_j \alpha_{h_j}(\sum_{i=1}^n \lambda_i \alpha_{g_i}(a)))\eta\| = \|\sum_{j=1}^m \mu_j u_{h_j}[(q_\varphi - \sum_{i=1}^n \lambda_i u_{g_i}) e_\varphi \pi_\varphi(a)\eta]\| < \epsilon.$$

Thus for any $b \in A$, we have

$$\begin{aligned} &|(\pi_\varphi(a) q_\varphi \pi_\varphi(b)\eta|\eta) - (\pi_\varphi(b) q_\varphi \pi_\varphi(a)\eta|\eta)| \\ &\leq 2\epsilon\|b\|\|\eta\| + |(\langle \pi_\varphi(\textstyle\sum_{j=1}^m \mu_j \alpha_{h_j}(\sum_{i=1}^n \lambda_i \alpha_{g_i}(a))), \pi_\varphi(b)\rangle\eta|\eta)|. \end{aligned}$$

Since the convex combination $\sum_{j=1}^m \mu_j u_{h_j}$ is arbitrary, by the assumption we have

$$([q_\varphi \pi_\varphi(a) q_\varphi, q_\varphi \pi_\varphi(b) q_\varphi]\eta|\eta) = 0.$$

As this is hold for all $\eta \in q_\varphi \mathcal{H}_\varphi$, it follows that $q_\varphi \pi_\varphi(A) q_\varphi$ is abelian.

Conversely, suppose that $q_\varphi \pi_\varphi(A) q_\varphi$ is abelian. Then for the above convex combination $\sum_{i=1}^n \lambda_i \alpha_{g_i}(a)$, we have

$$\begin{aligned} &|(\langle \pi_\varphi(\textstyle\sum_{i=1}^n \lambda_i \alpha_{g_i}(a)), \pi_\varphi(b)\rangle\eta|\eta)| \\ &\leq \|b\|\|\eta\|\|(\textstyle\sum_{i=1}^n \lambda_i u_{g_i} - q_\varphi) e_\varphi \pi_\varphi(a^*)\eta\| + \|(q_\varphi - \sum_{i=1}^n \lambda_i u_{g_i}) e_\varphi \pi_\varphi(a)\eta\|\|b\|\|\eta\|. \end{aligned}$$

Hence it follows from Proposition 4.3.4([1]) that (A, φ) is wG-abelian.

This completes the proof.

Thus we can state the following

THEOREM 2. *For $\varphi \in S_G$, the following conditions are equivalent :*

(1) (A, φ) *is wG-abelian ;*

(2) $q_\varphi \pi_\varphi(A) q_\varphi$ *is abelian ;*

(3) $\{\pi_\varphi(A), u_G\}'$ *is abelian ;*

(4) *there exists a unique maximal probability measure on S_G with barycenter φ .*

PROOF. (1) $\Leftrightarrow$ (2): By Proposition 1.

(2) $\Leftrightarrow$ (3): By Theorem 2 of [2].

(3) $\Leftrightarrow$ (4): By Proposition 4.3.7.(3) $\Leftrightarrow$ (4) of [1].

As an immediate consequence, we obtain the following

COROLLARY. *For $\varphi \in S_G$, if ξ_φ is separating for $\pi_\varphi(A)''$, then the following conditions are equivalent :*

(1) (A, φ) *is G-abelian ;*

(2) $p_\varphi \pi(A) p_\varphi$ *is abelian ;*

(3) $\{\pi_\varphi(A), u_G\}'$ *is abelian ;*

(4) *there exists a unique maximal probability measure on S_G with barycenter φ.* (cf.[1] Proposition 4.3.7.)

References

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