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ON THE 3-DIMENSIONAL PSEUDO-HERMITIAN SPACE FORMS AND OTHER GEOMETRIC STRUCTURES

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1. Introduction

A strictly pseudoconvex pseudo-Hermitian structure on a smooth manifold M is a contact form ω, equipped with a complex structure J on Null $\omega = \{X \in TM \mid \omega(X) = 0\}$. Null ω is a codimension 1 subbundle of the tangent bundle TM. It is called a contact subbundle. (Null ω, J) is called a CR structure. It is known that every orientable closed 3-manifold M admits a contact structure (cf. [10],[16],[46],[48],[63]). Since the contact subbundle Null ω is two dimensional, there exists a complex structure. Thus an orientable closed 3-manifold supports a pseudo-Hermitian structure. If $(M, (\omega, J))$ is a strictly pseudoconvex pseudo-Hermitian manifold, it admits a canonical Riemannian metric g^+ (resp. *Lorentz metric* g^-). (See §2.) Making use of the curvatures of these metrics, we can define a smooth function Λ on M^3 and then obtain a pseudo-Hermitian invariant on a compact 3-manifold M^3 with (ω, J),

$$\Lambda(M, (\omega, J)) = \frac{1}{2\pi}\int_M \Lambda dv$$

where $dv = d\text{vol}\ (M, g^+)$. If (ω, J) is a pseudo-Hermitian structure on M^3, then by definition it follows that $\omega \wedge (d\omega) \neq 0$ for every point of M. Thus there exists a vector field ξ dual to ω, *i.e.*, $\omega(\xi) = 1$ and $d\omega(\xi, X) = 0$ for all $X \in TM$ (cf. [10]). The vector field ξ is called a characteristic vector field of M. Let $\{\phi_t\}_{|t|<\epsilon}$ be a local one-parameter group of contact transformations of M induced by the characteristic vector field ξ. If for each t, ϕ_t is a CR automorphism of M (*i.e.*, $J \circ \phi_{t*} = \phi_{t*} \circ J$), then ξ is called a characteristic CR vector field. A *strictly pseudoconvex pseudo-Hermitian* manifold with a *characteristic CR vector field* is said to be *a standard pseudo-Hermitian* manifold.

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By definition, it is noted that a standard pseudo-Hermitian manifold used to be a Sasakian manifold (*i.e.*, possessing normal contact metric structures). (See [4],[57],[59].)

In this paper we examine the above invariant $\Lambda(M,(\omega,J))$ for compact standard pseudo-Hermitian 3-manifolds. We shall see that this invariant characterizes compact standard pseudo-Hermitian 3-manifolds whose underlying pseudo-Hermitian structure induces a *spherical CR structure*. A spherical CR structure on a $(2n+1)$-manifold is locally modelled on the sphere S^{2n+1} with respect to the group of CR transformations $PU(n+1,1)$. A spherical CR structure is said to be a $(PU(n+1,1),S^{2n+1})$-structure. More generally, let $\mathcal{G}$ be a finite dimensional Lie group with finitely many components and X an n-dimensional homogeneous space from $\mathcal{G}$. A $(\mathcal{G},X)$-structure (simply, geometric structure) on an n-manifold M is a maximal collection of charts $\{(\phi_\alpha,U_\alpha)\}_{\alpha\in\Lambda}$ satisfying that: (cf. [32],[41])

(1) $M=\bigcup_{\alpha\in\Lambda}U_\alpha,\quad \phi_\alpha:U_\alpha\rightarrow\phi_\alpha(U_\alpha)\quad(\subset X)$ is a homeomorphism,

(2) if $U_\alpha\cap U_\beta\neq\emptyset$, then the local change of coordinates $g_{\alpha\beta}=\phi_\beta\circ\phi_\alpha^{-1}$: $\phi_\alpha(U_\alpha\cap U_\beta)\rightarrow\phi_\beta(U_\alpha\cap U_\beta)$ extends to an element of $\mathcal{G}$.

The Thurston uniformization conjecture of compact 3-manifolds ([58],[62]) says that a closed irreducible 3-manifold decomposes into eight geometric pieces;

(1) an $(SO(4),S^3)$-structure (the spherical geometry),

(2) an $(\mathbf{R}^3\rtimes O(3),\mathbf{R}^3)$-structure (the Euclidean geometry),

(3) a $(PO(3,1),\mathbf{H}^3)$-structure (the hyperbolic geometry),

(4) a $(PO(1,1)\times O(3),\mathbf{R}^+\times S^2)$-structure (the Hopf geometry),

(5) an $(\mathbf{R}^1\times PO(2,1),\mathbf{R}^1\times\mathbf{H}^2)$-structure (the nonpositively curved geometry),

(6) an $(\mathbf{R}^1\underset{\mathbf{Z}}{\times}\widetilde{\mathrm{SL}_2}(\mathbf{R}),\tilde{\mathbf{H}}^{1,2})$-structure (the Lorentz standard geometry),

(7) an $(\mathcal{N}\rtimes U(1),\mathcal{N})$-structure (the nilgeometry),

(8) an $(\mathcal{S}\rtimes\mathbf{Z}/2,\mathcal{S})$-structure (the solvgeometry).

These $(\mathcal{G},X)$-structures are all Riemannian homogeneous geometries. However, most of $(\mathcal{G},X)$-structures naturally arise from non-Riemannian homogeneous spaces. In comparison to the Riemannian case, it is very difficult for a discrete group to act properly discontinuously because the stabilizer at each point of X is noncompact. In §5, we collect the current results which geometric piece admits a conformally flat structure, a spherical CR structure, or an affinely flat structure when we take those as a non-Riemannian homogeneous geometry. In §6, we give an example of closed 3-manifold which does not admit a conformally flat structure and a spherical CR structure.

2. Pseudo-Hermitian invariant

Let (ω, J) be a pseudo-Hermitian structure on a strictly pseudoconvex manifold M of dimension $2n+1$. We have the canonical metrics on M by setting $J\xi = 0$:

$$g^+(X,Y) = \omega(X)\cdot\omega(Y) + d\omega(JX,Y),\ g^-(X,Y) = -\omega(X)\cdot\omega(Y) + d\omega(JX,Y)$$

where $X, Y \in TM$. g^+ is a Riemannian metric; g^- is a (nondegenerate) Lorentz metric.

In the pseudo-Riemannian geometry, there exists a Levi-Cività connection on the frame bundle of M (cf. [66]). Thus if g is a pseudo-Riemannian metric then we can define the sectional curvature

$$K_p(X,Y,g) = -g_p(R(X,Y)X,Y)/g_p(X,X)g_p(Y,Y) - g_p(X,Y)^2$$

where $\{X,Y\}$ forms a nondegenerate plane section of T_pM for each $p \in M$. Let

$$K_p^+(X,Y) = K_p(X,Y,g^+),\ \ K_p^-(X,Y) = K_p(X,Y,g^-)$$

be the sectional curvatures respectively. We define a curvaturelike function Λ on Null ω as follows. Let (ω, J) be a strictly pseudoconvex pseudo-Hermitian structure on M. Put

$$\Lambda_p(X,Y;(\omega,J)) = \{K_p^+(X,Y) + K_p^-(X,Y)\}/2\left(1 + \frac{3d\omega(X,Y)^2}{d\omega(JX,X)d\omega(JY,Y) - d\omega(JX,Y)^2}\right)$$

for $X, Y \in$ Null ω_p. It is easy to check that the above form is independent of the choice of a basis which spans a plane section of Null ω at each $p \in M$. In particular if $\{X, JX\}$ forms an orthonormal basis with respect to g^+ and g^-, then the above formula reduces to

$$\Lambda_p(X,JX;(\omega,J)) = \frac{K_p^+(X,JX) + K_p^-(X,JX)}{2\cdot 4}.$$

Suppose that $\dim M = 3$. Then an orthonormal basis $\{X, JX\}$ spans Null ω at each point $p \in M$. Thus we have a smooth function $\Lambda(p) = \Lambda_p(X,JX;(\omega,J))$ at every $p \in M$.

Definition 2.1. *Let $(M,(\omega,J))$ be a compact pseudo-Hermitian 3-manifold.*

$$\Lambda(M,(\omega,J)) = \frac{1}{2\pi}\int_M \Lambda dv.$$

Here $dv = \omega \wedge d\omega$ is the volume element.

Thus we have defined a pseudo-Hermitian invariant on compact pseudo-Hermitian 3-manifolds.

3. Almost regular standard pseudo-Hermitian structure

Recall that a contact structure on M is *regular* (resp. *almost regular*) if *for every point p of M there exists a neighborhood U such that the integral curves of the characteristic vector field ξ passing through p pass through U exactly once* (resp. *finitely many times*).
Boothby and Wang [6] (after Thomas [61]) have shown that if M is a compact regular (resp. almost regular) manifold, then the characteristic vector field ξ generates a free circle action (resp. *almost free circle action*) on M and they have established the fibration theorem on compact regular contact manifolds. Here S^1-action is almost free if it has no fixed point.
Let (ω, J) be a pseudo-Hermitian structure on M. In order to generalize Boothby and Wang's result, the following condition is required that ξ is a characteristic CR vector field (cf. §1). Then a *regular standard pseudo-Hermitian* structure will be defined as a standard pseudo-Hermitian structure whose underlying contact structure is regular. Then Boothby and Wang's result will be generalized as follows:

Theorem 3.1. *A compact smooth $(2n+1)$-manifold admits a regular standard pseudo-Hermitian structure if and only if M is a principal circle bundle $\pi : M \to N$ over a Kähler manifold N whose fundamental 2-form Ω satisfies the following properties:*
(1) The Euler class of the bundle is represented by an integral cocycle $[\Omega] \in H^2(N; \mathbf{Z})$.
(2) $d\eta = \pi^*\Omega$ where η is a connection form of M.

This follows easily from the result of a standard pseudo-Hermitian structure (cf. [33]). Namely, if $(M, (\omega, J))$ is a compact standard pseudo-Hermitian manifold, then M is a Seifert fiber space $S^1 \to M\text{-}E \xrightarrow{\pi} M^*\text{-}E^*$ for which both g^+ and g^- induce a Kähler metric $\hat{g}$ on the space of principal orbits $M^*\text{-}E^*$. $M^*\text{-}E^*$ supports a Kähler structure $(\Omega, \hat{J})$ such that $\hat{g}_{p^*}(X, Y) = \Omega_{p^*}(\hat{J}X, Y)$ for $X, Y \in T_{p^*}(M^*\text{-}E^*)$.

We have three types of standard pseudo-Hermitian structures.

3.2.

(I) A *spherical* standard pseudo-Hermitian structure. (*i.e.*, the underlying standard pseudo-Hermitian structure (ω, J) induces a *spherical* CR structure.)

(II) An infraregular standard pseudo-Hermitian structure. (*i.e.*, some finite covering of M is regular.)

(III) An almost regular standard pseudo-Hermitian structure.

A compact standard pseudo-Hermitian manifold of constant curvature Λ is infraregular by the result of [33]. Moreover compact *spherical* standard pseudo-Hermitian manifolds M have been classified in [36],[33]. In fact, M is finitely covered by one of the following principal bundles:

(1) $S^1 \to S^{2n+1} \longrightarrow \mathbf{CP}^n$.

(2) $S^1 \to \mathcal{N}/\Gamma \longrightarrow T^n_{\mathbf{C}}$.

(3) $S^1 \to \tilde{\mathbf{H}}^{1,2n}/\tilde{\Gamma} \longrightarrow \mathbf{H}^n_{\mathbf{C}}/\Gamma^*$.

(4) $S^1 \to P(V^{2m+3}_{-1} \times S^{2(n-m)-1})/\Gamma \longrightarrow \mathbf{H}^{m+1}_{\mathbf{C}} \times \mathbf{CP}^{n-m-1}/\Gamma^*$.

Here Γ^* is a torsion free subgroup acting isometrically. In particular, $(I) \Rightarrow (II)$. Obviously, $(II) \Rightarrow (III)$.
We consider the converse of $(I) \Rightarrow (II) \Rightarrow (III)$ for 3-manifolds. Recall that a manifold is *aspherical* if the universal covering is contractible.

Proposition 3.3. *Let $(M, (\omega, J))$ be a compact standard pseudo-Hermitian 3-manifold. Then M is either an infraregular standard pseudo-Hermitian aspherical manifold or a standard pseudo-Hermitian spherical space form.*

Proof. Let $\{\phi_t\}_{|t|<\infty}$ be the one-parameter group generated by the characteristic CR vector field ξ. Suppose that T^k is a k-toruse which is the closure of $\{\phi_t\}_{|t|<\infty}$ in $\mathrm{Aut}_{CR}(M)$. When $k \geqq 2$, it follows from the classification of S^1-manifolds (e.g., [54],[55],[56]) that M is a lens space $L(p,q)$ $(p > 1)$ or the sphere S^3, otherwise M is finitely covered by $S^1 \times S^2$ or T^3. On the other hand, given a standard pseudo-Hermitian structure (ω, J, ξ), there exists an almost regular standard pseudo-Hermitian structure (η, J, ξ') on M which is arbitrary close to (ω, J, ξ) (cf. [33]). Let S^1 be the circle generated by the characteristic CR vector field ξ'. Put $M = S^1 \times S^2$ or T^3. As S^1 has no fixed point on M, there is an equivariant Seifert fibration of M associated with (η, J, ξ').

$$\begin{array}{ccccc}
\mathbf{Z} & \longrightarrow & \pi & \longrightarrow & Q \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{R}^1 & \longrightarrow & \tilde{M} & \xrightarrow{\tilde{\nu}} & \tilde{M}^* \\
\downarrow & & \mu\downarrow & & \downarrow \\
S^1 & \longrightarrow & M & \longrightarrow & M^*.
\end{array}$$

If $\tilde{g}^{\pm}$ is a lift of $g^{\pm}$ to the universal covering $\tilde{M}$, then as above there is a Kähler metric $\hat{g}^{*}$ on $\tilde{M}^{*}$ such that $\tilde{\nu}$ is a Riemannian submersion. In particular we note that $Q \subset \mathrm{Iso}\,(\tilde{M}^{*}, \hat{g}^{*})$ which acts properly discontinuously on $\tilde{M}^{*}$. Using Riemann's mapping theorem and Selberg's Lemma (or Fuchs' result in dimension two), there exists a normal subgroup Q' of finite index in Q which is torsion free. In particular Q' acts freely on $\tilde{M}^{*}$. Let $1 \to \mathbf{Z} \to \pi' \to Q' \to 1$ be the induced exact sequence. There assigns to it a principal bundle $S^1 \to M' \xrightarrow{\nu'} M'^{*}$ for which M' is a finite covering space of M. Let (η', J, S^1) be a lift of the pseudo-Hermitian structure (η, J, S^1) on M'. Obviously it induces a Kähler structure Ω' on M'^{*} such that $d\eta' = \nu'^{*}\Omega'$. Recall that the Euler class of the above principal bundle $S^1 \to M' \xrightarrow{\nu'} M'^{*}$ coincides with $[\Omega'] \in H^2(M'^{*}; \mathbf{Z})$ which is nonzero (cf. [38]). This implies that $M = S^1 \times S^2$ or T^3 never occur. Thus in the case that $\overline{\{\phi_t\}}_{|t|<\infty} = T^2$, M is a lens space $L(p,q)$ $(p > 1)$ or the sphere S^3.

Now, if $k = 1$, *i.e.*, $\overline{\{\phi_t\}}_{|t|<\infty} = S^1$, S^1 has no fixed point on M. Moreover the group $\mathrm{Psh}(M)$ of pseudo-Hermitian transformations of a strictly pseudo-convex pseudo-Hermitian manifold M of dimension $2n+1$ is a compact Lie group of dimension less than or equal to $(n+1)^2$ (cf. [64],[65]). In particular, $\mathrm{Psh}(S^3)$ is conjugate to a subgroup of $U(2)$. It follows again by the classification that M is an aspherical 3-manifold, or by the above argument M is finitely covered by S^3. M is a spherical space form S^3/π where $\pi \subset \mathrm{Psh}(S^3) \subset U(2)$ is a noncyclic subgroup. If M is aspherical, it is well known that some finite covering of M is a principal circle bundle (with nonzero Euler number). By definition, M is infraregular. □

Corollary 3.4. *Let M be a compact standard pseudo-Hermitian 3-manifold. Then M is a lens space $L(p,q)$ or the sphere S^3 such that $\mathrm{Aut}_{CR}(L(p,q))^0 = T^2$ $(p \geqq 2)$ or $\mathrm{Aut}_{CR}(S^3) = PU(2,1)$, otherwise M is an aspherical manifold or a spherical space form with noncyclic fundamental group such that $\mathrm{Aut}_{CR}(M)^0 = S^1$.*

Proof. The only compact connected Lie group that acts nontrivially on a closed aspherical manifold is a k-torus T^k (cf. [12],[13]). If M is aspherical, then $\mathrm{Aut}_{CR}(M)^0$ is compact (cf. [65],[31]). Thus $\mathrm{Aut}_{CR}(M)^0 = S^1$. □

Recall that compact *spherical* standard pseudo-Hermitian manifolds are infraregular (cf. (3.2)).

Proposition 3.5. *Let $(M, (\omega, J))$ be a closed standard pseudo-Hermitian 3-manifold. Put $\Lambda(M) = \Lambda(M, (\omega, J))$. Suppose that M is an infraregular pseudo-Hermitian manifold. Then*

(i) M is diffeomorphic to a spherical space form if $\Lambda(M) > 0$.

(ii) M is diffeomorphic to an infranilmanifold if $\Lambda(M) = 0$.

(iii) M is diffeomorphic to a Lorentz standard space form if $\Lambda(M) < 0$.

Proof. By the definition there exists a finite group F of order k for which a k-fold covering $\bar{M}$ is a principal bundle:

$$\begin{array}{ccccc} & & F & & \\ & & \downarrow & & \\ S^1 & \longrightarrow & \bar{M} & \xrightarrow{\nu} & N. \\ & & \pi\downarrow & & \\ & & M & & \end{array}$$

Let $\bar{\omega} = \pi^*\omega$ and $d\bar{v} = \bar{\omega} \wedge d\bar{\omega}$. Put $\pi(\bar{p}) = p$, $\pi_*(\bar{X}) = X$, $\pi_*(\bar{Y}) = Y$. Since $\int_{\bar{M}} \Lambda_{\bar{p}}(\bar{X}, \bar{J}\bar{X})d\bar{v} = k\int_M \Lambda_p(X, JX)dv$, it follows that

$$\Lambda(\bar{M}) = \frac{1}{2\pi}\int_{\bar{M}} \Lambda_{\bar{p}}(\bar{X}, \bar{J}\bar{X})d\bar{v} = k\Lambda(M).$$

On the other hand, for the above bundle $S^1 \to \bar{M} \xrightarrow{\nu} N$, the orthogonal complement of the tangent bundle TS^1 relative to $\bar{g}^{\pm}$ is Null $\bar{\omega}$ and $\nu_* :$ Null $\bar{\omega} \longrightarrow TN$ is an isometry for each $p \in \bar{M}$. Then O'Neill's formula (cf. [53]) implies that for $\bar{X}, \bar{Y} \in$ Null $\bar{\omega}$

$$4\hat{K}(\nu_*\bar{X}, \nu_*\bar{J}\bar{X}, \hat{g}) = K^{\pm}(\bar{X}, \bar{J}\bar{X}, \bar{g}^{\pm}) + \frac{3}{4}\frac{\bar{g}^{\pm}([\bar{X}, \bar{J}\bar{X}]^V, [\bar{X}, \bar{J}\bar{X}]^V)}{\bar{g}^{\pm}(\bar{X}, \bar{X})\bar{g}^{\pm}(\bar{J}\bar{X}, \bar{J}\bar{X}) - \bar{g}^{\pm}(\bar{X}, \bar{J}\bar{X})^2},$$

where Z^V is the vertical component of a vector field Z. Since $d\bar{\omega}(\bar{X}, \bar{Y}) = -\frac{1}{2}\bar{\omega}([\bar{X}, \bar{Y}])$ for $\bar{X}, \bar{Y} \in$ Null $\bar{\omega}$, it follows that

$$\bar{g}^{\pm}([\bar{X}, \bar{J}\bar{X}]^V, [\bar{X}, \bar{J}\bar{X}]^V) = \pm\bar{\omega}([\bar{X}, \bar{J}\bar{X}]) \cdot \bar{\omega}([\bar{X}, \bar{J}\bar{X}]) = \pm 4d\bar{\omega}(\bar{X}, \bar{J}\bar{X})^2.$$

Moreover as $\bar{g}^{\pm}(\bar{X}, \bar{X}) = 1$ and $\bar{g}^{\pm}(\bar{X}, \bar{J}\bar{X}) = 0$, we have that

$$4\hat{K}(\nu_*\bar{X}, \nu_*\bar{J}\bar{X}, \hat{g}) = K^{\pm}(\bar{X}, \bar{J}\bar{X}, \bar{g}^{\pm}) \pm 3d\bar{\omega}(\bar{X}, \bar{J}\bar{X})^2.$$

Therefore it follows that

$$4 \cdot 2\hat{K}(\nu_*\bar{X}, \nu_*\bar{Y}, \hat{g}) = K^{+}(\bar{X}, \bar{Y}, \bar{g}^{+}) + K^{-}(\bar{X}, \bar{Y}, \bar{g}^{-}).$$

By the definition (cf. §2), we obtain that

$$\Lambda_{\bar{p}}(\bar{X}, \bar{J}\bar{X}; (\bar{\omega}, \bar{J})) = \hat{K}_{\nu(\bar{p})}(\nu_*\bar{X}, \hat{J}\nu_*\bar{X}, \hat{g}).$$

Noting $d\bar{\omega} = \nu^*\bar{\Omega}$ and $d\bar{v} = \bar{\omega} \wedge d\bar{\omega}$, it follows that

$$\begin{aligned}\Lambda(\bar{M}) &= \frac{1}{2\pi}\int_{\bar{M}} \Lambda_{\bar{p}}(\bar{X}, \bar{J}\bar{X})d\bar{v} \\ &= \frac{1}{2\pi}\int_{S^1}\bar{\omega}\int_N \hat{K}_{\nu(\bar{p})}(\nu_*(\bar{X}), \hat{J}\nu_*(\bar{X}))\bar{\Omega} = \chi(N)\int_{S^1}\bar{\omega} = \chi(N).\end{aligned}$$

Hence $\Lambda(M) = \frac{1}{k}\Lambda(\bar{M}) = \frac{1}{k}\chi(N)$. Thus we conclude that;

$$\begin{aligned}\Lambda(M) &> 0 \text{ if and only if } N = \mathbf{CP}^1 \text{ and } \bar{M} \approx S^3. \\ \Lambda(M) &= 0 \text{ if and only if } N = T^2 \text{ and } \bar{M} \approx \mathcal{N}/\Delta. \\ \Lambda(M) &< 0 \text{ if and only if } N = S_g \text{ (closed orientable surface of genus } g \geqq 2) \text{ and} \\ &\quad\ \bar{M} \approx \tilde{H}^{1,2}/\Gamma.\end{aligned}$$

□

If M is a closed standard pseudo-Hermitian aspherical 3-manifold, then by Proposition 3.3 the characteristic CR vector field generates a circle action S^1 on M such that there is an equivariant principal circle bundle:

$$(\mathbf{Z}, \mathbf{R}) \to (\pi, \tilde{M}) \to (Q, W),$$

where $\mathbf{R}/\mathbf{Z} = S^1$, $\pi_1(M) = \pi$ and Q is the quotient group that acts properly discontinuously on a two dimensional contractible space W.

Lemma 3.6. *The center* $\mathcal{C}(\pi) = \mathbf{Z}$.

Proof. The uniformization theorem (cf. [66]) says that there is a conformal mapping $f : W \to \mathbf{R}^2$ (or $f : W \to \mathbf{H}^2$) such that $fQf^{-1} \subset E(2)^0$ (or $fQf^{-1} \subset PSL_2\mathbf{R}$) respectively. Thus we assume that (Q, W) is either a subgroup of $E(2)^0$ acting properly discontinuously on $\mathbf{R}^2$ or a subgroup of $PSL_2\mathbf{R}$ acting properly discontinuously on a hyperbolic plane $\mathbf{H}^2$.

Obviously, $\mathbf{Z} \subset \mathcal{C}(\pi)$, so put $\mathbf{A} = \mathcal{C}(\pi)/\mathbf{Z}$. Now $\mathbf{A}$ is a central subgroup of the discrete group Q lying in $E(2)^0$ or $PSL_2\mathbf{R}$. In particular, $\mathbf{A}$ is a finitely generated abelian group. Suppose that $\mathbf{A}$ has a torsion subgroup. Let $\mathbf{Z}/p$ be a cyclic group of prime order in the torsion group. Since W is contractible and Q is orientation-preseving, $\mathbf{Z}/p$ has a unique fixed point. As Q centralizes $\mathbf{Z}/p$, Q has a fixed point. Q being infinite, this contradicts that Q is proper. Hence $\mathbf{A}$ is a free abelian subgroup of rank i so that $\mathcal{C}(\pi) \approx \mathbf{Z} \oplus \mathbf{A}$ Note that rank $\mathcal{C}(\pi) = 1 + i \leqq 3$. If $i = 2$, put $\Delta = \mathcal{C}(\pi)$. When $i = 1$, choose an element $c \in \pi$ of infinite order which does not lie in $\mathcal{C}(\pi)$. Then the group Δ' generated by $\mathcal{C}(\pi)$ and c is isomorphic to $\mathbf{Z}^3$. Since both $\tilde{M}/\Delta$ and $\tilde{M}/\Delta'$ is homeomorphic to T^3, M is covered by a 3-torus T^3. Thus T^3 would be a standard pseudo-Hermitian manifold. As we know that T^3 does not admit a standard pseudo-Hermitian structure by Proposition 3.3, $\mathbf{A}$ is trivial or $\mathcal{C}(\pi) = \mathbf{Z}$. This proves the lemma. □

As an application, we have the following.

Corollary 3.7. *$\mathbf{A}(M)$ is a topological invariant among all closed standard pseudo-Hermitian aspherical 3-manifolds.*

Proof. Given two homotopic (homeomorphic) closed aspherical manifolds M_1, M_2 with pseudo-Hermitian structure $\{(\omega_i, J_i)\}_{i=1,2}$ respectively, let $h : \pi_1 \to \pi_2$ be an isomorphism between fundamental groups. There is an equivariant principal circle bundle: $(\mathbf{Z}, \mathbf{R}) \to (\pi_i, \tilde{M}_i) \to (Q_i, W_i)$. Since $h(\mathbf{Z}) = \mathbf{Z}$ by the above lemma, h induces an isomorphism $\hat{h} : Q_1 \to Q_2$. Let Q_1' be a torsion free normal subgroup of Q_1 of finite index. Put $Q_2' = \hat{h}(Q_1')$. The above group extension induces a group extension : $1 \to \mathbf{Z} \to \pi_i' \to Q_i' \to 1$ respectively. Put $M_i' = \tilde{M}_i/\pi_i'$. Then we have a principal circle bundle: $S^1 \to M_i' \to W_i/Q_i'$ $(i = 1, 2)$. In particular M_i is an infraregular pseudo-Hermitian manifold. M_i' is a k-fold covering of M_i where $|Q_i : Q_i'| = k$ for some k. As in the argument of Proposition 3.3 and noting that $Q_1' \approx Q_2'$, it follows that

$$\mathbf{A}(M_1) = \frac{1}{k}\chi(W_1/Q_1') = \frac{1}{k}\chi(W_2/Q_2') = \mathbf{A}(M_2).$$

Therefore $\mathbf{A}(M)$ is a homotopy invariant for closed standard *aspherical* pseudo-Hermitian 3-manifolds. □

We have shown in [33] that $\mathbf{A} > 0$ is an invariant for a pseudo-Hermitian spherical space form S^{2n+1}/F which induces a *spherical CR* structure. However, $\mathbf{A}(M, (\omega, J))$ is not a topological invariant for a spherical space form M admitting a standard pseudo-Hermitian structure. In fact, Proposition 3.5 fails if we relax *infraregular* to be *almost regular* for pseudo-Hermitian spherical 3-manifolds. Namely,

Theorem 3.8. *There exists an almost regular standard pseudo-Hermitian structure (η, J') on the sphere S^3 whose curvature $\mathbf{A}(S^3, (\eta, J')) < 0$.*

Proof. Such a pseudo-Hermitian structure is constructed as follows: Let M^3 be a closed Seifert fiber space over a hyperbolic orbifold with nonzero Euler number. Then there exists a standard almost regular pseudo-Hermitian structure (ω, J) on M^3 whose curvature $\mathbf{A} = \mathbf{A}_p(X, Y; (\omega, J)) = -1$.

Let us take a Brieskorn manifold $M(p,q,r)$ as M, where $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1<0$. (See [50].) Then there is the following Seifert fibration:

$$\begin{array}{ccccc} S^1 & \longrightarrow & M(p,q,r) & \longrightarrow & S^2 \\ {\scriptstyle /r}\downarrow & & {\scriptstyle /r}\downarrow & & \downarrow \\ S^1 & \longrightarrow & S^3 & \longrightarrow & S^2. \end{array}$$

Let $\nu : M(p,q,r) \to S^3$ be an r-fold branched covering with branched locus $L^1(p,q)$, which is a torus knot in S^3. The contact structure ω on $M(p,q,r)$ is invariant under the S^1-action.

Let $E = \{x \in M | S^1_x = \mathbf{Z}/r\}$ and $\mathcal{N}(E)$ be the tubular neighborhood of E in M. Then we have the commutative diagram of the branched covering:

$$\begin{array}{ccc} \mathcal{N}(E) & \xrightarrow{\tilde{f}} & S^1 \times D^2 \\ {\scriptstyle r/}\downarrow & & {\scriptstyle r/}\downarrow \\ \mathcal{N}(L^1(p,q)) & \xrightarrow{f} & S^1 \times D^2. \end{array}$$

Here f (resp. $\tilde{f}$) is an equivariant diffeomorphism such that $\omega|_{\mathcal{N}(E)} = \tilde{f}^*(C(r,1))$ where $C(r,1)$ is the canonical contact form on $S^1 \times D^2$ for a pair of integers $(r,1)$. (See [36] for details.) Now the contact form ω induces a contact form η on $S^3 - \mathcal{N}(L^1(p,q))$ because the projection $M(p,q,r) \to S^3$ is a regular r-fold covering outside $\mathcal{N}(L^1(p,q))$. It is noted that $\eta|\partial\mathcal{N}(L^1(p,q)) = f^*(C(1,1)|S^1 \times S^1)$ by the above diagram. We choose a contact form on $\mathcal{N}(L^1(p,q))$ to be $f^*(C(1,1))$. Put

$$\eta = \begin{cases} \eta & \text{on } S^3 - \mathcal{N}(L^1(p,q)) \\ f^*(C(1,1)) & \text{on } \mathcal{N}(L^1(p,q)). \end{cases}$$

Then we obtain a contact form η on S^3. Similarly the complex structure J induces a complex structure J' on Null η over $S^3 - \mathcal{N}(L^1(p,q))$. Since ξ is transverse to Null η, it is easy to extend this complex structure (i.e., a rotation) to the entire S^3. As $M(p,q,r) - \mathcal{N}(E) \xrightarrow{\nu} S^3 - \mathcal{N}(L^1(p,q))$ is a regular covering, we have

$$\Lambda_{\tilde{x}}(X,Y;(\omega,J)) = \Lambda_x(\nu_*(X),\nu_*(Y);(\eta,J')) = -1$$

for $\nu(\tilde{x}) = x \in S^3 - \mathcal{N}(L^1(p,q))$. It implies that $\Lambda(S^3) < 0$. □

4. Deformation of standard CR structure

It is known that there exists no nontrivial deformation of a contact structure on a compact manifold (cf. [24],[17]). We consider a nontrivial deformation of a CR structure on M. (See [4],[5],[13].)

Proposition 4.1. *Let $(M,(\omega,J))$ be a closed standard pseudo-Hermitian aspherical 3-manifold. There exists a family $\{(\omega_t,J_t)\}_{0\leqq t\leqq 1}$ of standard pseudo-Hermitian structures at $(\omega_0,J_0)=(\omega,J)$ for which a smooth map $\mathbf{A}_{(1)}$ is constant in curvatures $\mathbf{A}_{(t)}$ of (ω_t,J_t). In particular,* (Null ω_1, J_1) *is spherical.*

Proof. There exists a finite covering $\bar{M}$ of M such that $\bar{M}$ is a regular standard pseudo-Hermitian 3-manifold and

$$S^1 \to (F,\bar{M}) \xrightarrow{\nu} (F',N)$$

is an equivariant principal bundle where ν is equivariant with respect to finite groups F, F'. Let $(\bar{\omega},\bar{J})$ be a lift of (ω,J) to $\bar{M}$. $\bar{\omega}$ is invariant under F. It follows from Theorem 3.1 that
(1) $(N,(\Omega,\hat{J}),\hat{g})$ is a Kähler manifold.
(2) $d\bar{\omega}=\nu^*\Omega$ where $[\Omega]\in H^2(N;\mathbf{Z})$ represents the Euler class of the bundle.

We may assume that $[\Omega]$ is the generator of $H^2(N;\mathbf{Z})$. Let Y stand for $\mathbf{R}^2$ or $\mathbf{H}^1_{\mathbf{C}}$ endowed with the canonical Kähler metric $\hat{g}_c$ for $c=0,1$ respectively. By Riemann's mapping theorem (cf. [66]) there exist a Kähler manifold Y/Γ of constant holomorphic curvature and an equivariant holomorphic map $h:(F',N,\hat{g})\to(G',Y/\Gamma,\hat{g}_c)$ such that $\lambda(p)\cdot\hat{g}_p(X,Y)=(\hat{g}_c)_{h(p)}(h_*X,h_*Y)$ for some positive function $\lambda:N\to\mathbf{R}$ and that $G'\subset$ Iso (Y/Γ). As $\Omega(X,Y)=\hat{g}(X,\hat{J}Y)$, $\Omega_c(X,Y)=\hat{g}_c(X,\hat{J}_cY)$ and $h_*\circ\hat{J}=\hat{J}_c\circ h_*$, it follows that $\lambda\cdot\Omega=h^*\Omega_c$. In particular, $\hat{K}_q(X,\hat{J}X,\lambda\cdot\hat{g})=\hat{K}_{h(q)}(h_*X,h_*\hat{J}X,\hat{g}_c)=c$. On the other hand, since $[\lambda\cdot\Omega]=h^*[\Omega_c]$ is the generator of $H^2(N;\mathbf{Z})$, it follows that $[\lambda\cdot\Omega]=[\Omega]$. There is a 1-form θ on N such that $\lambda\cdot\Omega=\Omega+d\theta$. For $\alpha\in F'$, $\lambda(\alpha p)=\lambda(p)$ because Ω is F'-invariant. We can take θ as an F'-invariant 1-form. In fact, we may put $\theta'(X)=\sum_{\alpha\in F'}\theta(\alpha_*X)/|F'|$. Then $d\theta'=d\theta$. Put

$$\lambda_t(p)=t\lambda(p)+1-t \qquad (0\leqq t\leqq 1).$$

Then $\lambda_t>0$ and $\lambda_t\cdot\Omega$ is a Kähler form on N. We obtain a family of 1-forms $\{\bar{\eta}_t\}$ on $\bar{M}$ by setting $\bar{\eta}_t=\bar{\omega}+t\nu^*\theta$. Since $\bar{\omega}$ and $\nu^*\theta$ are both F-invariant, $\bar{\eta}_t$ is also F-invariant. Thus it induces a 1-form η_t on M. If ξ is the characteristic CR vector field for (ω,J), it is easy to see $\eta_t(\xi)=1$. Now, $d\bar{\eta}_t=\nu^*(\lambda_t\cdot\Omega)$, which implies that $\eta_t\wedge d\eta_t\neq 0$. Thus η_t is a contact form on M. Since $\nu_*:$ Null $\eta_t\to TN$ is an isomorphism, there is a complex structure J_t on Null η_t defined by $\nu_*\circ J_t=\hat{J}\circ\nu_*$. Note that $\eta_0=\omega$ and $\hat{J}_t=\hat{J}$ by the definition. We obtain a family of infraregular pseudo-Hermitian structures $\{(\eta_t,J_t)\}_{0\leqq t\leqq 1}$ starting at (ω,J), each of which has the characteristic CR vector field ξ. As ξ generates a one-parameter group $\{\phi_\theta\}_{|\theta|<\infty}$, it follows that

$$S^1=\{\phi_\theta\}_{|\theta|<\infty}\subset \mathrm{Aut}_{CR}(M,(\eta_t,J_t))^0.$$

Let $\bar{\eta}_1=\bar{\omega}+\nu^*\theta$ as above. As $d\bar{\eta}_1=\nu^*(\lambda\cdot\Omega)$, $\bar{\eta}_1$ induces the metric $\lambda\cdot\hat{g}$. Noting the proof of Theorem 3.3, for $\nu_*(\bar{X}) = X$ and each $p = \nu(\bar{p}) \in M$, we have that

$$\Lambda_{(1)}(\bar{p}) = \Lambda_{\bar{p}}(\bar{X}, J_1\bar{X}; (\eta_1, J_1)) = \hat{K}_p(X, \hat{J}X, \lambda \cdot \hat{g}) = c.$$

□

Corollary 4.2. *If the curvature $\Lambda = \Lambda_{(0)}$ is not constant, then the deformation of CR structures $\{(\mathrm{Null}\ \omega_t, J_t)\}_{0 \leqq t \leqq 1}$ is nontrivial.*

Proof. Suppose that there is a CR diffeomorphism $f : (M, (\mathrm{Null}\ \omega, J)) \to (M, (\mathrm{Null}\ \eta_1, J_1))$. Put $\xi' = f_*^{-1}(\xi)$. Then it follows that $\lambda \cdot \omega = f^*\eta_1$ where $\lambda(p) = 1/\omega_p(\xi'_p)$. Note that ξ' is the characteristic CR vector field for $f^*\eta_1$. So we have two standard pseudo-Hermitian structures (ω, J, ξ) and $(f^*\eta_1, J, \xi')$ representing the same CR structure. Then it follows from the result of [33] that $\Lambda_p(X, JX; (\omega, J)) = \lambda(p) \cdot \Lambda_p(X, JX; (f^*\eta_1, J))$. As $\Lambda_p(X, JX; (f^*\eta_1, J)) = \Lambda_{f(p)}(f_*X, J_1 f_*X; (\eta_1, J_1)) = \Lambda_{(1)}(p) = c$, we obtain that $\Lambda(p) = \Lambda_p(X, JX; (\omega, J)) = \lambda(p) \cdot c$. On the other hand, if $\{\phi'_\theta\}_{|\theta| < \infty}$ is a one-parameter group induced by ξ', then it follows that $f \circ \phi'_\theta(x) = \phi_\theta \circ f(x)$. Thus ϕ'_θ is a CR diffeomorphism preserving $(\mathrm{Null}\ \omega, J)$, *i.e.*, $\{\phi'_\theta\}_{|\theta|<\infty} \subset \mathrm{Aut}_{CR}(M, (\omega, J))^0$. If we note from Corollary 3.4 that $\mathrm{Aut}_{CR}(M, (\omega, J))^0 = S^1 = \{\phi_\theta\}_{|\theta|<\infty}$, then we have that $\xi' = a\xi$ for some contant number a. As $\lambda = 1/\omega(\xi') = 1/a$, $\Lambda(p) = \lambda(p) \cdot c = c/a$ for all $p \in M$. This contradicts the hypothesis that Λ is not constant. □

5. Geometric structure modelled on non-Riemannian homogeneous spaces

We collect the results concerning which piece in the geometric decomposition of a 3-manifold admits a conformally flat structure $(PO(n+1,1), S^n)$, a spherical CR structure $(PU(n+1,1), S^{2n+1})$ or an affinely flat structure $(A(n), \mathbf{R}^n)$ (cf. [22].[32],[58],[62]).

5.1. Conformally flat structure. First recall that a Riemannian manifold of constant curvature is a conformally flat manifold. From this, (1) a compact spherical space form, (2) a compact euclidean space form, (3) a compact hyperbolic space form admit a conformally flat structure. Moreover it is known that (4) a Hopf manifold, (5) $S^1 \times S_g$ admits a conformally flat structure. Here S_g is a closed surface of genus $g \geqq 2$. Note the following (cf. [18],[21],[51]).

Theorem 5.1.1. *If the holonomy group of a compact conformally flat manifold is virtually solvable, then M is finitely covered by the sphere, a torus or a Hopf manifold.*

From this, (7) an infranilmanifold, (8) an infrasolvmanifold do not admit a conformally flat structure. It is known that (6) a circle bundle over a closed surface S_g with nonzero Euler number e admit a conformally flat structure, but it is unknown whether g and e are chosen arbitrarily (cf. [25],[37],[40],[45]). See [1],[2],[3],[23],[29],[30],[42],[49] for related topics.

5.2. Spherical CR structure. We have shown that a standard pseudo-Hermitian manifold of constant Λ is a spherical CR manifold. Thus, (1) a compact spherical space form, (7) an infranilmanifold, (6) a circle bundle over a closed surface S_g with nonzero Euler number e admit a spherical CR structure. It follows from [51] that

Theorem 5.2.1. *If the holonomy group of a compact spherical CR manifold is virtually solvable, then M is finitely covered by the sphere, a Heisenberg nilmanifold, or a Hopf manifold.*

From this, (4) a Hopf manifold admits a spherical CR structure. (2) a compact euclidean space form, (8) an infrasolvmanifold do not admit a spherical CR structure. It is unknown whether (3) a compact hyperbolic space form or (5) $S^1 \times S_g$ admits a spherical CR structure.

Proposition 5.2.2. *Let M be a closed hyperbolic manifold or the product $S^1 \times S_g$ ($g \geqq 2$). If M admits a spherical CR structure, then the holonomy group is Zariski dense in $PU(2,1)$.*

Proof. We can assume that the holonmy group is neither finite nor stabilizes a finite number of points. Moreover, if the holonomy group leaves a totally geodesic subspace $\mathbf{H}^1_{\mathbf{C}}$ in $\mathbf{H}^2_{\mathbf{C}}$, then by the result of [23], M is finitely covered by a circle bundle over a closed surface S_g with nonzero Euler number. □

Concerning the existence of conformally flat structure or spherical CR structure of the above geometric pieces, we remark that the developing maps are not surjective (and so covering maps onto its image. cf. [30]) It is known that the developing map is necessarily a covering map if the holonomy group is virtually solvable (cf. [29],[51]). It is unknown that other geometric piece (*i.e.*, (3) a hyperbolic space form, (5) $S^1 \times S_g$, or (6) a Lorentz standard space form $\tilde{\mathbf{H}}^{1,2}/\Gamma$) admits a surjective developin map. Note that there is a one dimensional complex projective (conformal) structure on a closed surface S_g ($g \geqq 2$) whose developing map is surjective (cf. [22], [20]). We notice that there is an example of a surjective developing map on a connected sum of geometric pieces.

Let M_i be $S^1 \times S_g$ ($g \geqq 2$) or a principal circle bundle over S_g with nonzero Euler number for $i = 1, 2$ respectively. There is a canonical conformally flat (resp. spherical CR) structure on M_1 (resp. M_2). In this case the holonomy group Γ lies in $PO(2,1) \times S^1$ (resp. $U(1,1)$), where $PO(2,1)^0 \approx PU(1,1) \approx \mathrm{PSL}_2(\mathbf{R})$ and $S^1 \to U(1,1) \to PU(1,1)$ is the exact sequence. Let $C(\Gamma)$ be the central subgroup of Γ. Then the intersection of S^1 with $C(\Gamma)$ is an infinite group if and only if the holonomy group Γ is indiscrete. So we choose a conformally flat structure (spherical CR strucure) on M_i whose holonomy group is indiscrete.

Proposition 5.2.3. *Let M_i be a conformally flat manifold (spherical CR manifold) whose holonomy group is indiscrete as above.*

(1) *The developing map of a conformally flat structure on a connected sum $M_1 \sharp N$ for any closed conformally flat manifold N except for S^3 is surjective.*

(2) *The developing map of a spherical CR structure on a connected sum $M_2 \sharp L$ for any closed spherical CR manifold L except for S^3 is surjective.*

Proof. First note from [8],[41] that there is a canonical conformally flat structure or spherical CR structure on the connected sum $M_1 \sharp N$, $M_2 \sharp L$ respectively for which the holonomy of the fundamental group of M_i is a subgroup of the holonomy group G_i of $M_1 \sharp N$, $M_2 \sharp L$ respectively. Thus in our case the holonomy group G_i of $M_1 \sharp N$, $M_2 \sharp L$ is indiscete in $PO(4,1)$, $PU(2,1)$ respectively. Now suppose that the developing map of $M_1 \sharp N$, or $M_2 \sharp L$ is not surjective. Then it follows from [30] that the holonomy group G_i is discrete, or the developing map is a covering map onto S^3-S^1. In the latter case, this follows since the limit set $\Lambda(G_i) = \Lambda(\bar{G}_i^0) = S^1 = \partial \mathrm{dev}\ (X) \neq \emptyset$. Here X is the universal covering space of $M_1 \sharp N$ or $M_2 \sharp L$. See [23],[30]. Then it follows from the classification of [30],[36] that $M_1 \sharp N$ or $M_2 \sharp L$ is finitely covered by $S^1 \times S_g$ $(g \geqq 2)$ or a principal circle bundle over S_g with nonzero Euler number respectivly. Then each fundamental group has a normal subgroup isomorphic to $\mathbf{Z}$. But the fundamental group of $M_1 \sharp N$ or $M_2 \sharp L$ has no nontrivial normal subgroup. Hence this contradiction shows that the developing map of $M_1 \sharp N$ or $M_2 \sharp L$ is surjective. □

We refer to [20],[28],[64] for related topics.

5.3. Affinely flat structure. It is shown in [19] that

Theorem 5.3.1. *The fundamental group of a compact complete affinely flat 3-manifold is virtually solvable.*

Moreover there exists a simply transitive affine action of a connected simply connected, solvable Lie group on $\mathbf{R}^3$. We have that (2) a compact euclidean space form, (7) an infranilmanifold, (8) an infrasolvmanifold admit a complete affinely flat structure. On the other hand, both (4) a Hopf manifold and (5) $S^1 \times S_g$ admit an incomplete affinely flat structure. In fact, the universal cover of them are realized as a domain of $\mathbf{R}^3$. Obviously (1) a spherical space form does not admit an affinely flat structure. It is unknown whether (3) a compact hyperbolic space form, or (6) a circle bundle over a closed surface S_g with nonzero Euler number e admits an affinely flat structure. See [12],[26],[27],[42],[47],[60] for related works.

6. Holonomy groups of $PO(n+1,1)$, $PU(n+1,1)$

Let G be a connected subgroup of $PO(n+1,1) \approx \mathrm{Conf}\ (S^n)$. Denote $\overline{G}^0$ the identity component of the closure of G in $PO(n+1,1)$. If $\overline{G}^0$ is compact, then up to conjugacy $\overline{G}^0$

is contained in $O(n+1)$. If $\overline{G}^0$ has a noncompact radical, it can be shown that $\overline{G}^0$ fixes a point $\{\infty\}$ or exactly two points $\{0,\infty\}$ in S^n (cf. [30]). Then it follows that $\overline{G}^0 \subset \mathrm{Sim}(\mathbf{R}^n)$ where $\mathbf{R}^n = S^n - \{\infty\}$.

Lemma 6.1. *Suppose $\overline{G}^0$ is noncompact but has compact radical. Then $G = H \cdot K$ where K is a compact Lie group and H is a Lie subgroup that acts simply transitively on a totally geodesic subspace $\mathbf{H}^k_{\mathbf{R}}$ in $\mathbf{H}^{n+1}_{\mathbf{R}}$ where $2 \leqq k \leqq n+1$.*

Proof. It suffices to check that G does not have a fixed point and the limit set $L(G)$ contains more than two points; the lemma then follows from Lemma 4.4.5 of [9]. If G has a fixed point, then G is conjugate to a subgroup of $\mathrm{Sim}(\mathbf{R}^n)$. Since $\mathrm{Sim}(\mathbf{R}^n)$ is amenable (and so is any closed connected subgroup), $\overline{G}^0$ is an amenable Lie subgroup of $\mathrm{Sim}(\mathbf{R}^n)$ and thus an extension of a solvable Lie group by a compact Lie group. Since $\overline{G}^0$ has compact radical, $\overline{G}^0$ is itself compact, contradicting $\overline{G}^0$ being noncompact. If $L(G)$ consists of less than three points, then either $L(G) = \emptyset$ or $\overline{G}^0$ fixes a point. Since $L(\overline{G}) = L(G)$, $L(G) = \emptyset$ implies that $\overline{G}$ is compact. □

The similar result holds for spherical CR case; namely

Lemma 6.1′. *Let G be a connected subgroup of $PU(n+1,1)$ ($\approx \mathrm{Aut}_{CR}(S^{2n+1})$). Suppose that $\overline{G}^0$ is noncompact but has compact radical. Then $G = H \cdot K$ where K is a compact Lie group and H is a Lie subgroup that acts simply transitively on a totally geodesic subspace $\mathbf{H}^k_{\mathbf{C}}$ in $\mathbf{H}^{n+1}_{\mathbf{C}}$ where $1 \leqq k \leqq n+1$.*

Corollary 6.2. *If G is a connected subgroup of $PO(n+1,1)$, then G satisfies either one of the following:*

(1) *G is conjugate to a subgroup of $\mathrm{Sim}(\mathbf{R}^n)$ and $\overline{G}$ is noncompact.*

(2) *G has a unique fixed point in $\mathbf{H}^{n+1}_{\mathbf{R}}$, or a conjugate of G leaves fixed S^k where $0 \leqq k \leqq n-2$.*

(3) *G acts transitively on S^k where $1 \leqq k \leqq n$.*

Corollary 6.3. *If G is a connected subgroup of $PU(n+1,1)$, then G satisfies either one of the following:*

(1) *G is conjugate to a subgroup of $\mathrm{Sim}(\mathcal{N})$ and $\overline{G}$ is noncompact.*

(2) *G has a unique fixed point in $\mathbf{H}^{n+1}_{\mathbf{C}}$, or a conjugate of G leaves S^{2k+1} fixed where $0 \leqq k \leqq n-1$.*

(3) *G acts transitively on S^{2k+1} where* $0 \leqq k \leqq n$.

We shall study elliptic elements in $PO(n+1,1) \approx \mathrm{Conf}\ (S^n)$.

Proposition 6.4. *Let Γ be a subgroup of $PO(n+1,1)^0$. Suppose that every element in Γ is elliptic. Then the closure $\bar{\Gamma}$ does not contain a loxodromic element.*

Proof. If an element γ is loxodromic then it is conjugate to an element that fixes two points $\{0, \infty\}$. So we assume that γ fixes $\{0, \infty\}$. Then γ has the form;

$$\gamma = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & A \end{pmatrix},$$

where $x \in \mathbf{R}^+$, $A \in SO(n)$. Since every element of $SO(n)$ is conjugate to a matrix;

$$\begin{pmatrix} \begin{matrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{matrix} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \begin{matrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{matrix} \\ & & & (1) \end{pmatrix},$$

it follows that for all $B \in SO(n)$, $|\mathrm{tr}\ B| \leqq n+2$.

By our hypothesis $\lambda \neq 1$. Choosing some power if necessary, we may assume that $\lambda > 2n+5$. Therefore $\mathrm{tr}\ \gamma = \lambda + \lambda^{-1} + \mathrm{tr}\ A > n+3$. On the other hand if $\delta \in PO(n+1,1)^0$ is elliptic it is conjugate to an element of $SO(n+1)$. Hence $|\mathrm{tr}\ \delta| \leqq n+3$. Choosing a sequence $\{\gamma_i\}$ in Γ such that $\gamma = \lim \gamma_i$. Since each γ_i is elliptic, it follows that $\mathrm{tr}\ \gamma = \lim \mathrm{tr}\ (\gamma_i) \leqq n+3$, being a contradiction. □

Lemma 6.5. *Let Γ be a subgroup of $\mathrm{Sim}(\mathbf{R}^n)^0$ for which $E(n)^0 = \mathbf{R}^n \rtimes SO(n)$ is the euclidean subgroup. If $\gamma \in \Gamma$ is elliptic then $\gamma \in E(n)^0$. Moreover, $\gamma = (a, A)$ satisfies either one of the following:*

(1) $\gamma = A \in \{0\} \times SO(n)$.

(2) *Both* $a \neq 0$ *and* $\det\ (A - I) \neq 0$ *(i.e., A does not have 1 as an eigenvalue.)*

Proof. Let $< \gamma >$ be a subgroup generated by γ. Let $< \bar{\gamma} >$ be the closure of $\mathrm{Sim}(\mathbf{R}^n)^0$ and $\rho : \mathrm{Sim}(\mathbf{R}^n)^0 \to \mathbf{R}^+ \times SO(n)$ be the holonomy map. Since γ is elliptic, $< \bar{\gamma} >$ is compact and so $\rho(<\bar{\gamma}>)$ is also compact. If $\gamma=(a,\lambda A)$ then $\rho(<\bar{\gamma}>)=<\bar{\lambda A}>=<\lambda>\times<\bar{A}>$. It must be that $\lambda=1$, or $\gamma=(a,A)\in E(n)^0$. Now let p be a fixed point of the elliptic element γ. There exists a geodesic α passing through p such that $\alpha(\infty)=\{\infty\}$ on the boundary S^n. Since γ stabilizes α pointwisely, we have $\gamma(\alpha(-\infty))=\alpha(-\infty)$. Put $\alpha(-\infty)=y\in\mathbf{R}^n$ $(=S^n-\{\infty\}.)$ It follows that $Ay+a=y$, or it implies that either $a=0$ or $a\neq 0$ and det $(A-I)\neq 0$. □

Theorem 6.6. *Let Γ be a subgroup of $PO(n+1,1)^0$ consisting of elliptic elements. Then passing to a subgroup of finite index if necessary, one of the following is true;*

(i) Γ is conjugate to a subgroup of $E(n)^0$.

(ii) $\bar{\Gamma}$ is compact.

Proof. Let $\bar{\Gamma}$ be the closure of $PO(n+1,1)$. If the connected component $\bar{\Gamma}^0=\{1\}$, i.e., Γ being discrete, then Γ must be finite by the theorem of Tits. Suppose that $\bar{\Gamma}^0$ is nontrivial. Let $\mathcal{R}$ be the radical of $\bar{\Gamma}^0$.

Step 1. $\mathcal{R}$ is nontrivial. Since $\mathcal{R}$ is solvable, there are following possibilities;

(1) $\bar{\Gamma}^0$ contains a simply connected normal abelian subgroup V.

(2) $\bar{\Gamma}^0$ contains a normal subgroup T isomorphic to some k-torus.

For (1), the group V stabilizes the unique point $\{\infty\}$. Since V is normal in Γ, Γ leaves $\{\infty\}$ fixed and so $\Gamma\subset \mathrm{Sim}(\mathbf{R}^n)^0$ by passing to a subgroup of finite index. It then follows from Lemma 6.5 that $\Gamma\subset E(n)^0$.

For (2), Fix $(T,\mathbf{H}_{\mathbf{R}}^{n+1})$ is either a point $\{pt\}$ or isomorphic to some $\mathbf{H}_{\mathbf{R}}^{k+1}$. The first case implies that Γ fixes the point $\{pt\}$ or $L(\Gamma)=\emptyset$. The closure $\bar{\Gamma}$ is compact. The latter case shows that since Γ leaves S^n-S^k $(=\mathbf{H}_{\mathbf{R}}^{k+1}\times O(n-k))$, $\Gamma\subset PO(k+1,1)\times O(n-k)$ where $0\leqq k\leqq n$. Therefore we have $\bar{\Gamma}^0=H\times T$ for some $H\subset PO(k{+}1,1)$. It follows that $L(\bar{\Gamma}^0)=L(H)$. We may assume that $L(H)$ contains more than two points, otherwise it reduces to the previous cases. Then it follows by Lemma 6.1 that $H=PO(\ell,1)\cdot K$ in $PO(k+1,1)$. In particular $\bar{\Gamma}$ contains a loxodromic element (inside $PO(\ell,1)$). It is impossible by Proposition 6.4 so that $H\subset O(\ell)\times K$ being compact. And thus $\bar{\Gamma}^0$ is compact.

At this stage, we consider

Step 2. $\bar{\Gamma}^0$ has no radical. Then $\bar{\Gamma}^0$ is a simple Lie subgroup of $PO(n+1,1)$, it follows that $\bar{\Gamma}^0=PO(\ell,1)\cdot K$ up to conjugation. By the same argument as above it concludes that $\bar{\Gamma}^0$ is a compact subgroup.

Now, let Fix $(\bar{\Gamma}^0, \mathbf{H}_{\mathbf{R}}^{n+1}) = \mathbf{H}_{\mathbf{R}}^{m+1}$. As above, we have $\bar{\Gamma} \subset PO(m+1,1) \times O(n-m)$ and $\bar{\Gamma}^0 \subset O(n-m)$. Consider the exact sequence;

$$\begin{array}{ccccc} O(n-m) & \rightarrow & PO(m+1,1) \times O(n-m) & \xrightarrow{\rho} & PO(m+1,1) \\ \uparrow & & \uparrow & & \uparrow \\ O(n-m) \cap \bar{\Gamma} & \rightarrow & \bar{\Gamma} & \rightarrow & \rho(\bar{\Gamma}). \end{array}$$

It suffices to check that $\rho(\bar{\Gamma})$ is discrete. Since $O(n-m) \cap \bar{\Gamma}/\bar{\Gamma}^0$ is discrete and compact, it is finite. In view of the exact sequence $1 \rightarrow O(n-m) \cap \bar{\Gamma}/\bar{\Gamma}^0 \longrightarrow \bar{\Gamma}/\bar{\Gamma}^0 \longrightarrow \rho(\bar{\Gamma}) \rightarrow 1$, we see that $\rho(\bar{\Gamma})$ is discrete. In particular $\rho(\Gamma)$ is discrete in $PO(m+1,1)$.
Since Γ nomalizes $\bar{\Gamma}^0$, Γ leaves $\mathbf{H}_{\mathbf{R}}^{m+1}$ invariant while $O(n-m)$ fixes $\mathbf{H}_{\mathbf{R}}^{m+1}$ pointwisely. If $\gamma \in \Gamma$ is elliptic then $\rho(\gamma)$ is also elliptic in $\mathbf{H}_{\mathbf{R}}^{m+1}$. Hence $\rho(\Gamma)$ consists of elliptic elements. As we have shown that $\rho(\Gamma)$ is discrete, $\rho(\Gamma)$ is finite. And so $\rho(\bar{\Gamma})$ is still finite. The above exact sequence shows that $\bar{\Gamma}$ is compact. This completes the proof. □

Remark 6.7. *The similar results to Proposition 6.4, Lemma 6.5 and Theorem 6.6 hold for spherical CR structures.*

Suppose that a $(\mathcal{G}, X)$-structure is either conformally flat structure or spherical CR structure. A homomorphism ϕ of a group Γ into $\mathcal{G}$ is said to be a parabolic representation if a subgroup of finite index in $\phi(\Gamma)$ stabilizes the unique point $\{\infty\}$ in X. In other words, $\phi(\Gamma)$ is conjugate to a subgroup in $\mathrm{Sim}(\mathbf{R}^n)$ or $\mathrm{Sim}(\mathcal{N})$ respectively. It is known that a closed $(\mathcal{G}, X)$-manifold whose holonomy representation is parabolic is finitely covered by a sphere, a Hopf manifold, a 3-torus or a nilmanifold.

Problem. *Find a closed 3-manifold with (nontrivial) nonsolvable fundamental group such that any representaion into $\mathcal{G}$ is parabolic.*

If a compact manifold with such parabolic representations exists, then of course it does not admit the above $(\mathcal{G}, X)$-structure. We give an example of such a compact 3-manifold M. The manifold M obtained here has a torus-decomposition which decomposes into two Seifert pairs. This construction has been shown by Motegi [52]. See also [37].
Let N be a 3-sphere removed with the interor of the tubular neighborhood of a torus knot of type (p, q) where p, q are relatively prime integers greater than 1. Choose two such N_1, N_2 and glue them together along their torus boundaries by a certain homeomorphism. Let M be a resulting closed 3-manifold. The following is a generalization of Motegi when we view $\mathrm{PSL}\,(2, \mathbf{C}) \approx PO(3,1)^0$.

Theorem 6.8. *Put $\Gamma = \pi_1(M)$ for the above 3-manifold M. Then any representation of Γ into $PO(n+1,1)$ or $PU(n+1,1)$ $(n \geqq 1)$ is parabolic or sits in the maximal compact group $O(n+1)$ (resp. $U(n+1)$) up to conjugacy.*

Corollary 6.9. *The above 3-manifold M admits neither a conformally flat structure nor a spherical CR struture.*

Recall the construction: Let $S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \, ||z_1|^2 + |z_2|^2 = 1\}$. Put $L(p,q) = \{(z_1, z_2) \in S^3 \, |z_1^p + z_2^q = 0\}$. Then $L(p,q)$ is the torus knot of type (p,q). The complement $N = S^3 - \mathcal{N}(L(p,q))$ is a Seifert fiber space over a two ball with 2 distinguished points. The fundamental group $\pi_1(N)$ is isomorphic to

$$\{t, \alpha_1, \alpha_2 \mid [t, \alpha_1] = [t, \alpha_2] = 1, \alpha_1^p = t^{p'}, \alpha_2^q = t^{q'}\},$$

where $(p,p') = 1$, $(q,q') = 1$. Note that t is a central element of $\pi_1(N)$ which represents a regular fiber of N.

Let $N_i = S^3 - \mathcal{N}(L(p_i, q_i))$ be such a Seifert manifold for $i = 1, 2$. Let (m_1, ℓ_1), (m_2, ℓ_2) be the meridian-longitude pair of ∂N_i $(i = 1, 2)$ respectively. If t_i represents a central element of $\pi_1(N_i)$ as above, it follows that $t_i = m_i^{p_i q_i} \ell_i$ respectively. So instead of the basis (m_i, ℓ_i), we can choose the pair (m_i, t_i) $(i = 1, 2)$. Let $h : \partial N_1 \to \partial N_2$ be an orientation-reversing diffeomorphism whose induced homomorphism satisfies that $h_*(t_1) = m_2$, $h_*(m_1) = t_2$. Now glue N_1 and N_2 together along their boundary by h and let M be the resulting 3-manifold.

$\pi_1(M)$ is generated by the elements of $\pi_1(N_i)$ $(i = 1, 2)$ under the appropriate identification between (m_1, t_1) and (m_2, t_2).

Put $\Gamma = \pi_1(M)$ and $\mathcal{G} = PO(n+1, 1)$ or $PU(n+1, 1)$.

Proof of Theorem 6.8. Let ρ be a representation of Γ into $\mathcal{G}$. First suppose that $\rho(t_1) = 1$. Then by the definition of h, $\rho(m_2) = 1$. Recall that the fundamental group $\pi_1(N_i)$ is the normal closure of the meridian m_i $(i = 1, 2)$, *i.e.*, every element of $\pi_1(N_2)$ is generated by the elements of the form $gm_2^k g^{-1}$. Thus we have that $\rho(\pi_1(N_2)) = 1$. Since $h_*(m_1) = t_2$, it follows that $\rho(m_1) = 1$. Similary $\rho(\pi_1(N_1)) = 1$, or ρ is trivial. If $\rho(t_2) = 1$, then the similar argument shows that ρ is trivial.

So we suppose that both $\rho(t_1)$ and $\rho(t_2)$ are nontrivial. Suppose that $\rho(t_1)$ and $\rho(t_2)$ are elliptic. Then $\rho(t_1)$ has a unique fixed point x inside the hyperbolic space $\mathbf{H}_{\mathbf{R}}^{n+1}$ (resp. $\mathbf{H}_{\mathbf{C}}^{n+1}$). Since some powers of α_1, α_2 are generated by t_1, both $\rho(\alpha_1)$ and $\rho(\alpha_2)$ are also elliptic. Moreover they fix x. Thus every element of $\rho(\pi_1(N_1))$ fixes x and so $\rho(\pi_1(N_1))$ consists of elliptic elements. Similary $\rho(\pi_1(N_2))$ consits of elliptic elements because $\rho(t_2)$ is a nontrivial elliptic element. Hence $\rho(\Gamma)$ consists of elliptic elements. It follows from Theorem 6.6 that either $\rho(\Gamma) \subset E(n)$ up to conjugacy (resp. $\rho(\Gamma) \subset E(N) = \mathcal{N} \rtimes U(n)$), or the closure of $\rho(\Gamma)$ is compact. Thus in this case, ρ is a parabolic representaion or $\rho(\Gamma)$ is contained in the maximal compact group of $\mathcal{G}$.

Suppose that $\rho(t_1)$ is nonelliptic. $\rho(t_1)$ stabilizes the unique point $\{\infty\}$ or exactly two points $\{0,\infty\}$. Noting that $\rho(t_1)$ is a central element of $\rho(\pi_1(N_1))$, every element stabilizes $\{\infty\}$ or $\{0,\infty\}$. Since $h_*(m_1) = t_2$ and $\rho(t_2)$ is a central element of $\rho(\pi_1(N_2))$, every element of $\rho(\pi_1(N_2))$ stabilizes $\{\infty\}$ or $\{0,\infty\}$. As a consequence, every element of $\rho(\Gamma)$ stabilizes $\{\infty\}$ or $\{0,\infty\}$. Therefore we obtain that $\rho(\Gamma) \subset E(n)$, or $\rho(\Gamma) \subset O(n) \times \mathbf{R}^+$respectively. (Similary, $\rho(\Gamma) \subset E(\mathcal{N})$, or $\rho(\Gamma) \subset U(n) \times \mathbf{R}^+$.) □

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