A REMARK ON AN ABELIANNESS
OF INVARIANT STATES ON C*-DYNAMICS

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1. Introduction.

In [2], we considered a subspace $\mathcal{H}_\varphi'$ to get a characterization of the ergodicity of an invariant state $\varphi$ on a C*-dynamics $(A, G, \alpha)$, which makes the same role as $L^2(\varphi)$ in the case the C*-algebra $A$ is abelian. And, by the use of the subspace $\mathcal{H}_\varphi'$, we gave a characterization of an abelianness of invariant states on C*-dynamics.

In this note we will give another characterization of an abelianness of invariant states on C*-dynamics to claim that it is important to consider the subspace $\mathcal{H}_\varphi'$.

Let $A$ be a C*-algebra with unit element and $\alpha$ an action of a group $G$ on $A$. We say that $(A, G, \alpha)$ is a C*-dynamics. A state $\varphi$ on $A$ is said to be invariant if $\varphi(\alpha_g(x)) = \varphi(x)$ for $x \in A$ and $g \in G$. By $\mathcal{S}_G$, we denote the set of all invariant states on the C*-dynamics $(A, G, \alpha)$. Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the cyclic representation of $A$ induced by a state $\varphi$ on $A$. If $\varphi$ is an invariant state on $(A, G, \alpha)$, then it induces a unitary representation $u^\varphi$ (or simply $u$) of $G$ on the Hilbert space $\mathcal{H}_\varphi$ such that $u_g \pi_\varphi(x) u_g^* = \pi_\varphi(\alpha(x))$ for $x \in A$ and $g \in G$ and $u_g \xi_\varphi = \xi_\varphi$ for $g \in G$. In fact, it is defined by $u_g \pi_\varphi(x) = \pi_\varphi(\alpha_g(x)) \xi_\varphi$. By $\mathcal{H}_\varphi'$, we denote the closed subspace $[\pi_\varphi(A) \xi_\varphi]$ of $\mathcal{H}_\varphi$ which is the closed linear span of the subset $\pi_\varphi(A) \xi_\varphi$ and by $e_\varphi$, the projection onto $\mathcal{H}_\varphi'$. For $\varphi \in \mathcal{S}_G$, we denote $p_\varphi$ and $q_\varphi$ be the projections onto the subspaces $\{ \xi \in \mathcal{H}_\varphi : u_g \xi = \xi \ (g \in G) \}$ and $\{ \eta \in \mathcal{H}_\varphi' : u_g \eta = \eta \ (g \in G) \}$, respectively. The pair $(A, \varphi)$ is defined to be $G$-abelian if

$$\inf \{ \| \pi_\varphi([a', b]) \xi \| : a' \in Co(\alpha_G(a)) \} = 0$$

for all $a, b \in A$ and all $\xi \in p_\varphi \mathcal{H}_\varphi$, where $Co(\alpha_G(a))$ is the convex hull of $\{ \alpha_g(a) : g \in G \}$ ([1]). Moreover, we say that $(A, \varphi)$ is $wG$-abelian (weakly $G$-abelian) if

$$\inf \{ \| (\pi_\varphi(a'), \pi_\varphi(b)) \eta \eta \| : a' \in Co(\alpha_G(a)) \} = 0$$

for all $a, b \in A$ and all $\eta \in q_\varphi \mathcal{H}_\varphi$, where $(x, y) = xe_\varphi y - ye_\varphi x$. Then we will show that the following conditions are equivalent: (1) the pair $(A, \varphi)$ is $wG$-abelian; (2) $q_\varphi \pi_\varphi(A) q_\varphi$ is
abelian; (3) \( \{ \pi_\varphi(A), u_G \}^\prime \) is abelian (Theorem 2).

2. \( wG \)-abelian systems.

By a slight modification of the proof of the equivalence of the conditions (1) and (2) in Proposition 4.3.7([1]), we can prove the following

**Proposition 1.** Let \( \varphi \) be an invariant state on a \( C^* \)-dynamics \( (A, G, \alpha) \). Then \( (A, \varphi) \) is \( wG \)-abelian if and only if \( q_\varphi \pi_\varphi(A)q_\varphi \) is abelian, in the sense that the operators in \( q_\varphi \pi_\varphi(A)q_\varphi \) commute mutually.

**Proof.** Suppose that \( (A, \varphi) \) is \( wG \)-abelian. Given \( \epsilon > 0, a = a^* \) and \( \eta \in q_\varphi \mathcal{H}_\varphi \), there exists a convex combination \( \sum_{i=1}^n \lambda_i u_{g_i} \) of \( u^\varphi \) such that

\[
\| (\sum_{i=1}^n \lambda_i u_{g_i} - q_\varphi) e_\varphi \pi_\varphi(a) \eta \| < \epsilon
\]

([1] Proposition 4.3.4.). For any other convex combination \( \sum_{j=1}^m \mu_j u_{h_j} \) of \( u^\varphi \), we have

\[
\| q_\varphi \pi_\varphi(a) \eta - e_\varphi \pi_\varphi(\sum_{j=1}^m \mu_j \alpha_{h_j} (\sum_{i=1}^n \lambda_i \alpha_{g_i}(a))) \eta \| = \| \sum_{j=1}^m \mu_j u_{h_j} [(q_\varphi - \sum_{i=1}^n \lambda_i u_{g_i}) e_\varphi \pi_\varphi(a) \eta] \| < \epsilon.
\]

Thus for any \( b \in A \), we have

\[
|((\pi_\varphi(a) q_\varphi \pi_\varphi(b) \eta) | \eta \rangle - (\pi_\varphi(b) q_\varphi \pi_\varphi(a) \eta) | \eta \rangle | \\
\leq 2\epsilon \| \| b \| \| \| \eta \| \| + |((\pi_\varphi(\sum_{j=1}^m \mu_j \alpha_{h_j} (\sum_{i=1}^n \lambda_i \alpha_{g_i}(a))), \pi_\varphi(b)) \eta) \| |.
\]

Since the convex combination \( \sum_{j=1}^m \mu_j u_{h_j} \) is arbitrary, by the assumption we have

\[
((q_\varphi \pi_\varphi(a) q_\varphi, q_\varphi \pi_\varphi(b) q_\varphi) \eta) = 0.
\]

As this is hold for all \( \eta \in q_\varphi \mathcal{H}_\varphi \), it follows that \( q_\varphi \pi_\varphi(A)q_\varphi \) is abelian.

Conversely, suppose that \( q_\varphi \pi_\varphi(A)q_\varphi \) is abelian. Then for the above convex combination

\( \sum_{i=1}^n \lambda_i \alpha_{g_i}(a) \), we have

\[
|((\pi_\varphi(\sum_{i=1}^n \lambda_i \alpha_{g_i}(a)), \pi_\varphi(b)) \eta) | \\
\leq \| b \| \| \eta \| \| (\sum_{i=1}^n \lambda_i u_{g_i} - q_\varphi) e_\varphi \pi_\varphi(a^* \eta) \| + \| (q_\varphi - \sum_{i=1}^n \lambda_i u_{g_i}) e_\varphi \pi_\varphi(a) \eta \| \| b \| \| \eta \| \|
\]

Hence it follows from Proposition 4.3.4([1]) that \( (A, \varphi) \) is \( wG \)-abelian.

This completes the proof.

Thus we can state the following
THEOREM 2. For $\varphi \in S_G$, the following conditions are equivalent:

1. $(A, \varphi)$ is $wG$-abelian;
2. $q_\varphi \pi_\varphi(A)q_\varphi$ is abelian;
3. $\{\pi_\varphi(A), u_G\}'$ is abelian;
4. there exists a unique maximal probability measure on $S_G$ with barycenter $\varphi$.

PROOF. (1) $\Leftrightarrow$ (2): By Proposition 1.
(2) $\Leftrightarrow$ (3): By Theorem 2 of [2].
(3) $\Leftrightarrow$ (4): By Proposition 4.3.7. (3) $\Leftrightarrow$ (4) of [1].

As an immediate consequence, we obtain the following

COROLLARY. For $\varphi \in S_G$, if $\xi_\varphi$ is separating for $\pi_\varphi(A)''$, then the following conditions are equivalent:

1. $(A, \varphi)$ is $G$-abelian;
2. $p_\varphi \pi_\varphi(A)p_\varphi$ is abelian;
3. $\{\pi_\varphi(A), u_G\}'$ is abelian;
4. there exists a unique maximal probability measure on $S_G$ with barycenter $\varphi$. (cf.[1] Proposition 4.3.7.)

References


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