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NORM PRESERVING MULTIPLIERS ON A WEIGHTED SEGAL ALGEBRA

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ABSTRACT

In the present paper we prove a representation theorem for a norm preserving multiplier on a weighted Segal algebra defined on a locally compact abelian group G. In case G is compact and $r = 0$, then our theorem reduces to an earlier result of Tewari [3].

1. Introduction

Let G be a locally compact abelian group with Haar measure dx and let Γ be its dual. We suppose that Ω is the set of all continuous real valued even funtions ω on Γ such that

i) $\omega(\eta+\gamma) \leq \omega(\eta)\,\omega(\gamma)$

for all $\eta, \gamma \in \Gamma$

ii) For $0 \leq r \leq p$ and $1 \leq p < \infty$, the functions ω^r and ω^{-r} are locally integrable.

Let $M_{bd}(G)$ be the Banach space of all bounded regular complex-valued mesures on G with respect to the norm

$$\|\mu\| = |\mu|(G)$$
$$= \text{total variation of } \mu.$$

We now define $A^p_{\omega^r}(G)$ in the following way:

(1.1) $$A^p_{\omega^r}(G) = \left\{f \in L^1(G) : \hat{f} \in L^p_{\omega^r}(\Gamma)\right\},$$

where $\hat{f}(\gamma)$ be the Fourier transform of $f \in L^1(G)$ and $L^p_{\omega^r}(\Gamma)$ is the space of all complex-valued measurable functions $\hat{f}(\gamma)$ such that

(1.2) $$\|\hat{f}\|_{L^p_{\omega^r}} = \left(\int_\Gamma |\hat{f}(\gamma)|^p \omega^r(\gamma)\,d\gamma\right)^{1/p} < \infty.$$

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It can be easily verified that $A^p_{\omega^r}(G)$ is a Segal algebra with respect to the norm

$$\| f \|_{A^p_{\omega^r}} = \| f \|_1 + \| \hat{f} \|_{L^p_{\omega^r}} \qquad \big(f \in A^p_{\omega^r}(G)\big) \tag{1.3}$$

and convolution as multiplication.

We denote by $M(A^p_{\omega^r})$ the set of all multilpiers on $A^p_{\omega^r}$.

2. Tewari [3] has discussed the norm preserving multipliers of $A^p(G)$, when G is compact abelian group. He has shown that if G is a compact abelian group and T is norm preserving multiplier of $A^P(G)$, then there exist $\alpha \in G$ and a complex number λ of absolute value 1 such that $T = \lambda\tau_\alpha$, where τ_α denotes the operator of translation by amouny α.

In the present paper we generalize the above mentioned result of Tewari. Precisely, we shall prove the following:

Theorem: ***If G is a locally compact abelian group and T is a norm preserving multiplier of $A^p_{\omega^r}(G)$, $0 \le r \le p$, $1 \le p \le \infty$, then there exists $x \in G$ and a complex number λ of absolute value one such that $T = \lambda\tau_x$.***

3. The following lemma is essentioal for the proof of the theorem when G is either compact abelian group or locally compact abelian group.

Lemma 1. If T is a norm preserving multiplier of $A^p_{\omega^r}(G)$, then there exists $x \in G$ and a complex number λ of absolute value one such that $T = \lambda\tau x$.

For the proof see Wendel [4, theorem 3].

4. Proof of the Theorem. For Compact G.

Let γ be any element of Γ. Then we have

$$\gamma * \gamma = \gamma.$$

Hence we obtain

$$T(\gamma * \gamma) = T(\gamma) * \gamma = T(\gamma),$$

which implies that

$$T(\gamma) = \phi(\gamma)\,\gamma,$$

where $\phi(\gamma)$ is a complex number.

Since T is norm preserving, it follows that

$$|\phi(\gamma)| = 1, \qquad \gamma \in \Gamma.$$

We denote by $B(G)$ the set of all functions f in $L^1(G)$ whose Fourier transform has compact support.

Let $f = \sum_{i=1}^n a_i\gamma_i$ be a trigonometric polynomial in $B(G)$. Then we have

$$\begin{aligned} \left\| T\left\{\sum_{i=1}^n a_i\gamma_i\right\}\right\|_{a^p_{\omega^r}} &= \left\|\sum_{i=1}^n a_i T(\gamma_i)\right\|_{L^1} + \left\|\sum_{I=1}^n a_i\phi(\gamma_i)\,\omega(\gamma_i)\right\|_{L^p_{\omega^r}} \\ &= \left\| T\left\{\sum_{i=1}^n a_i\gamma_i\right\}\right\|_{L^1} + \left\|\sum_{i=1}^n a_i\omega(\gamma_i)\right\|_{L^p_{\omega^r}} . \end{aligned} \tag{4.1}$$

Also, since T is norm preserving, we have

$$\begin{aligned} \left\| T\left\{\sum_{i=1}^n a_i\gamma_i\right\}\right\|_{A^p_{\omega^r}} &= \left\|\sum_{i=1}^n a_i\,\omega(\gamma_i)\right\|_{A^p_{\omega^r}} \\ &= \left\|\sum_{i=1}^n a_i(\gamma_i)\right\|_{L^1} + \left\|\sum_{i=1}^n a_i(\gamma_i)\right\|_{L^p_{\omega^r}} . \end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), we get

$$\left\| T\left\{\sum_{i=1}^n a_i\gamma_i\right\}\right\|_{L^1} = \left\|\sum_{i=1}^n a_i\gamma_i\right\|_{L^1},$$

which implies that

$$\| Tf \|_1 = \| f \|_1 \qquad \text{for each } f \in B(G).$$

Since $B(G)$ is dense in $L^1(G)$, there exists a unique norm preserving multiplier T' of $L^1(G)$ onto itself such that

$$T'f = Tf \quad \text{for each} \quad f \in A^p_{\omega^r}(G).$$

Hence, using lemma 1, we infer that there exists $x \in G$ and a complex number λ with absolute value one such that

$$Tf = \lambda\tau_x f \quad \text{for each} \quad f \in A^p_{\omega^r}(G).$$

5. When G is a noncompact locally compact abelian group, the proof of the Theorem depends on the following lemmas:

Lemma 2. If G is a noncompact locally compact abelian group and $f \in L^p(G)$, then

$$\lim_{y\to\infty} \| f + \tau_y f \|_p = 2^{1/p} \| f \|_p.$$

For the proof see [2, p. 78–79].

Lemma 3. If G is a nondiscrete noncompact locally compact abelian group, then for every $T \in M\,(A^p_{\omega^r}(G))$ there exists a unique measure $\mu \in M_{bd}(G)$ such that

$$Tf = \mu * f$$

for all $f \in A^p_{\omega^r}(G)$. Also $M\,(A^p_{\omega^r}(G))$ is isometrically isomorphic to $M_{bd}(G)$.

Proof: On the lines of Keshava Murthy [1], we suppose that $f \in B(G)$. Hence there exists a compact set $\hat{K} \subset \Gamma$ such that

$$\hat{f}(\gamma) = 0 \qquad \text{for each } \gamma \notin \hat{K}.$$

Let $T \in M\,(A^p_{\omega^r})$.

Since G is nondiscrete and $A^p_{\omega^r}(G)$ is a Segal algebra on G, we have

$$\begin{aligned} \| Tf \| &\le \| Tf \|_{A^p_{\omega^r}} \\ &\le \| T \| \left(\| f_1 \| + \| \hat{f} \|_{L^p_{\omega^r}} \right). \end{aligned} \tag{5.1}$$

We shall prove the lemma for two cases

i) $2 \le p < \infty$

ii) $1 \le p < 2$

separately.

Case i) $2 \le p < \infty$.

We suppose that there exists a number q such that

$$1/p + 1/q = 1,$$

which implies that $1 < p \le 2$.

It is well known that $L^1(G) \cap L^q(G)$ is Segal algebra with respect to the norm

$$\| g \|_{1,q} = \| g \|_1 + \| g \|_q \qquad \text{for every } g \in L^1(G) \cap L^q(G).$$

Since $B(G)$ is contained in every Segal algebra $S(G)$ on G, we have

$$B(G) \subset L^1(G) \cap L^q(G).$$

Now, using Housdorff-Young inequality, we obtain

$$\| \hat{f} \|_p \leq \| f \|_q. \tag{5.2}$$

Also, for $f \in B(G)$, we see that

$$\begin{aligned}\int_\Gamma |\hat{f}|^p \, \omega^r(\gamma) \, d\gamma &= \int_{\hat{K}} |\hat{f}|^p \, \omega^r \, d\gamma \\ &\leq C(\hat{K}, \omega^r) \int_{\hat{K}} |\hat{f}|^p \, d\gamma \\ &= C(\hat{K}, \omega^r) \int_\Gamma |\hat{f}|^p \, d\gamma,\end{aligned}$$

where $C(\hat{K}, \omega^r)$ is a positive constant depending on the compact set $\hat{K}$, the weight function ω and the parameter r.

Hence we obtain

$$\begin{aligned}\| \hat{f} \|_{L^p_{\omega^r}} &\leq C(\hat{K}, \omega^r) \| \hat{f} \|_p \\ &\leq C(\hat{K}, \omega^r) \| \hat{f} \|_q\end{aligned} \tag{5.3}$$

by (5.2).

Combining (5.1) and (5.3), we get

$$\| Tf \|_1 \leq \| T \| \left(\| f \|_1 + C(\hat{K}, \omega^r) \| f \|_q \right). \tag{5.4}$$

Since G is a noncompact locally compact abelian group, applying Lemma 2, we have

$$\begin{aligned}2 \| Tf \|_1 &= \lim_{y \to \infty} \| Tf + \tau_y Tf \|_1 \\ &= \lim_{y \to \infty} \| T(f + \tau_y f) \|_1 \\ &\leq \lim_{y \to \infty} \| T \| \left(\| f + \tau_y f \|_1 + C(\hat{K}, \omega^r) \| f + \tau_y f \|_q \right) \\ &= 2 \| T \| \left(\| f \|_1 + C(\hat{K}, \omega^r) \, 2^{1/q - 1} \| f \|_q \right).\end{aligned}$$

Thus we get

$$\| Tf \|_1 \leq \| T \| \left(\| f \|_1 + C(\hat{K}, \omega^r) \, 2^{1/q-1} \| f \|_q \right).$$

Continuing this process n times, we obtain

$$\| Tf \|_1 \leq \| T \| \left(\| f \|_1 + C(\hat{K}, \omega^r) \, 2^{(1/q-1)n} \| f \|_q \right).$$

Taking limits on both the sides as $n \to \infty$, we get

$$\| Tf \|_1 \leq \| T \| \, \| f \|_1 \qquad \forall \, f \in B(G) \tag{5.5}$$

for $2^{(1/q-1)n} \to 0$ as $n \to \infty$.

This implies that T is a linear transformation from $B(G)$ to $L^1(G)$, which is bounded in L^1 norm and commutes with translation. Hence T is a multiplier from $B(G)$ to $L^1(G)$, Further, since $B(G)$ is dense in $L^1(G)$, hence T can be extended uniquely as a multiplier T^1 from $L^1(G)$ to $L^1(G)$.

Thus, from the definition of a multiplier on $L^1(G)$, we infer that there exists $\mu \in M_{bd}(G)$ satisfying the condition

$$
\begin{aligned}
& T_1 f = \mu * f && \text{for each } f \in L^1(G) \\
\Longrightarrow\ & T_1 f = \mu * f && \text{for each } f \in B(G) \\
\Longrightarrow\ & Tf = \mu * f && \text{for each } f \in B(G).
\end{aligned}
$$

Following Laarsen [2, p. 6], we have

$$\| T \| \geq \| \mu \|.$$

Since $B(G)$ is dense in $A^p_{\omega,r}(G)$, we infer that

$$Tf = \mu * f \qquad \forall\, f \in A^p_{\omega,r}(G).$$

With the property

$$\| T \| \geq \| \mu \|, \tag{5.6}$$

the equation

$$Tf = \mu * f$$

defines a bijective isomorphism between the multipliers for $A^p_{\omega,r}(G)$ and $M_{bd}(G)$. Since T corresponds $\mu \in M_{bd}(G)$ under this bijection, we have

$$\| T \| \leq \| \mu \|. \tag{5.7}$$

Combining (5.6) and (5.7), we obtain

$$\| T \| = \| \mu \|$$

Hence the theorem holds for $2 \leq p < \infty$.

case ii) $1 \leq p < 2$.

We now suppose that $s = 2/p$ such that

$$1/s + 2/s' = 1.$$

Since $1 \leq p < 2$, we have $s > 1$. Thus we have

$$\begin{aligned} \| \hat{f} \|_{L^p_{\omega^r}} &= \left(\int_\Gamma | \hat{f} |^p \omega^r \, d\gamma \right)^{1/p} \\ &= \left(\int_{\hat{K}} | \hat{f} |^p \omega^r \, d\gamma \right)^{1/p} \\ &\leq C\left(\hat{K}, \omega^{r/p}\right) \left(\int_{\hat{K}} |\hat{f}|^p \, d\gamma \right)^{1/p} \end{aligned}$$

for ω is locally bounded on Γ.

Applying Hölder's inequality, we obtain

$$\begin{aligned} \| \hat{f} \|_{L^p_{\omega^r}} &\leq C(\hat{K}, \omega^{r/p}) \left[\left(\int_{\hat{K}} | \hat{f} |^{ps} \, d\gamma \right)^{1/s} \left(\int_{\hat{K}} 1 \, d\gamma \right)^{1/s} \right]^{1/p} \\ &= C(\hat{K}, \omega^{r/p}) \left[\left(\int_{\hat{K}} | \hat{f} |^{ps} \, d\gamma \right)^{1/s} \right]^{1/p}, \end{aligned} \tag{5.8}$$

where $C(\hat{K}, \omega^{r/p})$ is a positive constant depending on the copmact set $\hat{K}$, the weight function ω and the parameters r and p. But $C(\hat{K}, \omega^{r/p})$ is not same at each occurrence.

From (5.8) it follows that

$$\begin{aligned} \| \hat{f} \|_{L^p_{\omega^r}} &\leq C(\hat{K}, \omega^{r/p}) \| \hat{f} \|_2 \\ &\leq C(\hat{K}, \omega^{r/p}) \| f \|_2 \end{aligned}$$

by Plancherels' theorem.

Thus, from (5.1), we have

$$\begin{aligned} \| Tf \|_1 &\leq \| T \| \left(\| f \|_1 + \| \hat{f} \|_{L^p_{\omega^r}} \right) \\ &\leq \| T \| \left(\| f \|_1 + C(\hat{K}, \omega^{r/p}) \| f \|_2 \right) \end{aligned}$$

Using Lemma 2 again, we obtain

$$\begin{aligned} 2\| Tf \|_1 &= \lim_{y\to\infty} \| Tf + \tau_y f \|_1 \\ &= \lim_{y\to\infty} \| T(f + \tau_y f) \|_1 \\ &\le \lim_{y\to\infty} \| T \| \left(\| f + \tau_y f \|_1 + C(\hat{K},\, \omega^{r/p}) \| f + \tau_y f \|_2 \right) \\ &= \| T \| \left(\| f \|_1 + C(\hat{K},\, \omega^{r/p})\, 2^{-1/2} \| f \|_2 \right). \end{aligned}$$

Continuing this process n times, we get

$$\| Tf \| \le \| T \| \left(\| f \| + C(\hat{K}, \omega^{r/p})\, 2^{-n/2} \| f \|_2 \right).$$

Hence, taking limit as $n \to \infty$, we get

$$\| Tf \| \le \| T \| \, \| f \|_1.$$

Now, using the arguments as in case i) the proof of the lemma follows for the case $1 \le p < 2$. Hence the lemma holds.

6. Proof of the Theorem for noncompact G.

By lemma 3 there exists a measure $\mu \in M_{bd}(G)$ such that

$$Tf = \mu * f \quad (f \in A^p_{\omega^r}(G))$$

with

$$\| \mu \| = 1.$$

Hence we see that

$$\begin{aligned} \| \mu * f \|_{A^p_{\omega^r}} &= \| \mu * f \|_1 + \| \hat{\mu} \hat{f} \|_{L^p_{\omega^r}} \\ &= \| f \|_{A^p_{\omega^r}} \\ &= \| f \|_1 + \| \hat{f} \|_{L^p_{\omega^r}}, \end{aligned}$$

which imlies that

$$\| \mu * f \|_1 = \| f \|_1 \qquad \forall\, f \in A^p_{\omega^r}(G).$$

Finally, since $A^p_{\omega^r}(G)$ is dense in $L^1(G)$, by lemma 1 the proof of the theorem is complete.

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