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A MODEL OF THE RANDOM MOTION OF MUTUALLY REFLECTING MOLECULES IN R^d

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Introduction

Models of the random motion of finitely many reflecting objects were constructed by Saisho [3] and Saisho and Tanaka [5]. In this paper we consider the random motion of M molecules in $\mathbf{R}^d$. We assume that (i) the k-th molecule consists of $n_k(\geq 1)$ atoms, (ii) two atoms in different molecules reflect when the distance between these two atoms equals to $\rho > 0$ and that (iii) the distance between any two atoms in the same molecule does not exceed $R > 0$. In our model, collisions between atoms belonging to the same molecule are not considered. In [3], (ii) was not considered and in [5] $n_k = 1$ for all k. The same model was discussed in [4] without proof.

Put $\Lambda = \{1, 2, \ldots, N\}, N = \sum_{k=1}^{M} n_k$ and

$$\Lambda_k = \{\sum_{i=1}^{k-1} n_i + 1, \sum_{i=1}^{k-1} n_i + 2, \ldots, \sum_{i=1}^{k} n_i\}, \quad k = 1, 2, \ldots, M.$$

Here we use the convention $\sum_{i=1}^{0} = 0$. We mean that Λ_k describes the set of indexes of atoms in the k-th molecule. We put $m(i) = k$ if $i \in \Lambda_k$. Then our model can be formulated as follows. For given $\boldsymbol{w} = (w_1, w_2, \ldots, w_N) \in C([0,\infty) \to \mathbf{R}^{Nd})$ satisfying

$$\begin{aligned} |w_i(0) - w_j(0)| \ &\leq R \quad \text{for all} \quad i, j \quad \text{with} \quad m(i) = m(j), \\ &\geq \rho \quad \text{for all} \quad i, j \quad \text{with} \quad m(i) \neq m(j), \end{aligned}$$

consider the equation

$$(0.1) \qquad \xi_i^R(t) = w_i(t) + \sum_{\substack{j=1 \\ (\neq i)}}^{N} \int_0^t (\xi_i^R(s) - \xi_j^R(s)) d\ell_{ij}^R(s), \quad i = 1, 2, \ldots, N,$$

under the conditions

(1) $\xi^R = (\xi_1^R, \xi_2^R, \ldots, \xi_N^R) \in C([0,\infty) \to \boldsymbol{R}^{Nd})$ with

$$
\begin{aligned}
|\xi_i^R(t) - \xi_j^R(t)| &\le R \quad \text{for all} \quad i,j \quad \text{with} \quad m(i) = m(j), \\
&\ge \rho \quad \text{for all} \quad i,j \quad \text{with} \quad m(i) \ne m(j), \quad t \ge 0,
\end{aligned}
$$

(2) ℓ_{ij}^R is a continuous function which is nonincreasing or nondecreasing according as $m(i) = m(j)$ or $m(i) \ne m(j)$ with $\ell_{ij}^R = \ell_{ji}^R$, $\ell_{ij}^R(0) = 0$ and

$$
\ell_{ij}^R(t) = \begin{cases} \int_0^t \mathbf{1}_{\{|\xi_i^R(s) - \xi_j^R(s)| = R\}}(s) d\ell_{ij}^R(s), & \text{if} \quad m(i) = m(j), \\ \int_0^t \mathbf{1}_{\{|\xi_i^R(s) - \xi_j^R(s)| = \rho\}}(s) d\ell_{ij}^R(s), & \text{if} \quad m(i) \ne m(j), \end{cases}
$$

where $\mathbf{1}_A$ denotes the indicator function of a set A.

Once we find a unique solution $\xi_i^R(t) = \xi_i^R(t, w_1, w_2, \ldots, w_N)$ of (0.1) for given $\boldsymbol{w} = (w_1, w_2 \ldots, w_N)$, we can define a stochastic process $\boldsymbol{X}(t) = (X_1(t), X_2(t), \ldots, X_N(t))$ by

$$
X_i(t) = \xi_i^R(t, W_1, W_2, \ldots, W_N), \quad W_i(t) = X_i(0) + B_i(t),
$$

where $B_i(t), i = 1, 2, \ldots, N$, are independent d-dimensional Brownian motions. Then $\boldsymbol{X}(t)$ satisfies the equation

$$
X_i(t) = W_i(t) + L_i(t), \quad i = 1, 2, \ldots, N,
$$

where $L_i(t)$ is a process of bounded variation which varies only when

(i)
$$
\begin{aligned}
R_i(t) &\equiv \max_{j: m(j) = m(i)} |X_i(t) - X_j(t)| = R \qquad \textit{or} \\
\rho_i(t) &\equiv \min_{j: m(j) \ne m(i)} |X_i(t) - X_j(t)| = \rho,
\end{aligned}
$$

so that

(ii)
$$
R_i(t) \le R, \quad \rho_i(t) \ge \rho, \quad t \ge 0, \quad i = 1, 2 \ldots, N.
$$

We call $\boldsymbol{X}(t)$ *the random motion of M molecules mutually reflecting in $\boldsymbol{R}^d$*.The first problem of this paper is to show that the equation (0.1) can be solved uniquely following the idea of [5]: we consider the (configulation) space

$$
\begin{aligned}
\mathcal{D}_R = \{x = (x_1, x_2, \ldots, x_N) \in \boldsymbol{R}^{Nd} : |x_i - x_j| < R \quad &\text{for} \quad \forall i,j \quad \text{with} \quad m(i) = m(j) \\
\text{and} \quad > \rho \quad &\text{for} \quad \forall i,j \quad \text{with} \quad m(i) \ne m(j)\}
\end{aligned}
$$

and show that $\mathcal{D}_R$ satisfies Conditions (A) and (B) (§ 2) which assure the existence of the unique solution of the Skorohod problem (abbreviated *SP*) $(w; \mathcal{D}_R)$ and then we see the equivalence of the equation (0.1) and the SP $(w; \mathcal{D}_R)$ (a precise formulation of Conditions (A), (B) and the SP are given in § 1).

Our second problem is to consider the convergence of ξ^R as R tends to 0 and determine the limiting function ξ^0. Roughly speaking, we show that ξ^0 describes the motion of mutually reflecting M hard balls of diameter ρ whose ratio of masses is $n_1 : n_2 : \cdots : n_M$ (§§ 3, 4).

Next we consider the stochastic differential equation (abbreviated *SDE*)

$$\begin{cases} dX_i^R(t) = \sigma(X_i^R(t))dB_i(t) + b(X_i^R(t))dt \\ \qquad\qquad + \sum_{\substack{j=1 \\ (\neq i)}}^{N} (X_i^R(t) - X_j^R(t))d\ell_{ij}^R(t), \quad i = 1, 2, \dots, N, \\ X^R(0) \in \overline{\mathcal{D}_R}, \end{cases}$$

under conditions similar to (1) and (2) (§ 5).

§1. Skorohod problem

Let $D \subset \boldsymbol{R}^n$ be a domain and we call a member $\boldsymbol{n}$ of the set $\mathcal{N}_x \equiv \mathcal{N}_x(D) = \{\boldsymbol{n} : |\boldsymbol{n}| = 1, B(x - r\boldsymbol{n}, r) \cap D = \emptyset, r > 0\}$ *an inward unit normal vector* at $x \in \partial D$, where $B(y,r) = \{z \in \boldsymbol{R}^n : |y - z| < r\}$. We also denote $\mathcal{N}_{x,r} \equiv \mathcal{N}_{x,r}(D) = \{\boldsymbol{n} : |\boldsymbol{n}| = 1, B(x - r\boldsymbol{n}, r) \cap D = \emptyset\}$, $r > 0$.

Remark 1.1. Let $\boldsymbol{n}$ be a unit vector in $\boldsymbol{R}^n$. Then the following two statements are equivalent:

(1) $\boldsymbol{n} \in \mathcal{N}_{x,r}$,

(2) $\langle y - x, \boldsymbol{n}\rangle + \frac{1}{2r}|y - x|^2 \geq 0$ for all $y \in \overline{D}$,

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\boldsymbol{R}^n$ (see [1], Remark 1.2).

Now we pose the following two conditions on D.

Condition (A). *There exists a positive constant r_D such that*

$$\mathcal{N}_x = \mathcal{N}_{x,r_D} \neq \emptyset \quad \textit{for all} \quad x \in \partial D.$$

Condition (B). *There exist constants $\delta > 0$ and $\beta \in [1, \infty)$ with the following property; for any $x \in \partial D$ there exists a unit vector $\boldsymbol{e}_x$ such that*

$$\langle \boldsymbol{e}_x, \boldsymbol{n}\rangle \geq 1/\beta \quad \textit{for all} \quad \boldsymbol{n} \in \bigcup_{y \in B(x,\delta)\cap\partial D} \mathcal{N}_y.$$

Then we pose the following problem introduced by Lions and Sznitman [1] and Saisho [2].

Skorohod problem$(w;D)$. For given $w \in C([0,\infty) \to \boldsymbol{R}^n)$, $w(0) \in \overline{D}$, find a pair(ξ,ℓ) of functions satisfying the equation

$$(1.1) \qquad \xi(t) = w(t) + \int_0^t \boldsymbol{n}(s)d\ell(s)$$

under the conditions:

(i) $\xi \in C([0,\infty) \to \overline{D})$,

(ii) ℓ is a continuous nondecreasing function with $\ell(0) = 0$ and

$$\ell(t) = \int_0^t 1_{\partial D}(\xi(s))d\ell(s),$$

(iii) $\boldsymbol{n}(s) \in \mathcal{N}_{\xi(s)}$ if $\xi(s) \in \partial D$.

We call (1.1) *the Skorohod equation* (abbreviated *SE*) *for* $(w;D)$.

Theorem 1.1([2]). *If D satisfies Conditions* (A) *and* (B), *for any $w \in C([0,\infty) \to \boldsymbol{R}^n)$ with $w(0) \in \overline{D}$, there exists a unique solution (ξ,ℓ) of the SP $(w;D)$ and ξ is continuous in (t,w).*

Suppose that $\mathcal{D} \subset \boldsymbol{R}^n$ is a domain written in the form $\mathcal{D} = \bigcap_{i=1}^p \mathcal{D}_i$ (finite intersection), where each $\mathcal{D}_i$ is a smooth domain in $\boldsymbol{R}^n$ satisfying Conditions (A) and (B) with $r_i \equiv r_{\mathcal{D}_i}$. Furthermore we assume the following conditions on $\mathcal{D}$.

Condition (B_0). *There exists $\beta_0 \in [1,\infty)$ with the following property: for any $x \in \partial\mathcal{D}$ there exists a unit vector $\mathbf{e}_x^0$ such that*

$$\langle \mathbf{e}_x^0, \boldsymbol{n} \rangle \geq 1/\beta_0 \quad \text{for all} \quad \boldsymbol{n} \in \mathcal{N}_x.$$

Condition (C). *There exist constants $\gamma_c \in (-1,1)$ and $\delta_c > 0$ such that*

$$\langle \boldsymbol{n}_i(x), \boldsymbol{n}_j(y) \rangle \geq \gamma_c,\ x \in \partial\mathcal{D}_i \cap \partial\mathcal{D},\ y \in \partial\mathcal{D}_j \cap \partial\mathcal{D},\ 1 \leq i,j \leq p,\ |x-y| < \delta_c.$$

Now we put $I(x) = \{1 \leq i \leq p :\ x \in \partial\mathcal{D}_i\}$, $\alpha(x) = \sharp I(x)$, and

$$\mathcal{N}'_x(\mathcal{D}) = \left\{ \boldsymbol{n} \in \boldsymbol{R}^n : |\boldsymbol{n}| = 1, \quad \boldsymbol{n} = \sum_{i \in I(x)} c_i \boldsymbol{n}_i(x), \quad c_i \geq 0 \right\}, \quad x \in \partial\mathcal{D}.$$

Then we have the following lemma.

Lemma 1.1. $\mathcal{N}_x(\mathcal{D}) = \mathcal{N}'_x(\mathcal{D})$, $\forall x \in \partial\mathcal{D}$.

Proof. $\mathcal{N}_x(\mathcal{D}) \supset \mathcal{N}'_x(\mathcal{D})$ is clear from the fact

$$\langle y-x, \boldsymbol{n}_i(x)\rangle + \tfrac{1}{2r_i}|y-x|^2 \geq 0, \quad \forall y \in \overline{\mathcal{D}}, i \in I(x).$$

For the converse inclusion we note that for any $\alpha\,(0<\alpha<1)$ there exists $\delta'_i > 0$ such that

$$C(x, \boldsymbol{n}_i(x) : \alpha) \cap B(x, \delta'_i) \subset \mathcal{D}_i \cup \{x\},$$

where $C(x, \boldsymbol{n}_i(x):\alpha) = \{y \in \boldsymbol{R}^n : \langle y-x, \boldsymbol{n}_i(x)\rangle > \alpha\,|y-x|\}$. Thus, setting $\delta' \equiv \bigwedge_{i\in I(x)} \delta'_i$, we have

$$\bigcap_{i\in I(x)} C(x, \boldsymbol{n}_i(x):\alpha) \cap B(x,\delta') \subset \mathcal{D} \cup \{x\},$$

which implies

$$x - \mathcal{N}_x(\mathcal{D}) \subset \bigcap_{0<\alpha<1} \overline{\sum_{i\in I(x)} C\,(x, \boldsymbol{n}_i\,(x):\alpha)^*}. \tag{1.2}$$

Here, $C\,(x, \boldsymbol{n}_i\,(x):\alpha)^*$ is the dual cone of $C\,(x, \boldsymbol{n}_i\,(x):\alpha)$ defined by

$$C\,(x, \boldsymbol{n}_i\,(x):\alpha)^* = \{z : \langle z-x, y-x\rangle \leq 0,\ y \in C(x, \boldsymbol{n}_i(x):\alpha)\}.$$

(1.2) implies $\mathcal{N}_x(\mathcal{D}) \subset \mathcal{N}'_x(\mathcal{D})$ (see [5], p.736). ∎

Proposition 1.1. $\mathcal{D}$ satisfies Condition (A) with $\mathcal{N}_x(\mathcal{D}) = \mathcal{N}'_x(\mathcal{D})$.

The proof is easy from Condition (A) for $\mathcal{D}_i$, $1 \leq i \leq p$, Condition (C) and Lemma 1.1 and so, it is omitted.

Proposition 1.2. $\mathcal{D}$ satisfies Condition (B).

Proof. We define a unit vector $\boldsymbol{e}_x$ in Condition (B) by the following manner. By smoothness of $\mathcal{D}_i$, $1 \leq i \leq p$, we easily see that there exists a constant $\delta' > 0$ such that for each $x \in \partial\mathcal{D}$ there exists $z \in B(x,\delta') \cap \partial\mathcal{D}$ with $I(z) \supset I(y)$ for $\forall y \in B(x,\delta') \cap \partial\mathcal{D}$. Then, we define $\boldsymbol{e}_x = \boldsymbol{e}^0_z$. By the smoothness of $\mathcal{D}_i$ again, we see that for any $0<c<1$, there exists a constant $\delta'' > 0$ such that for any $x, y \in \partial\mathcal{D}_i$, $|x-y| < \delta''$, $1 \leq i \leq p$, we have $\langle \boldsymbol{n}_x, \boldsymbol{n}_y\rangle > c$ with $\boldsymbol{n}_x \in \mathcal{N}_x(\mathcal{D}_i), \boldsymbol{n}_y \in \mathcal{N}_y(\mathcal{D}_i)$. Thus, combining this with Lemma 1.1 and Condition (B_0) for $\mathcal{D}$, we get Condition (B) for $\mathcal{D}$. ∎

Now let $(\Omega, \mathcal{F}, P)$ be some probability space with a right continuous filtration $(\mathcal{F}_t)$. We assume that $\mathcal{F}_t$ contains all P-null sets. Then consider the following SDE of Skorohod type for D on $(\Omega, \mathcal{F}, P)$:

$$\begin{cases} dX(t) = \sigma(X(t))dB(t) + b(X(t))dt + \boldsymbol{n}(t)d\ell(t), \\ X(0) \in \overline{D}. \end{cases} \tag{1.3}$$

Here $\sigma : \overline{D} \to \boldsymbol{R}^n \otimes \boldsymbol{R}^n$, $b : \overline{D} \to \boldsymbol{R}^n$ are bounded Lipschitz continuous functions and $X(0)$ is an $\mathcal{F}_0$-measurable random variable and $\{B(t)\}$ is an n-dimensional Brownian motion with $B(0) = 0$. A solution (X, ℓ) of (1.3) should be found under the following conditions:

(i) X is a $\overline{D}$-valued $(\mathcal{F}_t)$-adapted continuous process,

(ii) ℓ is a continuous non-decreasing process with $\ell(0) = 0$ and

$$\ell(t) = \int_0^t 1_{\partial D}(X(s))d\ell(s),$$

(iii) $\boldsymbol{n}(s) \in \mathcal{N}_{X(s)}$ if $X(s) \in \partial D$.

Theorem 1.2([2]). *If D satisfies Conditions* (A) *and* (B), *there exists a unique* (*strong*) *solution of* (1.2) *for any initial value $X(0) \in \overline{D}$.*

§2. Existence and uniqueness of a solution of the equation (0.1)

In this section we first prove that the domain $\mathcal{D}_R$ satisfies Conditions (A)and (B). If we set

$$\mathcal{D}_{ij} = \{x \in \boldsymbol{R}^{Nd} : |x_i - x_j| > \rho\}, \quad m(i) = m(j),$$
$$\mathcal{D}^{ij} = \{x \in \boldsymbol{R}^{Nd} : |x_i - x_j| < R\}, \quad m(i) \neq m(j),$$

for $1 \leq i < j \leq N$, we immediately get

$$\mathcal{D}_R = \bigcap_{(i,j):m(i)=m(j)} \mathcal{D}_{ij} \cap \bigcap_{(i,j):m(i)\neq m(j)} \mathcal{D}^{ij}.$$

Thus, Proposition 1.1 yields immediately the following proposition.

Proposition 2.1. *$\mathcal{D}_R$ satisfies Condition* (A) *with*

$$\mathcal{N}_x(\mathcal{D}_R) = \left\{ \boldsymbol{n} : |\boldsymbol{n}| = 1, \quad \boldsymbol{n} = \sum_{(i,j)\in \boldsymbol{J}_x} c_{ij}\boldsymbol{n}_{ij}(x), \quad c_{ij} \geq 0 \right\}, \quad x \in \partial\mathcal{D}_R,$$

where $\boldsymbol{J}_x = \{(i,j) : x \in \partial\mathcal{D}_{ij}$ or $x \in \partial\mathcal{D}^{ij}\}$ and

$$\boldsymbol{n}_{ij}(x) = \begin{cases} (0,\ldots,0, \underset{(i-\text{th})}{\frac{x_j - x_i}{\sqrt{2}R}}, 0,\ldots,0, \underset{(j-\text{th})}{\frac{x_i - x_j}{\sqrt{2}R}}, 0,\ldots,0), & \text{if} \quad m(i) = m(j), \\ (0,\ldots,0, \underset{(i-\text{th})}{\frac{x_i - x_j}{\sqrt{2}\rho}}, 0,\ldots,0, \underset{(j-\text{th})}{\frac{x_j - x_i}{\sqrt{2}\rho}}, 0,\ldots,0), & \text{if} \quad m(i) \neq m(j). \end{cases}$$

For $x \in \boldsymbol{R}^{Nd}$ and $I \subset \Lambda \equiv \{1,2,\ldots,N\}$, we denote $x_I = \{x_i : i \in I\}$,

$$g(I) \equiv g(I,x) = \frac{1}{\sharp I}\sum_{i\in I} x_i,$$

and define $x^g = (x_1^g, x_2^g, \ldots, x_M^g) \in \boldsymbol{R}^{Md}$ by $x_k^g = g(\Lambda_k, x)$, where $\sharp I$ is the number of elements in I. We also denote $g_i = x_{m(i)}^g, i \in \Lambda$. We note $|g_i - x_i| \le R$. If $I = \bigcup_{i=1}^p \Lambda_{k_i}$, we denote the number p of molecules included in I by $\natural I$.

Definition 2.1. Suppose $x \in \boldsymbol{R}^{Nd}$.

(1) x_I and x_J $(I, J \subset \Lambda)$ are said to be *2ρ-separated* if $|x_i - x_j| \ge 2\rho$ for all $i \in I$ and $j \in J$.

(2) When $I = \bigcup_{i=1}^p \Lambda_{k_i}$, $2 \le p \le M$, x_I is called *a cluster* if for any $\Lambda_k, \Lambda_h \subset I$ there exists a sequence of indexes $i_0(= k), i_1, \ldots, i_q(= h)$ such that $\Lambda_{i_\lambda} \subset I$, $\lambda = 0, 1, \ldots, q$ and $|x_i - x_j| < 2\rho$ for some $i \in \Lambda_{i_\lambda}$ and $j \in \Lambda_{i_{\lambda+1}}$, $\lambda = 0, 1, \ldots, q-1$.

For each $x \in \partial\mathcal{D}_R$, we classify the index set Λ into four classes:

$$\begin{aligned}
\Gamma_a \equiv \Gamma_a(x) = \{i \in \Lambda :\ & |x_j - x_k| \ge 2\rho \quad \text{for any} \quad j \in \Lambda_{m(i)}, k \notin \Lambda_{m(i)} \\
& \text{and} \quad |x_j - x_k| > R/2 \quad \text{for some} \quad j, k \in \Lambda_{m(i)}\}, \\
\Gamma_b \equiv \Gamma_b(x) = \{i \in \Lambda :\ & |x_j - x_k| < 2\rho \quad \text{for some} \quad j \in \Lambda_{m(i)}, k \notin \Lambda_{m(i)} \\
& \text{and} \quad |x_j - x_k| > R/2 \quad \text{for some} \quad j, k \in \Lambda_{m(i)}\}, \\
\Gamma_c \equiv \Gamma_c(x) = \{i \in \Lambda :\ & |x_j - x_k| < 2\rho \quad \text{for some} \quad j \in \Lambda_{m(i)}, k \notin \Lambda_{m(i)} \\
& \text{and} \quad |x_j - x_k| \le R/2 \quad \text{for any} \quad j, k \in \Lambda_{m(i)}\}, \\
\Gamma_s \equiv \Gamma_s(x) = \Lambda \setminus & (\Gamma_a \cup \Gamma_b \cup \Gamma_c).
\end{aligned}$$

Remark 2.1. For $x \in \partial\mathcal{D}_R$ we can write

$$\{x_1, x_2, \ldots, x_N\} = \bigcup_{k=1}^{p} x_{I_k} \cup x_{\Gamma_a} \cup x_{\Gamma_s}, \tag{2.1}$$

where $x_{I_k}, k = 1, 2, \ldots, p$, are mutually 2ρ-separated clusters and the convention $\bigcup_{k=1}^0 = \emptyset$ is used. For $i \in \Gamma_b \cup \Gamma_c$ there exists a unique $k\,(1 \le k \le p)$ such that $i \in I_k$ and we say that x_{I_k} is *the maximal cluster* including x_i.

Hereafter, we fix $x \in \partial\mathcal{D}_R$ throughout this section.

Remark 2.2. Suppose that x_I is a cluster and $i \in I$. Then Definition 2.1 yields $|x_i - g(I)| < (\natural I)R + 2(\natural I - 1)\rho$.

Now we define $u = (u_1, u_2, \ldots, u_N) \in \boldsymbol{R}^{Nd}$ by

$$u_i = \begin{cases} g_i, & \text{if} \quad i \in \Gamma_a, \\ 2g_i - g(I(i)), & \text{if} \quad i \in \Gamma_b \cup \Gamma_c, \\ x_i, & \text{if} \quad i \in \Gamma_s, \end{cases}$$

where $x_{I(i)}$, $I(i) \subset \Lambda$, is the maximal cluster including x_i. Then we have the following lemma.

Lemma 2.1. *We have* $|u-x| < \{(M+2)R + 2(M-1)\rho\}\sqrt{N}$.

Proof. Suppose $i \in \Gamma_b \cup \Gamma_c$ and that x_I is the maximal cluster including x_i. Then, by Remark 2.2, we have

$$|u_i - x_i| = |2g_i - g(I) - x_i| < 2R + (\natural I)R + 2(\natural I - 1)\rho.$$

For $i \in \Gamma_a(x)$, we have $|u_i - x_i| = |g_i - x_i| \le R$, and for $i \in \Gamma_s(x)$, $|u_i - x_i| = 0$. Thus, by (2.1) we have

$$\begin{aligned} |u-x|^2 &\le \sum_{k=1}^{p} \sharp I_k \{(\natural I_k + 2)R + 2(\natural I_k - 1)\rho\}^2 + (\sharp\Gamma_a)R^2 \\ &< \{(M+2)R + 2(M-1)\rho\}^2(\sharp I) + (\sharp\Gamma_a)R^2 \\ &< \{(M+2)R + 2(M-1)\rho\}^2 N. \qquad \blacksquare \end{aligned}$$

Now we define

$$\begin{aligned} I(x) = \{(i,j) : 1 \le i < j \le N, \quad & |x_i - x_j| = R,\ m(i) = m(j) \\ \text{or} \quad & |x_i - x_j| = \rho,\ m(i) \ne m(j)\}, \end{aligned}$$

and $e_x^0 = (u-x)/|u-x|$. Here we remark that the i-th component $(e_x^0)_i$ of e_x^0 is given by

$$(e_x^0)_i = \begin{cases} (g_i - x_i)/|u-x|, & \text{if} \quad i \in \Gamma_a, \\ (2g_i - g(I(i)) - x_i)/|u-x|, & \text{if} \quad i \in \Gamma_b \cup \Gamma_c, \\ 0, & \text{if} \quad i \in \Gamma_s. \end{cases}$$

Then we have the following lemma.

Lemma 2.2. *If* $(i,j) \in I(x)$ *and* $\rho > 4R$, *we have* $\langle e_x^0, n_{ij}(x)\rangle > 1/\beta_0$,*where* $\beta_0 \equiv \sqrt{2N}\{(M+2)R + 2(M-1)\rho\}/\{R \wedge (\rho - 4R)\}$.

Proof. Suppose $m(i) = m(j)$. Then, $i, j \in \Gamma_a(x)$ or $\Gamma_b(x)$ and in both cases we have

$$\langle e_x^0, n_{ij}(x)\rangle = \langle x_i - x_j, x_i - x_j\rangle / \sqrt{2}R|u-x| = R/\sqrt{2}|u-x|. \tag{2.2}$$

Next, we assume $m(i) \ne m(j)$. Then we have $i, j \in \Gamma_b \cup \Gamma_c$ and

$$\begin{aligned} \langle e_x^0, n_{ij}(x)\rangle &= \langle 2g_i - 2g_j - x_i + x_j, x_i - x_j\rangle / \sqrt{2}\rho\,|u-x| \\ &= \{\rho^2 + 2\langle g_i - g_j - x_i + x_j, x_i - x_j\rangle\}/\sqrt{2}\rho\,|u-x| \\ &\ge (\rho - 4R)/\sqrt{2}|u-x|. \end{aligned} \tag{2.3}$$

By (2.2), (2.3) and Lemma 2.1, we have

$$\begin{aligned}\langle e_x^0, n_{ij}(x)\rangle &\geq \{R\wedge(\rho-4R)\}/\sqrt{2}|u-x| \\ &> \frac{\{R\wedge(\rho-4R)\}}{\sqrt{2N}}\{(M+2)R+2(M-1)\rho\}^{-1} \\ &= 1/\beta_0,\end{aligned}$$

which complete the proof of Lemma 2.2. ∎

By Lemma 2.2 and Proposition 2.1, we see that $\mathcal{D}_R$ satisfies Condition (B_0) with β_0. Thus, Proposition 1.2 yields the following proposition.

Proposition 2.2. *If $\rho > 4R$, $\mathcal{D}_R$ satisfies Condition* (B).

Theorem 2.1. *The equation* (0.1) *has a unique solution.*

Proof. Consider the SP $(w;\mathcal{D}_R)$:

$$\xi(t) = w(t) + \int_0^t n(s)d\ell(s), \tag{2.4}$$

where $n(s) \in \mathcal{N}_{\xi(s)}(\mathcal{D}_R)$ can be written in the form

$$n(s) = \sum_{1\leq i<j\leq N} c_{ij}(s)n_{ij}(s), \quad c_{ij}(s) \geq 0.$$

If we put

$$\ell_{ij}(t) = \begin{cases} \dfrac{-1}{\sqrt{2}R}\displaystyle\int_0^t c_{ij}(s)d\ell(s), & \text{if} \quad m(i)=m(j), \\ \dfrac{1}{\sqrt{2}\rho}\displaystyle\int_0^t c_{ij}(s)d\ell(s), & \text{if} \quad m(i)\neq m(j), \end{cases}$$

and

$$c_{ij}(t) = \begin{cases} c_{ji}(t), & \text{if} \quad (j,i)\in J_{\xi(t)}, \\ 0, & \text{if} \quad (i,j),(j,i)\notin J_{\xi(t)}, \quad \xi(t)\in\partial\mathcal{D}_R, \end{cases}$$

it is easy to see that (2.4) yields (0.1).

For the converse, we see that (0.1) also implies (2.4) with

$$n(t) = a(t)/|a(t)|, \quad d\ell(t) = \sqrt{2}|a(t)|d\ell'(t),$$

where

$$\begin{aligned} a(t) &= \sum_{\substack{1\leq i<j\leq N \\ m(i)\neq m(j)}} \rho d_{ij}(t)n_{ij}(\xi(s)) - \sum_{\substack{1\leq i<j\leq N \\ m(i)=m(j)}} R d_{ij}(t)n_{ij}(\xi(s)), \\ \ell'(t) &= \sum_{\substack{1\leq i<j\leq N \\ m(i)\neq m(j)}} \ell_{ij}(t) - \sum_{\substack{1\leq i<j\leq N \\ m(i)=m(j)}} \ell_{ij}(t), \end{aligned}$$

and $d_{ij}(t) = d\ell_{ij}(t)/d\ell'(t)$ (Radon-Nikodym derivative). ∎

We now define

$$\mathcal{D}_\infty = \left\{x = (x_1, x_2, \dots, x_N) \in \boldsymbol{R}^{Nd} : |x_i - x_j| > \rho,\ m(i) \neq m(j)\right\},$$
$$\mathcal{O} = \left\{x = (x_1, x_2, \dots, x_M) \in \boldsymbol{R}^{Md} : |x_i - x_j| > \rho,\ i \neq j\right\},$$

and remark the following for the latter use.

Remark 2.3. (1) ([3]) $\mathcal{D}_\infty$ satisfies Conditions (A) and (B) with $r_\infty \equiv r_{\mathcal{D}_\infty} = \rho\{8(N-1)^{\frac{3}{2}}\}^{-1}$ and

$$\mathcal{N}_x(\mathcal{D}_\infty) = \left\{\boldsymbol{n} : |\boldsymbol{n}| = 1,\ \boldsymbol{n} = \sum_{(i,j)\in \boldsymbol{J}_x^\infty} c_{ij}\boldsymbol{n}_{ij}(x), \quad c_{ij} \geq 0\right\}, \quad x \in \partial\mathcal{D}_\infty,$$

where $\boldsymbol{J}_x^\infty = \{(i,j) : 1 \leq i < j \leq N,\ |x_i - x_j| = \rho,\ m(i) \neq m(j)\}$.

(2) ([3],[5]) $\mathcal{O}$ satisfies Conditions (A) and (B) with $r_{\mathcal{O}} = \rho\{8(M-1)^{\frac{3}{2}}\}^{-1}$ and

$$\mathcal{N}_x(\mathcal{O}) = \left\{\boldsymbol{n} : |\boldsymbol{n}| = 1,\ \boldsymbol{n} = \sum_{(k,h)\in \boldsymbol{J}_x^{\mathcal{O}}} c_{kh}\boldsymbol{m}_{kh}(x), \quad c_{kh} \geq 0\right\}, \quad x \in \partial\mathcal{O},$$

where $\boldsymbol{J}_x^{\mathcal{O}} = \{(k,h) : 1 \leq k < h \leq M,\ |x_k - x_h| = \rho\}$ and

$$\boldsymbol{m}_{kh}(x) = \Big(0, \dots, 0, \underset{(k-th)}{\frac{x_k - x_h}{\sqrt{2}\rho}}, 0, \dots, 0, \underset{(h-th)}{\frac{x_h - x_k}{\sqrt{2}\rho}}, 0, \dots, 0\Big).$$

§3. Convergence of ξ^R as R tends to 0

Let $\xi^R(t) = w(t) + \int_0^t \boldsymbol{n}(s)d\ell^R(s)$ be a SE for $(w; \mathcal{D}_R)$. Then by Proposition 2.1, we can write

$$\xi^R(t) = w(t) + \psi^R(t) + \varphi^R(t), \tag{3.1}$$

where

$$\psi^R(t) = \int_0^t \sum_{\substack{1\leq i<j\leq N \\ m(i)=m(j)}} c_{ij}(s)\boldsymbol{n}_{ij}(s)d\ell^R(s),$$
$$\varphi^R(t) = \int_0^t \sum_{\substack{1\leq i<j\leq N \\ m(i)\neq m(j)}} c_{ij}(s)\boldsymbol{n}_{ij}(s)d\ell^R(s).$$

Thus, setting

$$\begin{aligned} m(s) &= \sum_{\substack{1\le i<j\le N \\ m(i)\ne m(j)}} c'_{ij}(s)n_{ij}(s), \\ c'_{ij}(s) &= \frac{c_{ij}(s)}{\left|\sum_{\substack{1\le i<j\le N \\ m(i)\ne m(j)}} c_{ij}(s)n_{ij}(s)\right|}, \\ d\widetilde{\ell^R}(s) &= \left|\sum_{\substack{1\le i<j\le N \\ m(i)\ne m(j)}} c_{ij}(s)n_{ij}(s)\right| d\ell^R(s), \end{aligned}$$

we have $\varphi^R(t) = \int_0^t m(s)d\widetilde{\ell^R}(s), m(s) \in \mathcal{N}_{\xi^R(s)}(\mathcal{D}_\infty)$, and we see the following remark.

Remark 3.1. (3.1) is regarded as a SE for $(w + \psi^R; \mathcal{D}_\infty)$.

For any $x \in \boldsymbol{R}^{Md}$ we define $\overline{\overline{x}} = (\overline{\overline{x}}_1, \overline{\overline{x}}_2, \ldots, \overline{\overline{x}}_N) \in \boldsymbol{R}^{Nd}$ by $\overline{\overline{x}}_i = x_k$ if $m(i) = k$, and for a function $f : [0, \infty) \to \boldsymbol{R}^{Md}$ we define $\overline{\overline{f}} : [0, \infty) \to \boldsymbol{R}^{Nd}$ by $\overline{\overline{f}}(t) = \overline{\overline{f(t)}}$, $t \in [0, \infty)$. We denote $\eta^R = (\xi^R)^g$, $\Phi^R = (\varphi^R)^g$. Clearly, η^R describes the motion of the center of gravity of each molecule. Using this notation, (3.1) yields $\eta^R(t) = w^g(t) + \Phi^R(t)$.

Remark 3.2. For each $x \in \partial\mathcal{D}_\infty$, we can represent uniquely as

$$\{x_1, x_2, \ldots, x_N\} = \bigcup_{k=1}^p x_{I_k} \cup x_{\{i:m(i)\in \boldsymbol{K}_s\}},$$

where $0 \le p \le [M/2]$,

$$\boldsymbol{K}_s \equiv \boldsymbol{K}_s(x) = \{1 \le k \le M : |x_i - x_j| \ge 2\rho \quad \text{for any } i \in \Lambda_k \text{ and } j \in \Lambda_h,\ h \ne k\},$$

and x_{I_k}, $k = 1, 2, \ldots, p$, are mutually 2ρ-separated clusters. We also define

$$\boldsymbol{K}_c \equiv \boldsymbol{K}_c(x) = \{1 \le k \le M : |x_i - x_j| < 2\rho \quad \text{for some } i \in \Lambda_k \text{ and } j \in \Lambda_h,\ h \ne k\}.$$

Then, if $|x_i - x_j| < 2\rho$ for some $i \in \Lambda_k$ and $j \notin \Lambda_k$, we have $\Lambda_k \subset I_h$ for some unique $h\,(1 \le h \le p)$. We denote this h by $a(k) \equiv a(k, x)$. Clearly, $|x_i - x_j| < 2\rho$ implies $a(m(i)) = a(m(j))$.

We define $\boldsymbol{v}_x = (v_1(x), v_2(x), \ldots, v_N(x)) \in \boldsymbol{R}^{Nd}$, $x \in \partial\mathcal{D}_\infty$ by

$$v_i(x) = \begin{cases} 0, & \text{if } m(i) \in \boldsymbol{K}_s, \\ x^g_{m(i)} - \mathcal{G}(m(i); x), & \text{if } m(i) \in \boldsymbol{K}_c, \end{cases}$$

where $\mathcal{G}(k; x) = \frac{1}{\sharp I_{a(k)}} \sum_{h:a(h)=a(k)} x^g_h$, $k = 1, 2, \ldots, M$. Then put $\widetilde{\boldsymbol{e}}_x = \boldsymbol{v}_x/|\boldsymbol{v}_x|$, $x \in \partial\mathcal{D}_\infty$.

Hereafter, we assume $R < \rho/8$ and prepare some lemmas.

Lemma 3.1. *Assume that* $|y^g - x^g| < \widetilde{\delta}$, $0 < \widetilde{\delta} < \rho/3\sqrt{2}$, *and* $x, y \in \overline{\mathcal{D}_R} \cap \partial\mathcal{D}_\infty$. *Then if* $|y_i - y_j| = \rho$, *we have* $\langle \widetilde{e}_x, n_{ij}(y)\rangle > 1/\sqrt{N}$.

Proof. Set $k = m(i)$, $h = m(j)$. Noting $\mathcal{G}(k;x) = \mathcal{G}(h;x)$, we have

$$\begin{aligned}\langle v_x, n_{ij}(y)\rangle &= \left\langle x_k^g - \mathcal{G}(k;x), \frac{y_i - y_j}{\sqrt{2}\rho}\right\rangle + \left\langle x_h^g - \mathcal{G}(h;x), \frac{y_j - y_i}{\sqrt{2}\rho}\right\rangle \\ &= \langle x_k^g - x_h^g, y_i - y_j\rangle / \sqrt{2}\rho.\end{aligned} \tag{3.2}$$

Since $|y_k^g - y_i|$, $|y_h^g - y_j| \le R$, we have

$$\begin{aligned}&\langle x_k^g - x_h^g, y_i - y_j\rangle \\ &\ge |x_k^g - x_h^g|^2 - |x_k^g - x_h^g| \cdot (|y_k^g - x_k^g| + |y_h^g - x_h^g|) - 2R|x_k^g - x_h^g| \\ &\ge |x_k^g - x_h^g|^2 - (\sqrt{2}\widetilde{\delta} + 2R)|x_k^g - x_h^g| \\ &= \left\{|x_k^g - x_h^g| - (\sqrt{2}\widetilde{\delta} + 2R)/2\right\}^2 - \left\{(\sqrt{2}\widetilde{\delta} + 2R)/2\right\}^2.\end{aligned}$$

Thus, using $|x_k^g - x_h^g| \ge 2\rho - 2R$, we easily get

$$\langle x_k^g - x_h^g; y_i - y_j\rangle \ge 2\rho^2. \tag{3.3}$$

On the other hand, we have

$$|v_x| \le \left(\sum_{i=1}^N \left|x_{m(i)}^g - \mathcal{G}(k;x)\right|^2\right)^{1/2} \le (\rho + 2R)\sqrt{N}.$$

Hence, combining this with (3.2) and (3.3), we have

$$\langle \widetilde{e}_x, n_{ij}(y)\rangle \ge \frac{\sqrt{2}\rho}{\sqrt{N}(\rho + 2R)} > \frac{1}{\sqrt{N}}. \quad ∎$$

We use the following notation; for a continuous function u defined on $[0,\infty)$, we set

$$\begin{aligned}&\Delta_{s,t}(u) = \sup\{|u(t_1) - u(t_2)| : s \le t_1 < t_2 \le t\}, \\ &\Delta_{s,t,h}(u) = \sup\{|u(t_1) - u(t_2)| : s \le t_1 < t_2 \le t,\ |t_1 - t_2| \le h\},\ h > 0, \\ &\|u\|_t = \sup\{|u(s)| : 0 \le s \le t\}, \\ &| u |_t = \text{the total variation of } u \text{ on } [0,t], \\ &| u |_t^s = | u |_t - | u |_s,\ 0 \le s \le t.\end{aligned}$$

Remark 3.3. For any $u \in C([0,\infty) \to \boldsymbol{R}^{Md})$, we have
(1) $|\overline{u}(t) - \overline{u}(s)|/\sqrt{N} \le |u(t) - u(s)| \le |\overline{u}(t) - \overline{u}(s)|$, $s, t \ge 0$.
(2) $\Delta_{s,t}(\overline{u})/\sqrt{N} \le \Delta_{s,t}(u) \le \Delta_{s,t}(\overline{u})$, $0 \le s < t$.

Remark 3.4. By the definition of $\overline{\Phi^R}$, we see $\left|\overline{\Phi^R}\right|_t^s \le \left|\varphi^R\right|_t^s$, $0 \le s \le t$.

Lemma 3.2. *Suppose that ξ^R and $\xi^{R'}$ solve the SE's*

$$\xi^R(t) = w(t) + \psi^R(t) + \varphi^R(t),$$
$$\xi^{R'}(t) = w(t) + \psi^{R'}(t) + \varphi^{R'}(t),$$

for $(w + \psi^R; \mathcal{D}_\infty)$ and $(w + \psi^{R'}; \mathcal{D}_\infty)$, respectively in the sence of (3.1). Then,

$$(3.4)\quad \left|\overline{\eta^R}(t) - \overline{\eta^{R'}}(t)\right|^2 \le \frac{2}{r_\infty}\int_0^t \left|\overline{\eta^R}(s) - \overline{\eta^{R'}}(s)\right|^2 \left(d\left|\varphi^R\right|_s + d\left|\varphi^{R'}\right|_s\right)$$
$$+ 2(R+R')\left\{\sqrt{N} + \frac{1}{r_\infty}(R+R')N\right\}\left(\left|\varphi^R\right|_t + \left|\varphi^{R'}\right|_t\right),\quad t \ge 0,$$

$$(3.5)\quad \left|\overline{\eta^R}(t) - \overline{\eta^R}(s)\right|^2 \le \Delta^2_{s,t}(\overline{w}) + 2\left\{\Delta_{s,t}(\overline{w^g}) + 2R\left(\frac{2}{r_\infty}RN + \sqrt{N}\right)\right\}\left|\varphi^R\right|_t^s$$
$$+ \frac{2}{r_\infty}\int_s^t \left|\overline{\eta^R}(s) - \overline{\eta^R}(u)\right|^2 d\left|\varphi^R\right|_u,\quad s, t \ge 0.$$

Proof. For (3.4), we have

$$(3.6)\quad \int_0^t \left\langle \overline{\eta^R}(s) - \overline{\eta^{R'}}(s), d\overline{\Phi^R}(s)\right\rangle$$
$$= \int_0^t \sum_{k=1}^M \frac{1}{n_k}\sum_{j\in\Lambda_k}\left\langle \sum_{i\in\Lambda_k}(\xi_i^R(s) - \xi_i^{R'}(s)), d\varphi_j^R(s)\right\rangle$$
$$= \int_0^t \sum_{k=1}^M \frac{1}{n_k}\sum_{j\in\Lambda_k}\left\{\left\langle n_k(\xi_j^R(s) - \xi_j^{R'}(s)), d\varphi_j^R(s)\right\rangle\right.$$
$$\left. + \left\langle \sum_{i\in\Lambda_k}(\xi_i^R(s) - \xi_i^{R'}(s)) - n_k(\xi_j^R(s) - \xi_j^{R'}(s)), d\varphi_j^R(s)\right\rangle\right\}$$
$$= \int_0^t \left\langle \xi^R(s) - \xi^{R'}(s), d\varphi^R(s)\right\rangle$$
$$+ \int_0^t \sum_{k=1}^M \sum_{j\in\Lambda_k}\left\langle \frac{1}{n_k}\sum_{i\in\Lambda_k}(\xi_i^R(s) - \xi_i^{R'}(s)) - (\xi_j^R(s) - \xi_j^{R'}(s)), m_j^R(s)\right\rangle d\widetilde{\ell^R}(s).$$

Similarly,

$$\int_0^t \left\langle \overline{\eta^R}(s) - \overline{\eta^{R'}}(s), d\overline{\Phi^{R'}}(s)\right\rangle$$

$$= \int_0^t \left\langle \xi^R(s) - \xi^{R'}(s), d\varphi^{R'}(s) \right\rangle$$

$$+ \int_0^t \sum_{k=1}^{M} \sum_{j\in\Lambda_k} \left\langle \frac{1}{n_k} \sum_{i\in\Lambda_k} (\xi_i^R(s) - \xi_i^{R'}(s)) - (\xi_j^R(s) - \xi_j^{R'}(s)), m_j^{R'}(s) \right\rangle d\widetilde{\ell^{R'}}(s).$$

Thus,

$$\text{(3.7)} \qquad \left| \overline{\eta^R}(t) - \overline{\eta^{R'}}(t) \right|^2 = 2 \int_0^t \left\langle \overline{\eta^R}(s) - \overline{\eta^{R'}}(s), d\overline{\Phi^R}(s) - \overline{\Phi^{R'}}(s) \right\rangle$$

$$\le 2 \int_0^t \left\langle \xi^R(s) - \xi^{R'}(s), d\varphi^R(s) - d\varphi^{R'}(s) \right\rangle$$

$$+ (R+R') \left\{ \int_0^t \sum_{j=1}^{N} \left| m_j^R(s) \right| d\widetilde{\ell^R}(s) + \int_0^t \sum_{j=1}^{N} \left| m_j^{R'}(s) \right| d\widetilde{\ell^{R'}}(s) \right\}$$

$$\le 2 \int_0^t \left\langle \xi^R(s) - \xi^{R'}(s), d\varphi^R(s) - d\varphi^{R'}(s) \right\rangle$$

$$+ (R+R')\sqrt{N} \left(\left| \varphi^R \right|_t + \left| \varphi^{R'} \right|_t \right).$$

By the assumption, Remark 1.1 and Condition (A) for $\mathcal{D}_\infty$, we have

$$\text{(3.8)} \qquad \int_0^t \left\langle \xi^R(s) - \xi^{R'}(s), d\varphi^R(s) - d\varphi^{R'}(s) \right\rangle$$

$$\le \frac{1}{2r_\infty} \int_0^t \left| \xi^R(s) - \xi^{R'}(s) \right|^2 \left(d\left| \varphi^R \right|_s + d\left| \phi^{R'} \right|_s \right).$$

On the other hand, we get

$$\text{(3.9)} \qquad \left| \xi^R(s) - \xi^{R'}(s) \right|^2$$

$$\le \sum_{i=1}^{N} \left\{ \left| \overline{\eta_i^R}(s) - \overline{\eta_i^{R'}}(s) \right| + \left| \xi_i^R(s) - \overline{\eta_i^R}(s) \right| + \left| \xi_i^{R'}(s) - \overline{\eta_i^{R'}}(s) \right| \right\}^2$$

$$\le 2 \left| \overline{\eta^R}(s) - \overline{\eta^{R'}}(s) \right|^2 + 2(R+R')^2 N.$$

Hence, combining this with (3.7) and (3.8), we have

$$\left| \overline{\eta^R}(t) - \overline{\eta^{R'}}(t) \right|^2 \le \frac{1}{r_\infty} \int_0^t \left| \xi^R(s) - \xi^{R'}(s) \right|^2 \left(d\left| \varphi^R \right|_s + d\left| \varphi^{R'} \right|_s \right)$$

$$+ (R+R')\sqrt{N} \left(\left| \varphi^R \right|_t + \left| \varphi^{R'} \right|_t \right)$$

$$\le \frac{2}{r_\infty} \int_0^t \left| \overline{\eta^R}(s) - \overline{\eta^{R'}}(s) \right|^2 \left(d\left| \varphi^R \right|_s + d\left| \varphi^{R'} \right|_s \right)$$

$$+2\left\{(R+R')\sqrt{N}+\frac{1}{r_\infty}(R+R')^2N\right\}\left(\Big|\,\varphi^R\,\Big|_t+\Big|\,\varphi^{R'}\,\Big|_t\right),$$

which proves (3.4).

Next we prove (3.5). By the similar calculation to (3.6), we have

$$\begin{aligned}&\int_s^t\left\langle\overline{\eta^R}(u)-\overline{\eta^R}(s),d\overline{\Phi^R}(u)\right\rangle\\&=\int_s^t\langle\xi^R(u)-\xi^R(s),d\varphi^R(u)\rangle\\&\quad+\int_s^t\sum_{k=1}^M\sum_{j\in\Lambda_k}\left\langle\frac{1}{n_k}\sum_{i\in\Lambda_k}\left(\xi_i^R(u)-\xi_i^R(s)\right)-\left(\xi_j^R(u)-\xi_j^R(s)\right),m_j(u)\right\rangle d\widetilde{\ell^R}(u)\\&\le\int_s^t\langle\xi^R(u)-\xi^R(s),d\varphi^R(u)\rangle+2R\sqrt{N}\,\Big|\,\varphi^R\,\Big|_t^s.\end{aligned}$$

By the assumption, Remark 1.1 and Condition (A) for $\mathcal{D}_\infty$,

$$\int_s^t\langle\xi^R(s)-\xi^R(u),d\varphi^R(u)\rangle+\frac{1}{2r_\infty}\int_s^t\left|\xi^R(s)-\xi^R(u)\right|^2d\,\Big|\,\varphi^R\,\Big|_u\ge 0.$$

Thus, we have

$$\begin{aligned}(3.10)\qquad&\int_s^t\left\langle\overline{\eta^R}(u)-\overline{\eta^R}(s),d\overline{\Phi^R}(u)\right\rangle\\&\le\frac{1}{2r_\infty}\int_s^t|\xi^R(s)-\xi^R(u)|^2d\,\Big|\,\varphi^R\,\Big|_u+2R\sqrt{N}\,\Big|\,\varphi^R\,\Big|_t^s.\end{aligned}$$

By the similar calculation to (3.9), we have

$$(3.11)\qquad |\xi^R(s)-\xi^R(u)|^2\le 2\left|\overline{\eta^R}(s)-\overline{\eta^R}(u)\right|^2+8NR^2.$$

Hence, (3.10) and (3.11) yield

$$\begin{aligned}(3.12)\qquad&\int_s^t\left\langle\overline{\eta^R}(u)-\overline{\eta^R}(s),d\overline{\Phi^R}(u)\right\rangle\\&\le\frac{1}{r_\infty}\int_s^t\left|\overline{\eta^R}(s)-\overline{\eta^R}(u)\right|^2d\,\Big|\,\varphi^R\,\Big|_u+2R\left(\frac{2}{r_\infty}RN+\sqrt{N}\right)\Big|\,\varphi^R\,\Big|_t^s.\end{aligned}$$

Thus, by (3.12) and Remark 3.4, we have

$$\left|\overline{\eta^R}(t)-\overline{\eta^R}(s)\right|^2=|\overline{w^g}(t)-\overline{w^g}(s)|^2+2\int_s^t\left\langle\overline{\eta^R}(u)-\overline{\eta^R}(s),d\overline{\Phi^R}(u)\right\rangle$$

$$+2\int_s^t \left\langle \overline{w^g}(t)-\overline{w^g}(u), d\overline{\Phi^R}(u)\right\rangle$$

$$\leq \Delta_{s,t}^2(\overline{w^g}) + 2\left\{\Delta_{s,t}(\overline{w^g}) + 2R\left(\frac{2}{r_\infty}RN+\sqrt{N}\right)\right\} \mid \varphi^R \mid_t^s$$

$$+\frac{2}{r_\infty}\int_s^t \left|\overline{\eta^R}(s)-\overline{\eta^R}(u)\right|^2 d \mid \varphi^R \mid_u.$$

The proof is finished. ∎

Now we define

$$\begin{aligned} T_0^R &= \inf\{t \geq 0 : \xi^R(t) \in \partial\mathcal{D}_\infty\}, \\ t_n^R &= \inf\left\{t > T_{n-1} : \left|\overline{\eta^R}(t)-\overline{\eta^R}(T_{n-1})\right| \geq \widetilde{\delta}/2\right\}, \\ T_n^R &= \inf\{t \geq t_n : \xi^R(t) \in \partial\mathcal{D}_\infty\}, \quad n=1,2,3,\dots. \end{aligned}$$

Hereafter we omit the superscript 'R' if there is no possibility of confusion.

Lemma 3.3. *We have*

$$\mid \varphi \mid_t^s < N\left(\Delta_{s,t}(\overline{\eta}) + \Delta_{s,t}(\overline{w^g})\right), \quad s,t \in [T_{n-1}, T_n].$$

Proof. Let $T_{n-1} \leq s < t \leq t_n$ and set $\widetilde{e} = \widetilde{e}_{\xi(T_{n-1})}$. Then, since $\langle \widetilde{e}, \psi(t)-\psi(s)\rangle = 0$, Lemma 3.1 implies

$$\begin{aligned} \langle \widetilde{e}, \xi(t)-\xi(s)\rangle &= \langle \widetilde{e}, w(t)-w(s)\rangle + \langle \widetilde{e}, \varphi(t)-\varphi(s)\rangle \\ &> \langle \widetilde{e}, w(t)-w(s)\rangle + \mid \varphi \mid_t^s/\sqrt{N}, \end{aligned}$$

$$\mid \varphi \mid_t^s < \sqrt{N}\big\{|\langle \widetilde{e}, \xi(t)-\xi(s)\rangle| + |\langle \widetilde{e}, w(t)-w(s)\rangle|\big\}. \tag{3.13}$$

If we put $x = \xi(T_{n-1})$, $v = v_{\xi(T_{n-1})}$ and $\Delta\xi = \xi(t)-\xi(s)$, etc., we get

$$\begin{aligned} |\langle \widetilde{e}, \Delta\xi\rangle| &= \left|\sum_{k\in K_c}\sum_{i\in\Lambda_k}\langle x_k^g - \mathcal{G}(k;x), \Delta\xi_i\rangle/|v|\right| \\ &= \left|\sum_{k\in K_c} n_k \langle x_k^g - \mathcal{G}(k;x), \Delta\eta_k\rangle/|v|\right| \\ &\leq \left(\sum_{k\in K_c}\frac{n_k^2|x_k^g-\mathcal{G}(k;x)|^2}{|v|^2}\right)^{\frac12}\left(\sum_{k\in K_c}|\Delta\eta_k|^2\right)^{\frac12} \\ &\leq \frac{\sqrt{N}}{|v|}\left(\sum_{k\in K_c} n_k|x_k^g-\mathcal{G}(k;x)|^2\right)^{\frac12}|\Delta\eta| \end{aligned}$$

$$= \sqrt{N}|\Delta\eta| \le \sqrt{N}\Delta_{s,t}(\eta).$$

Similarly, $|\langle\widetilde{e}, \Delta w\rangle| \le \sqrt{N}\Delta_{s,t}(w^g)$.Thus, by (3.13), we have

$$\mid \varphi \mid_t^s < N\left(\Delta_{s,t}(\eta) + \Delta_{s,t}(w^g)\right), \quad s,t \in [T_{n-1}, t_n].$$

Since $\varphi(t) = constant$, $t \in (t_n, T_n]$, by Remark 3.2 we consequently have

$$\begin{aligned} \mid \varphi \mid_t^s &< N\left(\Delta_{s,t}(\eta) + \Delta_{s,t}(w^g)\right) \\ &\le N\left(\Delta_{s,t}(\overline{\eta}) + \Delta_{s,t}(\overline{w^g})\right), \quad s,t \in [T_{n-1}, T_n]. \end{aligned}$$ ∎

Lemma 3.4. *Let T be any finite fixed time. Then for any $\varepsilon > 0$ we have*

$$\Delta_{s,t}(\overline{\eta}) \le \left[(1+\varepsilon^{-1})\left\{\Delta_{s,t}(\overline{w^g}) + 2NR\left(1+\frac{2R}{r_\infty}\right)\right\} + \varepsilon \mid \varphi \mid_t^s\right] \times\exp\left(\mid \varphi \mid_t^s / r_\infty\right), \quad s,t \in [0,T]. \tag{3.14}$$

Proof. By (3.5) we have

$$|\overline{\eta}(t) - \overline{\eta}(s)|^2 \le \Delta_{s,t}^2(\overline{w^g}) + 2\left\{\Delta_{s,t}(\overline{w^g}) + 2NR\left(1+\frac{2R}{r_\infty}\right)\right\} \mid \varphi \mid_t^s + \frac{2}{r_\infty}\int_s^t |\overline{\eta}(s) - \overline{\eta}(u)|^2 d \mid \varphi \mid_u .$$

Using Gronwall's lemma, we have

$$\begin{aligned} |\overline{\eta}(t) - \overline{\eta}(s)|^2 &\le \left[\Delta_{s,t}^2(\overline{w^g}) + 2\left\{\Delta_{s,t}(\overline{w^g}) + 2NR\left(1+\frac{2R}{r_\infty}\right)\right\} \mid \varphi \mid_t^s\right] \\ &\qquad \times\exp\left(2 \mid \varphi \mid_t^s / r_\infty\right) \\ &< \left[\left\{\Delta_{s,t}(\overline{w^g}) + 2NR\left(1+\frac{2R}{r_\infty}\right)\right\}^2 + 2\left\{\Delta_{s,t}(\overline{w^g}) + 2NR\left(1+\frac{2R}{r_\infty}\right)\right\} \mid \varphi \mid_t^s\right] \\ &\qquad \times\exp\left(2 \mid \varphi \mid_t^s / r_\infty\right) \\ &\le \left[(1+\varepsilon^{-2})\left\{\Delta_{s,t}(\overline{w^g}) + 2NR\left(1+\frac{2R}{r_\infty}\right)\right\}^2 + \left(\varepsilon \mid \varphi \mid_t^s\right)^2\right] \\ &\qquad \times\exp\left(2 \mid \varphi \mid_t^s / r_\infty\right), \end{aligned}$$

which yields (3.14). ∎

Lemma 3.5. *For any finite $T > 0$, there exist positive constants K_1', K_2' such that*

$$\Delta_{s,t}(\overline{\eta}) \le K_1'\Delta_{s,t}(\overline{w^g}) + K_2'R\left(1+\frac{R}{r_\infty}\right), \quad s,t \in [T_{n-1}, T_n], \tag{3.15}$$

provided that $T_n \le T$. *Here* K_1', K_2' *depend only on* $N, \rho, T, \|\overline{w^g}\|_T$ *and are independent of* $R, \|w\|_T$.

Proof. By Lemmas 3.3 and 3.4, we have

$$\triangle_{s,t}(\overline{\eta}) \le \left[(1+\varepsilon^{-1})\left\{\triangle_{s,t}(\overline{w^g}) + 2NR\left(1+\frac{2R}{r_\infty}\right)\right\} + N\varepsilon\left(\triangle_{s,t}(\overline{\eta}) + \triangle_{s,t}(\overline{w^g})\right)\right]$$
$$\times\exp\left\{N\left(\triangle_{s,t}(\overline{\eta}) + \triangle_{s,t}(\overline{w^g})\right)/r_\infty\right\}, \quad s,t \in [T_{n-1}, T_n].$$

Since $\triangle_{s,t}(\overline{\eta}) \le \widetilde{\delta}$ for $s,t \in [T_{n-1}, t_n]$, we have

$$\triangle_{s,t}(\overline{\eta}) \le \left\{(1+\varepsilon^{-1}+N\varepsilon)\triangle_{s,t}(\overline{w^g}) + 2NR\left(1+\frac{2R}{r_\infty}\right)(1+\varepsilon^{-1})\right.$$
$$\left.+N\varepsilon\triangle_{s,t}(\overline{\eta})\right\}\exp\left\{2N\left(\widetilde{\delta}+\|\overline{w^g}\|_T\right)/r_\infty\right\}, \quad s,t \in [T_{n-1}, T_n].$$

Thus, for $0 < \varepsilon < \exp\{-2N(\widetilde{\delta}+\|\overline{w^g}\|_T)/r_\infty\}/N$,

$$\triangle_{s,t}(\overline{\eta}) \le K_1^\varepsilon \triangle_{s,t}(\overline{w^g}) + K_2^\varepsilon R\left(1+\frac{2R}{r_\infty}\right), \quad s,t \in [T_{n-1}, t_n], \tag{3.16}$$

where

$$K_1^\varepsilon = \frac{(1+\varepsilon^{-1}+N\varepsilon)\exp\{2N(\widetilde{\delta}+\|\overline{w^g}\|_T)/r_\infty\}}{1-N\varepsilon\exp\{2N(\widetilde{\delta}+\|\overline{w^g}\|_T)/r_\infty\}},$$
$$K_2^\varepsilon = \frac{2N(1+\varepsilon^{-1})}{1+\varepsilon^{-1}+N\varepsilon}K_1^\varepsilon.$$

On the other hand, if $t_n < T_n$, we have

$$|\overline{\eta}(t) - \overline{\eta}(s)| = |\overline{w^g}(t) - \overline{w^g}(s)|, \quad s,t \in [t_n, T_n].$$

Thus, $\triangle_{s,t}(\overline{\eta}) = \triangle_{s,t}(\overline{w^g})$ for $s,t \in [t_n, T_n]$. Combining this with (3.16), we get

$$\triangle_{s,t}(\overline{\eta}) \le (K_1^\varepsilon + 1)\triangle_{s,t}(\overline{w^g}) + K_2^\varepsilon R\left(1+\frac{2R}{r_\infty}\right), \quad s,t \in [T_{n-1}, T_n].$$

Therefore we have (3.15) with

$$K_1' = \inf\left\{K_1^\varepsilon + 1 : 0 < \varepsilon < \exp\{-2N(\widetilde{\delta}+\|\overline{w^g}\|_T)/r_\infty\}/N\right\},$$
$$K_2' = \inf\left\{K_2^\varepsilon : 0 < \varepsilon < \exp\{-2N(\widetilde{\delta}+\|\overline{w^g}\|_T)/r_\infty\}/N\right\}. \qquad \blacksquare$$

Proposition 3.1. *Let $T > 0$ be any finite time. Then for sufficiently small $R > 0$, there exist positive constants K_1, K_2 such that*

$$(3.17)\qquad |\varphi|_t^s \leq K_1 \Delta_{s,t}(\overline{w^g}) + K_2 R\left(1 + \frac{R}{r_\infty}\right), \quad 0 \leq s < t \leq T,$$

where K_1, K_2 depend only on $N, \rho, T, \|\overline{w^g}\|_T$ and the modulus of uniform continuity of $\overline{w^g}$.

Proof. By Lemma 3.5 we have

$$\tilde{\delta}/2 = \left|\overline{\eta}(t_n) - \overline{\eta}(T_{n-1})\right| \leq K_1' \Delta_{T_{n-1},t_n}(\overline{w^g}) + K_2' R\left(1 + \frac{R}{r_\infty}\right),$$

$$(3.18)\qquad \Delta \leq \Delta_{T_{n-1},t_n}(\overline{w^g}) + \frac{K_2'}{K_1'} R\left(1 + \frac{R}{r_\infty}\right),$$

where $\Delta \equiv \tilde{\delta}/2K_1'$. On the other hand, if we take R sufficiently small so that

$$\frac{K_2'}{K_1'} R\left(1 + \frac{R}{r_\infty}\right) < \Delta/2,$$

by the continuity of $\overline{w^g}$ in t, there exists a positive constant h such that $\Delta_{0,T,h}(\overline{w^g}) < \Delta/2$, where h depends only on the modulus of uniform continuity of $\overline{w^g}$, T and Δ. Thus, if $T_n \leq T$ we have $T_n - T_{n-1} \geq h$. Indeed, if we suppose $T_n - T_{n-1} < h$, we have

$$\Delta_{T_{n-1},t_n}(\overline{w^g}) \leq \Delta_{0,T,h}(\overline{w^g}) < \Delta/2,$$

which contradicts (3.18). So, we have $T_n > T$ for $n > T/h$. On the other hand, by Lemmas 3.3 and 3.5, we have

$$|\varphi|_t^s \leq N(K_1' + 1)\Delta_{s,t}(\overline{w^g}) + NK_2' R\left(1 + \frac{R}{r_\infty}\right), \quad s, t \in [T_{n-1}, T_n].$$

Therefore, we consequently have

$$|\varphi|_t^s \leq K_1 \Delta_{s,t}(\overline{w^g}) + K_2 R\left(1 + \frac{R}{r_\infty}\right), \quad 0 \leq s < t \leq T,$$

where

$$K_1 = N(\frac{T}{h} + 1)(K_1' + 1), \quad K_2 = N(\frac{T}{h} + 1)K_2'.$$

The proof of Proposition 3.1 is finished. ∎

Remark 3.5. For sufficiently small $R > 0$, Proposition 3.1 and Remark 3.4 imply that $|\varphi^R|_t$, $|\overline{\Phi^R}|_t$ are uniformly bounded in R for any finite $t > 0$.

Remark 3.6. We easily see that the constants K_1, K_2 in (3.17) are continuous in $\|\overline{w^g}\|_T$ and the modulus of uniform continuity of $\overline{w^g}$.

Theorem 3.1. *Let $T > 0$ be any finite time. Then $\overline{\eta^R}$ converges uniformly in $t \in [0,T]$ as R tends to* 0.

Proof. Let $0 < R_1 < R_2 < R$ and suppose that R is sufficiently small. We denote a upper bound of $\mid \varphi^R \mid_T$ by C. Then, Lemma 3.2 implies

$$\left|\overline{\eta^{R_1}}(t) - \overline{\eta^{R_2}}(t)\right|^2 \leq 8CR(\sqrt{N} + 2RN/r_\infty) + \frac{2}{r_\infty}\int_0^t \left|\overline{\eta^{R_1}}(s) - \overline{\eta^{R_2}}(s)\right|^2 \left(d \mid \varphi^{R_1} \mid_s + d \mid \varphi^{R_2} \mid_s\right).$$

By Gronwall's lemma,

$$\left|\overline{\eta^{R_1}}(t) - \overline{\eta^{R_2}}(t)\right|^2 \leq 8CR(\sqrt{N} + 2RN/r_\infty)e^{4Ct/r_\infty}.$$

Thus,

$$\left\|\overline{\eta^{R_1}} - \overline{\eta^{R_2}}\right\|_t^2 \leq 8CR(\sqrt{N} + 2RN/r_\infty)e^{4CT/r_\infty} \longrightarrow 0 \quad \text{as} \quad R \downarrow 0. \tag{3.19}$$

The proof is finished. ∎

The following theorem is immediate from Theorem 3.1 and the fact $\left|\xi_i^R(t) - \overline{\eta_i^R}(t)\right| \leq R,\ 1 \leq i \leq N$.

Theorem 3.2. *For any finite $T \geq 0$, ξ^R converges uniformly in $t \in [0,T]$ as R tends to* 0.

Hereafter we denote the limit functions of $\overline{\eta^R}$ and $\overline{\Phi^R}$ as $R \downarrow 0$ by ξ^0 and $\overline{\Phi^0}$, respectively.

§4. Characterization of the limiting function

In this section we prove that the limiting function ξ^0 solves the SP $(\overline{w^g}; \mathcal{D}_\infty)$. To show this, we prepare the following lemma.

Lemma 4.1. *Let $T > 0$ be a finite fixed time. Then for any $\zeta \in C([0,\infty) \to \overline{\mathcal{D}_\infty})$ we have*

$$\int_s^t \left\langle \zeta(u) - \overline{\eta^R}(u), d\overline{\Phi^R}(u)\right\rangle + \frac{1}{r_\infty}\int_s^t \left|\zeta(u) - \overline{\eta^R}(u)\right|^2 d \mid \varphi^R \mid_u + \left\{((\zeta^*(T) + R)\sqrt{N} + NR^2/r_\infty\right\} \mid \varphi^R \mid_t^s \geq 0,\ 0 \leq s \leq t \leq T,$$

where $\zeta^*(t) \equiv \sup_{1\le k\le M} \sup_{i\in\Lambda_k} \|\zeta_i - \zeta_k^g\|_t$.

Proof. By Remarks 1.1, 2.3 and 3.1, we have

$$
\begin{aligned}
0 &\le \langle \zeta(u) - \xi^R(u), m(u)\rangle + \frac{1}{2r_\infty}\left|\zeta(u)-\xi^R(u)\right|^2 \\
&\le \sum_{k=1}^{M}\sum_{i\in\Lambda_k}\left\{\left\langle \zeta_i(u) - \overline{\eta_i^R}(u), m_i(u)\right\rangle + \left\langle \overline{\eta_i^R}(u) - \xi_i^R(u), m_i(u)\right\rangle\right\} \\
&\qquad + \frac{1}{2r_\infty}\sum_{i=1}^{N}\left\{\left|\zeta_i(u) - \overline{\eta_i^R}(u)\right| + \left|\overline{\eta_i^R}(u) - \xi_i^R(u)\right|\right\}^2 \\
&\le \sum_{k=1}^{M}\sum_{i\in\Lambda_k}\left\{\left\langle \zeta_k^g(u) - \overline{\eta_i^R}(u), m_i(u)\right\rangle + \langle \zeta_i(u) - \zeta_k^g(u), m_i(u)\rangle\right\} \\
&\qquad + \sum_{i=1}^{N} R|m_i(u)| + \frac{1}{r_\infty}\left|\zeta(u) - \overline{\eta^R}(u)\right|^2 + NR^2/r_\infty \\
&\le \sum_{k=1}^{M} n_k\left\langle \zeta_k^g(u) - \eta_k^R(u), \frac{\sum_{i\in\Lambda_k} m_i(u)}{n_k}\right\rangle + \zeta^*(u)\sum_{i=1}^{N}|m_i(u)| \\
&\qquad + R\sqrt{N} + \frac{1}{r_\infty}\left|\zeta(u) - \overline{\eta^R}(u)\right|^2 + NR^2/r_\infty \\
&\le \sum_{k=1}^{M}\sum_{j\in\Lambda_k}\left\langle \zeta_j(u) - \eta_k^R(u), \frac{\sum_{i\in\Lambda_k} m_i(u)}{n_k}\right\rangle + \frac{1}{r_\infty}\left|\zeta(u) - \overline{\eta^R}(u)\right|^2 \\
&\qquad + (\zeta^*(u) + R)\sqrt{N} + NR^2/r_\infty,\ d\,|\,\varphi^R\,|_u\text{-a.e.}
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\int_s^t \sum_{k=1}^{M}\sum_{j\in\Lambda_k}\left\langle \zeta_j(u) - \eta_k^R(u), \frac{\sum_{i\in\Lambda_k} m_i(u)}{n_k}\right\rangle d\widetilde{\ell^R}(u) \\
&+\frac{1}{r_\infty}\int_s^t \left|\zeta(u) - \overline{\eta^R}(u)\right|^2 d\,|\,\varphi^R\,|_u + \left\{(\zeta^*(T) + R)\sqrt{N} + NR^2/r_\infty\right\}|\,\varphi^R\,|_t^s \ge 0.
\end{aligned}
$$

Since

$$
\sum_{k=1}^{M}\sum_{j\in\Lambda_k}\left\langle \zeta_j(u) - \eta_k^R(u), \frac{\sum_{i\in\Lambda_k} m_i(u)}{n_k}\right\rangle d\widetilde{\ell^R}(u) = \left\langle \zeta(u) - \overline{\eta^R}(u), d\overline{\Phi^R}(u)\right\rangle,
$$

we have

$$
\int_s^t \left\langle \zeta(u) - \overline{\eta^R}(u), d\overline{\Phi^R}(u)\right\rangle + \frac{1}{r_\infty}\int_s^t \left|\zeta(u) - \overline{\eta^R}(u)\right|^2 d\,|\,\varphi^R\,|_u
$$

$$+\left\{(\zeta^*(T)+R)\sqrt{N}+NR^2/r_\infty\right\} \mid \varphi^R \mid_t^s \geq 0.$$

The proof is finished. ∎

Theorem 4.1. *ξ^0 solves the SP $(\overline{w^g};\mathcal{D}_\infty)$, that is, $\xi^0(t)=\overline{w^g}(t)+\overline{\Phi^0}(t)$ is a Skorohod equation.*

Proof. By Theorem 3.1, we have $\overline{\eta^R}\to\overline{\eta^0}$, $\overline{\Phi^R}\to\overline{\Phi^0}$ uniformly in $t\in[0,T]$ as R tends to 0 for each finite $T>0$. Thus, all we have to show is the following (1) and (2).

$$\text{(1)}\qquad d\overline{\Phi^0}(u)=\widetilde{n}(u)d\left|\overline{\Phi^0}\right|_u, \quad \widetilde{n}(u)\in\mathcal{N}_{\xi^0(u)}(\mathcal{D}_\infty) \text{ if } \xi^0(u)\in\partial\mathcal{D}_\infty.$$

$$\text{(2)}\qquad \int_0^t \widetilde{n}(u)d\left|\overline{\Phi^0}\right|_u=\overline{\Phi^0}(t).$$

For the proof we adopt the similar procedure to that of [1: Theorem 4.1]. Let ζ be any function in $C([0,\infty)\to\overline{\mathcal{D}_\infty})$ and put

$$I_1=\int_s^t\left\langle\zeta(u)-\overline{\eta^R}(u),d\overline{\Phi^R}(u)\right\rangle,$$

$$I_2=\int_s^t|\zeta(u)-\overline{\eta^R}(u)|^2 d\mid\varphi^R\mid_u,\ 0\leq s\leq t\leq T.$$

By Remark 3.5, there exists a constant $C>0$ which is independent of R with

$$\mid\Phi^R\mid_T\leq\left|\overline{\Phi^R}\right|_T\leq\mid\varphi^R\mid_T<C,$$

$$\mid\Phi^0\mid_T\leq\left|\overline{\Phi^0}\right|_T\leq\underline{\lim}_{R\downarrow0}\left|\overline{\Phi^R}\right|_T<C.$$

If we put $\widetilde{\zeta}=\zeta-\xi^0$, we have

$$I_1=\int_s^t\left\langle\widetilde{\zeta}(u),d\overline{\Phi^R}(u)\right\rangle+\int_s^t\left\langle\xi^0(u)-\overline{\eta^R}(u),d\overline{\Phi^R}(u)\right\rangle.$$

It is easy to see that

$$\left\|\int_s^\cdot\left\langle\xi^0(u)-\overline{\eta^R}(u),d\overline{\Phi^R}(u)\right\rangle\right\|_T\leq\left\|\xi^0-\overline{\eta^R}\right\|_T\cdot C\to0 \quad\text{as}\quad R\downarrow0.$$

Let $s=t_0<t_1<\cdots<t_n=t$ be an equi partition of $[s,t]$ and define $\widetilde{\zeta}^n$ by $\widetilde{\zeta}^n(u)=\widetilde{\zeta}(t_k)$ for $t_k<u\leq t_{k+1}$, $k=0,1,2,\ldots,n-1$. For any fixed $\varepsilon>0$ we take n so that $\left\|\widetilde{\zeta}^n-\widetilde{\zeta}\right\|_t\leq\varepsilon$ holds. Then we have

$$\left|\int_s^t\left\langle\widetilde{\zeta}(u)-\widetilde{\zeta}^n(u),d\overline{\Phi^R}(u)\right\rangle\right|\leq\varepsilon C,$$

$$\left|\int_s^t \left\langle \widetilde{\zeta}(u) - \widetilde{\zeta}^n(u), d\overline{\Phi^0}(u) \right\rangle\right| \le \varepsilon C.$$

Thus,

$$\left|\int_s^t \left\langle \widetilde{\zeta}(u), d\overline{\Phi^R}(u) \right\rangle - \int_s^t \left\langle \widetilde{\zeta}(u), d\overline{\Phi^0}(u) \right\rangle\right| \le 2\varepsilon C + o(1), \quad R \downarrow 0.$$

Therefore we have

$$I_1 \longrightarrow \int_s^t \left\langle \zeta(u) - \xi^0(u), d\overline{\Phi^0}(u) \right\rangle \quad \text{uniformly in } t \in [s,T] \text{ as } R \downarrow 0.$$

Next, let da_u be any weak limit on $[0,T]$ of $\left| \varphi^R \right|_u$ as $R \downarrow 0$ via some subsequence $(R_k) : R_1 > R_2 > \cdots \to 0$. Then,

$$\begin{aligned}
&\left|\int_0^t |\zeta(u) - \overline{\eta^R}(u)|^2 d\left| \varphi^R \right|_u - \int_0^t |\zeta(u) - \xi^0(u)|^2 da_u\right| \\
\le\ &\left|\int_0^t |\zeta(u) - \overline{\eta^R}(u)|^2 d\left| \varphi^R \right|_u - \int_0^t |\zeta(u) - \xi^0(u)|^2 d\left| \varphi^R \right|_u\right| \\
&+\left|\int_0^t |\zeta(u) - \xi^0(u)|^2 d\left| \varphi^R \right|_u - \int_0^t |\zeta(u) - \xi^0(u)|^2 da_u\right|.
\end{aligned}$$

It is clear that the second term of the right-hand side of the above inequality tends to 0 as $R \downarrow 0$ via (R_k) by the definition of da_u. Setting $\widetilde{\zeta^R} = \zeta - \overline{\eta^R}$, we have

$$\begin{aligned}
&\left|\int_0^t |\zeta(u) - \overline{\eta^R}(u)|^2 d\left| \varphi^R \right|_u - \int_0^t |\zeta(u) - \xi^0(u)|^2 d\left| \varphi^R \right|_u\right| \\
\le\ &\left\| \left|\widetilde{\zeta^R}(\cdot)\right|^2 - |\widetilde{\zeta}(\cdot)|^2 \right\|_T \cdot C \longrightarrow 0 \quad \text{as} \quad R \downarrow 0.
\end{aligned}$$

Hence, we have $I_2 \to \int_s^t |\zeta(u) - \xi^0(u)|^2 da_u$ as $R \downarrow 0$ via (R_k). Since $d\left| \overline{\Phi^0} \right|_u \ll da_u$ (absolutely continuous), there exists a bounded measurable function $h : [0,T] \to \boldsymbol{R}^{Nd}$ such that $d\overline{\Phi^0}(u) = h(u)da_u$. By Lemma 4.1, for any $\zeta' \in C([0,\infty) \to \overline{\mathcal{D}_\infty})$ with $\zeta_i' = \zeta_j', m(i) = m(j)$, we have

$$\int_s^t \langle \zeta'(u) - \xi^0(u), h(s)da_u \rangle + \frac{1}{r_\infty} \int_s^t \left|\zeta'(u) - \xi^0(u)\right|^2 da_u \ge 0, \quad 0 \le s \le t \le T,$$

$$\text{(4.1)} \qquad \left\langle \zeta'(u) - \xi^0(u), h(u) \right\rangle + \frac{1}{r_\infty} \left|\zeta'(u) - \xi^0(u)\right| \ge 0, \quad da_u\text{–a.e.}$$

On the other hand, for a function $\chi \in C(\boldsymbol{R}^{Nd} \to [0,1])$ with

$$\chi = \begin{cases} 1 & \text{on a compact set included in } \mathcal{D}_\infty, \\ 0 & \text{on } \boldsymbol{R}^{Nd} \setminus \mathcal{D}_\infty, \end{cases}$$

we have

$$\begin{aligned} &\left| \int_0^t \chi(\xi^0(u)) d \left| \varphi^R \right|_u - \int_0^t \chi(\overline{\eta^R}(u)) d \left| \varphi^R \right|_u \right| \\ \leq\ & \left\| \chi(\xi^0(\cdot)) - \chi(\overline{\eta^R}(\cdot)) \right\|_T \cdot C \longrightarrow 0, \ R \downarrow 0. \end{aligned}$$

Moreover, since $\left|\xi_i^R(t) - \xi_j^R(t)\right| = \rho$, $m(i) \neq m(j)$ implies

$$\rho - 2R \leq \left| \overline{\eta_i^R}(t) - \overline{\eta_j^R}(t) \right| = \left| \eta_{m(i)}^R(t) - \eta_{m(j)}^R(t) \right| \leq \rho + 2R,$$

we easily have

$$\begin{aligned} 0 &= \lim_{R \downarrow 0} \int_0^t \chi(\overline{\eta^R}(u)) d \left| \varphi^R \right|_u \\ &= \lim \int_0^t \chi(\xi^0(u)) d \left| \varphi^R \right|_u \\ &= \int_0^t \chi(\xi^0(u)) da_u, \end{aligned}$$

where 'lim' means the limit in $R \downarrow 0$ via (R_k). Letting χ increase to $1_{\mathcal{D}_\infty}$, we have $\xi^0(u) \in \partial\mathcal{D}_\infty$, da_u–a.e. Thus, by (4.1), Remarks 1.1 and 2.3, there exist $\lambda(u) \geq 0$ and $\widetilde{\boldsymbol{n}}(u) \in \widetilde{\mathcal{N}}_{\xi^0(u)}(\mathcal{D}_\infty)$ such that $h(u) = \lambda(u)\widetilde{\boldsymbol{n}}(u)$, da_u–a.e., where

$$\widetilde{\mathcal{N}}_{\overline{x}}(\mathcal{D}_\infty) \equiv \left\{ v \in \mathcal{N}_{\overline{x}}(\mathcal{D}_\infty) : v = \overline{\boldsymbol{u}}/|\overline{\boldsymbol{u}}|,\ \boldsymbol{u} \in \mathcal{N}_x(\mathcal{O}) \right\}, \ x \in \partial\mathcal{O}.$$

Hence, if $\xi^0(u) \in \partial\mathcal{D}_\infty$, we have

$$d \left| \overline{\Phi^0} \right|_u = |h(u)| da_u = \lambda(u) da_u,$$

$$\begin{aligned} d\overline{\Phi^0}(u) &= h(u) da_u \\ &= \lambda(u)\widetilde{\boldsymbol{n}}(u) da_u \\ &= \widetilde{\boldsymbol{n}}(u) d \left| \overline{\Phi^0} \right|_u, \ \ \widetilde{\boldsymbol{n}}(u) \in \mathcal{N}_{\xi'(\sqcap)}(\mathcal{D}_\infty), \end{aligned}$$

and $\int_0^t \widetilde{\boldsymbol{n}}(u) d \left| \overline{\Phi^0} \right|_u = \overline{\Phi^0}(t)$. This completes the proof of the theorem. ∎

Remark 4.1. $\widetilde{n} \in \widetilde{\mathcal{N}}_{\overline{x}}(\mathcal{D}_\infty)$, $x \in \partial\mathcal{O}$ can be written in the form

$$\widetilde{n}_i = \sum_{\substack{j=1 \\ (\neq i)}}^{N} \widetilde{c}_{ij}(\overline{x}_i - \overline{x}_j),\ \widetilde{c}_{ij} \geq 0,\ i = 1, 2, \ldots, N,$$

where we note that $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N) \in \partial\mathcal{D}_\infty$ and $\overline{x}_i = \overline{x}_j$ if $m(i) = m(j)$.

We now define $\eta^0 = (\eta^0_1, \eta^0_2, \ldots, \eta^0_M) \in \boldsymbol{R}^{Md}$ by $\eta^0_k = \xi^0_i$ for i with $m(i) = k$, $k = 1, 2, \ldots, M$. Then, it is easy to see that $\overline{\eta^0} = \xi^0$.

By Theorem 4.1 and Remark 4.1, if we put $k = m(i)$, we can write

$$\begin{aligned} \overline{\Phi^0_i}(t) &= \int_0^t \widetilde{n}_i(s) d \left| \overline{\Phi^0} \right|_s \\ &= \int_0^t \sum_{\substack{j=1 \\ (\neq i)}}^{N} \widetilde{c}_{ij}(s) \big(\overline{\eta^0}_i(s) - \overline{\eta^0}_j(s) \big) d \left| \overline{\Phi^0} \right|_s,\quad i = 1, 2, \ldots, N. \end{aligned}$$

Setting $\widetilde{\ell^0_{ij}}(t) = \int_0^t \widetilde{c}_{ij}(s) d \left| \overline{\Phi^0} \right|_s$ and then

$$\ell^0_{kh}(t) = \widetilde{\ell^0_{ij}}(t) \quad \text{if} \quad k = m(i) \quad \text{and} \quad h = m(j),$$

we easily have the following theorem.

Theorem 4.2. *$\{\eta^0(t)\}$ satisfies the following equation:*

$$\eta^0_k(t) = w^g{}_k(t) + \sum_{\substack{h=1 \\ (\neq k)}}^{M} n_h \int_0^t \big(\eta^0_k(s) - \eta^0_h(s) \big) d\ell^0_{kh}(s),\ k = 1, 2, \ldots, M, \tag{4.2}$$

under the conditions

(1) $\eta^0 = (\eta^0_1, \eta^0_2, \ldots, \eta^0_M) \in \mathrm{C}([0, \infty) \to \boldsymbol{R}^{Md})$ *and* $\left| \eta^0_k(t) - \eta^0_h(t) \right| \geq \rho$ if $k \neq h$,

(2) ℓ^0_{kh} *is a continuous nondecreasing function with* $\ell^0_{kh} = \ell^0_{hk}$, $\ell^0_{kh}(0) = 0$, *and*

$$\ell^0_{kh}(t) = \int_0^t \mathbf{1}_{\{|\eta^0_k(s) - \eta^0_h(s)| = \rho\}}(s) d\ell^0_{kh}(s).$$

In particular, if $n_1 = n_2 = \cdots = n_M$, $\{\eta^0(t)\}$ solves the SP $(w^g; \mathcal{O})$.

Remark 4.2. The existence of the unique solution of (4.2) is easily derived from Theorem 3.1 in [3].

§5. SDE representing the motion of mutually reflecting molecules

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)$. We assume that $\mathcal{F}_t$ contains all P-null sets and $\mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$. We also assume that there exist independent d-dimensional Brownian motions $\{B_i(t)\}$, $1 \leq i \leq N$, with $B_i(0) = 0$.

For given $\sigma : \boldsymbol{R}^d \to \boldsymbol{R}^d \otimes \boldsymbol{R}^d$ and $b : \boldsymbol{R}^d \to \boldsymbol{R}^d$, we consider the SDE

$$(5.1) \qquad \begin{aligned} dX_i^R(t) &= \sigma(X_i^R(t))dB_i(t) + b(X_i^R(t))dt \\ &\quad + \sum_{\substack{j=1 \\ (\neq i)}}^{N} (X_i^R(t) - X_j^R(t))d\ell_{ij}^R(t), \quad i = 1, 2, \ldots, N, \end{aligned}$$

under the conditions

(i) $\{X_i^R\}$ is an $(\mathcal{F}_t)$-adapted $\boldsymbol{R}^d$-valued continuous process satisfying

$$\begin{aligned} |X_i^R(t) - X_j^R(t)| &\leq R \qquad \text{for } \forall j \quad \text{with} \quad m(j) = m(i), \\ &\geq \rho \qquad \text{for } \forall j \quad \text{with} \quad m(j) \neq m(i), \ t \geq 0, \end{aligned}$$

(ii) $\{\ell_{ij}^R\}$ is an $(\mathcal{F}_t)$-adapted continuous nonincreasing or nondecreasing process according as $m(i) = m(j)$ or $m(i) \neq m(j)$, with $\ell_{ij} = \ell_{ji}$, $\ell_{ij}(0) = 0$ and

$$\ell_{ij}(t) = \begin{cases} \int_0^t \mathbf{1}_{\left\{\left|X_i^R(s)-X_j^R(s)\right|=R\right\}}(s)d\ell_{ij}^R(s), & \text{if} \quad m(i) = m(j), \\ \int_0^t \mathbf{1}_{\left\{\left|X_i^R(s)-X_j^R(s)\right|=\rho\right\}}(s)d\ell_{ij}^R(s), & \text{if} \quad m(i) \neq m(j). \end{cases}$$

Here we always assume that the initial values $X_i^R(0) \equiv X_i$, $i = 1, 2, \ldots, N$, are $\boldsymbol{R}^d$-valued $\mathcal{F}_0$-measurable random variables satisfying

$$(5.2) \qquad \begin{aligned} |X_i - X_j| &\leq R \qquad \text{if} \quad m(i) = m(j), \\ &\geq \rho \qquad \text{if} \quad m(i) \neq m(j). \end{aligned}$$

The following theorem is the immediate consequence of Theorem 1.2 and [5: Theorem 5.1].

Theorem 5.1. Assume that σ and b are bounded and Lipschitz continuous functions. Then for any $\mathcal{F}_0$-measurable initial values satisfying (5.2), there exists a unique strong solution of the SDE (5.1).

Indeed, setting

$$\sigma(x)=\begin{bmatrix} \sigma(x_1) & & & 0 \\ & \sigma(x_2) & & \\ & & \ddots & \\ 0 & & & \sigma(x_N) \end{bmatrix}, \quad b(x)=\begin{bmatrix} b(x_1) \\ \vdots \\ b(x_N) \end{bmatrix}, \quad x=(x_1,\dots,x_N)\in \mathbf{R}^{Nd},$$

and

$$L^R(t) = \int_0^t n(s)d\ell^R(s),$$

$$W^R(t) = X(0) + \int_0^t \sigma(X^R(s))dB(s) + \int_0^t b(X^R(s))ds, \tag{5.3}$$

we see that the equation (5.1) is equivalent to the Skorohod SDE $X^R(t) = W^R(t) + L^R(t)$ for $(W^R; \mathcal{D}_R)$, where $\{B(t)\}$ is an Nd-dimensional $\mathcal{F}_t$-Brownian motion with $B(0) = 0$.

Finally we consider the convergence problem as $R \downarrow 0$. Let $T > 0$ be any fixed time and P^R the probability measure on $\mathrm{C}([0,T] \to \mathbf{R}^{Nd} \times \mathbf{R}^{Nd})$ introduced by $\{(B(t), W^R(t)) : 0 \leq t \leq T\}$. Then we get the following lemma. The proof is essentially the same as that of [2: Lemma 5.1] and so, is omitted.

Lemma 5.1. *The family* $\{P^R, R > 0\}$ *is tight.*

Remark 5.1. Lemma 5.1 yields that there exists $R_1 > R_2 > \cdots$ such that P^{R_n} converges weakly as $n \to \infty$. If we put $P^n = P^{R_n}$, Skorohod's realization theorem of almost sure convergence implies that we can find, on a suitable probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, a sequence of processes (B_n, W_n), $n \geq 1$, with the following conditions:

(1°) For each n, $\{(B_n(t), W_n(t)), 0 \leq t \leq T\}$ is equivalent in law to $\{(B(t), W^R(t)), 0 \leq t \leq T\}$

(2°) B_n and W_n converge uniformly in $t \in [0,T]$(a.s.) as $n \to \infty$ to some processes $\mathcal{B}$ and $\mathcal{W}$, respectively.

Remark 5.2. Let $X_n(t) = W_n(t) + L_n(t)$ be the SP for $(W_n; \mathcal{D}_{R_n})$. Then Remark 5.1 (1°) and (5.3) imply

$$W_n(t) = X_n(0) + \int_0^t \sigma(X_n(s))dB_n(s) + \int_0^t b(X_n(s))ds.$$

Now, we prepare some Skorohod equations. Let

$$X'(t) = \overline{\mathcal{W}^g}(t) + L'(t),$$
$$X_n^R(t) = W_n(t) + L_n^R(t),$$
$$X_n^0(t) = \overline{W_n{}^g}(t) + L_n^0(t),$$

be SP's for $(\overline{\mathcal{W}^g}; \mathcal{D}_\infty)$, $(W_n; \mathcal{D}_R)$ and $(\overline{W_n{}^g}; \mathcal{D}_\infty)$, respectively. We denote $Y_n \equiv X_n{}^g$ and $Y_n^R \equiv (X_n^R)^g$. Then we have the following lemma.

Lemma 5.2. *For any $\varepsilon > 0$, there exist $R_0 \equiv R_0(\omega)$ and $n_0 \equiv n_0(\omega)$ such that*

$$\|\overline{Y_n} - \overline{Y_n^R}\|_T < \varepsilon \quad \text{for} \quad \forall R < R_0,\ \forall n > n_0,$$

almost surely.

Proof. We write $X_n = W_n + \psi_n + \varphi_n$ and $X_n^R = W_n + \psi_n^R + \varphi_n^R$ in the sense of (3.1). Then we easily see that Remark 5.1 (2°), Proposition 3.1 and Remark 3.6 imply $\mid \varphi_n \mid_T$ and $\mid \varphi_n^R \mid_T$ are uniformly bounded in n. Hence, by Lemma 3.2 and Gronwall's lemma, we have

$$\|\overline{Y_n} - \overline{Y_n^R}\|_T^2 \leq (R + R_n)C',$$

with some $C' \equiv C'(\omega)$ depending only on N, ρ, T, $\left\|\overline{\mathcal{W}^g}\right\|_T$, the modulus of uniform continuity of $\overline{\mathcal{W}^g}$ and ω. ∎

The following lemma is immediate from Proposition 3.1, Remark 3.6, (3.19) and Theorem 4.1.

Lemma 5.3. *For any $\varepsilon > 0$, there exist $R_0' \equiv R_0'(\omega)$ and $n_0' \equiv n_0'(\omega)$ such that*

$$\|\overline{Y_n^R} - X_n^0\|_T < \varepsilon \quad \text{for} \quad \forall R < R_0', \forall n (> n_0'),$$

almost surely, where R_0' can be taken uniformly in $n > n_0'$.

By Remark 5.1 (2°) and the result on continuity in Theorem 1.1, we have the following lemma.

Lemma 5.4. *For any $\varepsilon > 0$, there exists $n_0'' \equiv n_0''(\omega) > 0$ such that*

$$\|X_n^0 - X'\|_T < \varepsilon \quad \text{for} \quad \forall n > n_0'',$$

almost surely.

Lemmas 5.2, 5.3 and 5.4 yield the following proposition.

Proposition 5.1. *For any $\varepsilon > 0$, there exists $n_0''' \equiv n_0'''(\omega) > 0$ such that*

$$\|\overline{Y_n} - X'\|_T < \varepsilon \quad \text{for} \quad \forall n > n_0''',$$

almost surely.

Thus, noting Remark 5.1 (1°) and Remark 5.2, we get the following theorem.

Theorem 5.2. *(X', L') solves the Skorohod equation*

$$X'(t) = X'(0) + \int_0^t \sigma(X'(s))dB(s) + \int_0^t b(X'(s))ds + L'(t)$$

for $\mathcal{D}_\infty$.

The proof is done by the same procedure as that of Lemma 5.2 in [2] .

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