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ANALYTIC SOLUTIONS OF A DEGENERATED SYSTEM OF TWO NONLINEAR EQUATIONS AT AN IRREGULAR TYPE SINGULARITY

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I. Introduction

1. Preliminary reduction. Consider a system of nonlinear equations

(1.1) $$\begin{cases} x^2y' = S(x,y,z), & ' = \dfrac{d}{dx}, \\ xz' = W(x,y,z), \end{cases}$$

where x is a complex variable and $S(x,y,z)$ and $W(x,y,z)$ are analytic function in a neighborhood of $(0,0,0)$:

(1.2) $$|x| < \delta_0, \quad |y| < \delta_1, \quad |z| < \delta_2, \quad (\delta_0, \delta_1, \delta_2 : \text{positive constants}),$$

satisfying

(1.3) $$S(0,0,0) = 0, \qquad W(0,0,0) = 0.$$

The point $x = 0$ is called an *irregular type singularity* of (1.1). There are many studies of (1.1) at an irregular type singularity when $S_y(0,0,0) \neq 0$, $W_z(0,0,0) \neq 0$, (e.g. see P. F. Hsieh [Hs1, Hs2], M. Hukuhara [Hu1] and M. Iwano [I1, I2, I3] and their references). However, the study of (1.1) at an irregular type singularity when $W_z(0,0,0) = 0$ was virtually not done, except a formal solution was obtained in P. F. Hsieh and J. J. Przybylski [HP]. This is the degenerated case of (1.1). In this paper, the analytic solution of (1.1) for the degenerated case is to be studied by means of a different formal solution.

By (1.3), the system (1.1) can be written as

(1.4) $$\begin{cases} x^2y' = \lambda y + \mu z + \nu x + \displaystyle\sum_{i+j+k=2}^{\infty} S_{ijk}x^iy^jz^k, \\ xz' = \alpha y + \beta z + \gamma x + \displaystyle\sum_{i+j+k=2}^{\infty} W_{ijk}x^iy^jz^k, \end{cases}$$

where the right hand sides are convergent series in (1.2).

PROPOSITION 1-1. *Under the assumption that*

$$\lambda \neq 0, \qquad \lambda\beta - \mu\alpha = 0, \tag{1.5}$$

(1.4) can be reduced analytically to

$$\begin{cases} x^2y' = \lambda y + f(x,y,z), \\ xz' = g(x,y,z). \end{cases} \tag{E.1}$$

where $f(x,y,z)$ and $g(x,y,z)$ are power series in the form:

$$\begin{cases} f(x,y,z) = \sum_{i+j+k=2}^{\infty} f_{ijk}x^iy^jz^k, \\ g(x,y,z) = \sum_{i+j+k=2}^{\infty} g_{ijk}x^iy^jz^k. \end{cases} \tag{1.6}$$

with the right hand sides convergent in a neighborhood of (0, 0, 0).

It is clear from (1.6) that

$$f(0,0,0) = g(0,0,0) = 0, \tag{1.7}$$

and

$$f_y(0,0,0) = f_z(0,0,0) = 0, \qquad g_y(0,0,0) = g_z(0,0,0) = 0. \tag{1.8}$$

This proposition indicates that the study of a degenerated sytem of two equations with an irregular type singularity at $x = 0$ can be done on (E.1) without loss of generality. We will study (E.1) with (1.6) in this paper.

PROOF OF PROPOSITION 1-1: First put

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\mu}{\lambda} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix}. \tag{1.9}$$

Then, (1.1) is reduced to

$$\begin{cases} x^2Y' = \lambda Y + \nu x + \sum_{i+j+k=2}^{\infty} \overset{o}{S}_{ijk}x^iY^jZ^k, \\ xZ' = \alpha Y + \gamma x + \sum_{i+j+k=2}^{\infty} \overset{o}{W}_{ijk}x^iY^jZ^k. \end{cases} \tag{1.10}$$

Secondly, put

$$Y = Y_1 - \frac{\nu}{\lambda}x, \tag{1.11}$$

then, (1.10) is reduced to

$$\begin{cases} x^2Y_1' = \lambda Y_1 + \sum_{i+j+k=2}^{\infty} \hat{S}_{ijk}x^iY_1^jZ^k, \\ xZ' = \alpha Y_1 + \left(\gamma - \frac{\nu}{\lambda}\alpha\right)x + \sum_{i+j+k=2}^{\infty} \hat{W}_{ijk}x^iY_1^jZ^k. \end{cases} \tag{1.12}$$

Finally, put

$$Z = \frac{\alpha}{\lambda}xY_1 + Z_1 + (\gamma - \frac{\alpha\nu}{\lambda})x, \tag{1.13}$$

then, (1.9) is reduced to

$$\begin{cases} x^2Y_1' = \lambda Y_1 + \sum_{i+j+k=2}^{\infty} f_{ijk}x^iY_1^jZ_1^k, \\ xZ_1' = \sum_{i+j+k=2}^{\infty} g_{ijk}x^iY_1^jZ_1^k. \end{cases} \tag{1.14}$$

Evidently, the transformations (1.9), (1.11) and (1.13) are analytic and the right hand sides of (1.14) are convergent in a neighborhood of (0, 0, 0). **q.e.d.**

2. The Main Reuslts. In this paper, we will study (E.1) with (1.6) and the following assumption:

Assumption I. $\lambda \neq 0, \quad g_{002} \neq 0.$

We will prove first the following

PROPOSITION 2-1. *Under* Assumption I, *there exists a formal transformation*

$$\begin{cases} y = u + \sum_{i+j+k=2}^{\infty} p_{ijk}x^iu^jv^k, \\ z = v + \sum_{i+j+k=2}^{\infty} q_{ijk}x^{i+1}u^jv^k, \quad (q_{i0k} = 0), \end{cases} \tag{2.1}$$

such that (E.1) is reduced to

$$\begin{cases} x^2u' = [\lambda + \lambda_0(v) + \lambda_1(v)x]u, \\ xv' = v^2(b + \hat{b}v), \qquad (b = g_{002}), \end{cases} \tag{E.2}$$

where $\lambda_0(v)$ is a convergent series in v, $\lambda_0(0) = 0$, $\lambda_1(v)$ is linear in v and b and $\hat{b}$ are constants with $b \neq 0$.

The proof of this proposition is to be given in Part II.

In order to state the main theorem, for two small constants, ρ and ϵ, consider

$$r(\theta;\rho,\epsilon) = \begin{cases} \rho\dfrac{\sin(\theta+\arg b)}{\sin(\frac{\pi}{2}+\arg b)}, & \dfrac{\pi}{2}-\arg b < \theta < \pi - \arg b - \epsilon, \\ \rho, & -\dfrac{\pi}{2}-\arg b \le \theta \le \dfrac{\pi}{2}-\arg b, \\ \rho\dfrac{\sin(\theta+\arg b)}{\sin(-\frac{\pi}{2}+\arg b)}, & -\pi-\arg b+\epsilon < \theta < -\dfrac{\pi}{2}-\arg b. \end{cases} \tag{2.2}$$

We assume further the following

Assumption II. $f_{i0k} = 0$ for all $i + k \geq 2$.

Using Proposition 2-1, we will prove the following main theorem in this paper.

THEOREM M. *Assume that (E.1) with (1.6) satisfies* Assumptions I *and* II. *Let*

$$\begin{aligned} &|x| < \rho_1, && \arg\lambda - \frac{\pi}{2} + \epsilon_1 < \arg x < \arg\lambda + \frac{\pi}{2} - \epsilon_1, \\ &|v| < r(\arg v;\rho_2,\epsilon_2), && -\pi - \arg b + \epsilon_2 < \arg v < \pi - \arg b - \epsilon_2, \\ &|u| < \rho_3, \end{aligned} \tag{D}$$

be a domain in (x,v,u)-space, where ρ_1,ρ_2 and ρ_3 are suitable small constants ($\rho_1 < 1$) and ϵ_1 and ϵ_2 are preassigned sufficiently small constants. Let (x_0,v_0,u_0) be an arbitrary point in (D), and $\{U(x),V(x)\}$ be the solution of (E.2) such that $U(x_0) = u_0$, $V(x_0) = v_0$. Then, (E.1) has an analytic solution $\{P(x,V(x),U(x)),Q(x,V(x),U(x))\}$ where

$$\begin{aligned} P(x,V(x),U(x)) &= \sum_{k=1}^{\infty} P_k(x,V(x))U(x)^k, \\ Q(x,V(x),U(x)) &= Q_0(x,V(x)) + x\sum_{k=1}^{\infty} Q_k(x,V(x))U(x)^k, \end{aligned} \tag{2.3}$$

convergent uniformly for $U(x)$ in

$$|u| < \rho_3. \tag{2.4}$$

Here $Q_0(x,v)$ and $P_k(x,v)$ and $Q_k(x,v)$ $(k = 1,2,\cdots)$ are analytic in

$$\begin{aligned} |x| < \rho_1, \quad & \arg\lambda - \frac{\pi}{2} + \epsilon_1 < \arg x < \arg\lambda + \frac{\pi}{2} - \epsilon_1 \\ |v| < r(\arg v;\rho_2,\epsilon_2), \quad & -\pi - \arg b + \epsilon_2 < \arg v < \pi - \arg b - \epsilon_2, \end{aligned} \tag{2.5}$$

and admit asymptotic expansions in the form

$$(2.6)\qquad \begin{aligned} P_k(x,v) &\simeq \sum_{\ell=0}^{\infty} P_{k\ell}(v)x^{\ell}, \qquad (k=1,2,\cdots),\\ Q_k(x,v) &\simeq \sum_{\ell=0}^{\infty} Q_{k\ell}(v)x^{\ell}, \qquad (k=0,1,2,\cdots), \end{aligned}$$

uniformly in

$$(2.7)\qquad |v| < r(\arg v; \rho_2, \epsilon_2), \quad -\pi - \arg b + \epsilon_2 < \arg v < \pi - \arg b - \epsilon_2,$$

as x tends to 0 in the sector

$$(2.8)\qquad \arg\lambda - \frac{\pi}{2} + \epsilon_1 < \arg x < \arg\lambda + \frac{\pi}{2} - \epsilon_1$$

with $Q_{00}(v)$ analytic in a neighborhood of $v = 0$ and other coefficients $P_{k\ell}(v)$ and $Q_{k\ell}(v)$ admitting asymptotic expansions

$$(2.9)\qquad \begin{aligned} P_{k\ell}(v) &\simeq \sum_{j=0}^{\infty} P_{k\ell j}v^{j}, \qquad (k=1,2,3,\cdots;\ell=0,1,2,\cdots),\\ Q_{k\ell}(v) &\simeq \sum_{j=0}^{\infty} Q_{k\ell j}v^{j}, \qquad (k=0,1,2,\cdots;\ell=0,1,2,\cdots;\ell\neq 0 \text{ when } k=0), \end{aligned}$$

as v tends to 0 in the sector (2.7).

Here $\lambda_0(v) = \dfrac{\partial f}{\partial y}(0,0,Q_{00}(v))$ *and $\lambda_1(v)$ is the first two terms of the expansion* $\dfrac{\partial^2 f}{\partial x\partial y}(0,0,Q_{00}(v)) + \dfrac{\partial^2 f}{\partial z\partial y}(0,0,Q_{00}(v))Q_{01}(v)$ *in the powers of v.*

This theorem is to be proved in Part IV and V. The domain (D) is called the *stable domain* of (E.2). We will develop it in Part III.

In order to establish Theorem M, we are indeed looking for an analytic transformation

$$(\mathrm{T})\qquad \begin{cases} y = P(x,v,u),\\ z = Q(x,v,u), \end{cases}$$

such that (E.1) is reduced to (E.2), where $P(x,v,u)$ and $Q(x,v,u)$ are given by (2.3) with $V(x)$ and $U(x)$ replaced by v and u, respectively. In order to find the stable domain (D) with non-empty sector in x–plane so that (T) is analytic in (D), we have to impose Assumption II and find a transformation (T) satisfying $P(x,v,0) \equiv 0$. To find a more general transformation (T) which allows wider sectors in x-plane and v-plane for the stable domain than those in (D) for a more general system of equations is an open problem.

II. Formal Reduction

3. Finite reductions. In this part, we will prove Proposition 2-1; namely to investigate the formal transformation of (2.1) to assure that it is possible to reduce (E.1) to (E.2). First, we consider the following transformation:

$$(3.1)\qquad \begin{cases} y = \eta + \sum\limits_{(N)} P_{ijk} x^i \eta^j \zeta^k, \\ z = \zeta + \sum\limits_{(N)} Q_{ijk} x^{i+1} \eta^j \zeta^k, \end{cases}$$

where $\sum\limits_{(N)}$ denote the sum of nonnegative intgers i, j, k such that $i + j + k = N \geq 2$. The coefficients P_{ijk} and Q_{ijk} are to be chosen suitably that the reduced system is as simple as possible.

Note that the inverse transformation of (3.1) is

$$(3.2)\qquad \begin{cases} \eta = y - \sum\limits_{(N)} P_{ijk} x^i y^j z^k + [x, y, z]_{2N-1}, \\ \zeta = z - \sum\limits_{(N)} Q_{ijk} x^{i+1} y^j z^k + [x, y, z]_{2N}, \end{cases}$$

where $[x, y, z]_m$ denotes a convergent power series in (x, y, z) with the sum of powers in (x, y, z) at least m.

Differentiate (3.2), by (E.1) and (1.6), we have

$$(3.3)\qquad \begin{aligned} x^2\eta' &= x^2 y' - \sum_{(N)} P_{ijk}\{ix + j\frac{x^2 y'}{y} + kx\frac{xz'}{z}\} x^i y^j z^k + [x, y, z]_{N+1}, \\ &= \lambda y + \sum_{i+j+k=2}^{\infty} f_{ijk} x^i y^j z^k - \sum_{(N)} P_{ijk}\{ix + \lambda j + \cdots\} x^i y^j z^k + \cdots \\ &= \lambda(\eta + \sum_{(N)} P_{ijk} x^i \eta^j \zeta^k) + \sum_{i+j+k=2}^{\infty} f_{ijk} x^i \eta^j \zeta^k + [x, \eta, \zeta]_{N+1} \\ &\quad - \sum_{(N)} \lambda j P_{ijk} x^i \eta^j \zeta^k + [x, \eta, \zeta]_{N+1}. \end{aligned}$$

Write the equation in η as

$$(3.4)\qquad x^2\eta' = \lambda\eta + \sum_{i+j+k=2}^{\infty} A_{ijk} x^i \eta^j \zeta^k,$$

then,

$$(3.5)\qquad \begin{cases} A_{ijk} = f_{ijk}, & i+j+k < N, \\ A_{ijk} = f_{ijk} + (1-j)\lambda P_{ijk}, & i+j+k = N, \\ \cdots. \end{cases}$$

In a similar way,

$$(3.6)\qquad \begin{aligned} x\zeta' &= \sum_{i+j+k=2}^{\infty} g_{ijk}x^i y^j z^k - \sum_{(N)} Q_{ijk}\{(i+1) + x^{-1}(\lambda j + \cdots)\}x^{i+1}y^j z^k + \cdots \\ &= \sum_{i+j+k=2}^{\infty} g_{ijk}x^i \eta^j \zeta^k - \sum_{(N)} \lambda j Q_{ijk}x^i\eta^j\zeta^k + [x,\eta,\zeta]_{N+1}. \end{aligned}$$

Write the equation in ζ as

$$(3.7)\qquad x\zeta' = \sum_{i+j+k=2}^{\infty} B_{ijk}x^i\eta^j\zeta^k,$$

then,

$$(3.8)\qquad \begin{cases} B_{ijk} = g_{ijk}, & i+j+k < N, \\ B_{ijk} = g_{ijk} - j\lambda Q_{ijk}, & i+j+k = N, \\ \cdots. \end{cases}$$

Thus, we can get

$$(3.9)\qquad \begin{cases} A_{ijk} = 0 & \text{for } j \neq 1, \\ B_{ijk} = 0 & \text{for } j \neq 0, \\ Q_{ijk} = 0 & \text{for } j = 0, \end{cases} \quad \text{and} \quad \begin{cases} A_{ijk} = f_{ijk} & \text{for } j = 1, \\ B_{ijk} = g_{ijk} & \text{for } j = 0. \end{cases}$$

Combine the tranformation (3.1) for $N = 2, 3, \cdots$, we can get a formal transformation

$$(3.10)\qquad \begin{cases} y = \eta + \displaystyle\sum_{i+j+k=2}^{\infty} P_{ijk}x^i\eta^j\zeta^k, \\ z = \zeta + \displaystyle\sum_{i+j+k=2}^{\infty} Q_{ijk}x^{i+1}\eta^j\zeta^k, \quad Q_{i0k} = 0, \end{cases}$$

such that (E.1) is transformed to

$$(3.11)\qquad \begin{cases} x^2\eta' = \eta\Big[\lambda + \displaystyle\sum_{i+k=1}^{\infty} a_{ik}x^i\zeta^k\Big], \\ x\zeta' = \displaystyle\sum_{i+k=2}^{\infty} b_{ik}x^i\zeta^k, \end{cases}$$

with

$$b_{02} = g_{002} = b \neq 0. \tag{3.12}$$

4. Simplification of the second equation. The formal reduction of the second equation of (3.11) was done by M. Hukuhara [Hu2]. We will apply his method here. In order to simplify the second equation of (3.11), put

$$\zeta = V + \sum_{(N)} Q_{ik} x^i V^k. \tag{4.1}$$

Note that the inverse transformation of (4.1) is

$$V = \zeta - \sum_{(N)} Q_{ik} x^i \zeta^k + [x, \zeta]_{2N-1}. \tag{4.2}$$

Differentiating (4.2), by (3.11), we have

$$\begin{aligned} xV' &= x\zeta' - \sum_{(N)} Q_{ik}(i + k\frac{x\zeta'}{\zeta}) x^i \zeta^k + [x, \zeta]_{2N-1} \\ &= \sum_{i+k=2}^{\infty} b_{ik} x^i \zeta^k - \sum_{(N)} iQ_{ik} x^i \zeta^k + [x, \zeta]_{N+1} \\ &= \sum_{i+k=2}^{\infty} b_{ik} x^i V^k - \sum_{(N)} iQ_{ik} x^i V^k + [x, V]_{N+1}. \end{aligned} \tag{4.3}$$

Write (4.3) as

$$xV' = \sum_{i+k=2}^{\infty} B_{ik} x^i V^k, \tag{4.4}$$

then,

$$\begin{cases} B_{ik} = b_{ik}, & i + k < N, \\ B_{ik} = b_{ik} - iQ_{ik}, & i + k = N, \\ \quad\cdots . \end{cases} \tag{4.5}$$

Thus, if $i \neq 0$, we can choose Q_{ik} such that $B_{ik} = 0$. Combine (4.1) for $N = 2, 3, \cdots$, we have a formal transformation

$$\zeta = V + \sum_{i+k=2}^{\infty} Q_{ik} x^i V^k, \tag{4.6}$$

which reduces the second equation of (3.11) to

$$xV' = \sum_{k=2}^{\infty} b_k V^k. \tag{4.7}$$

By Assumption I, (3.12) and (4.5), we have

$$b_2 = b \neq 0. \tag{4.8}$$

To reduce (4.7) further, consider the transformation

$$V = v + Q_N v^N, \qquad (N \geq 2). \tag{4.9}$$

Then,

$$v = V - Q_N V v^N + [V]_{2N-1}. \tag{4.10}$$

Differentiate (4.10), we have

$$\begin{aligned} xv' &= xV' - NQ_N V^{N-1} \cdot xV' + [V]_{2N} \\ &= bV^2 + \sum_{k=3}^{\infty} b_k V^k - NQ_N(bV^{N+1} + \cdots) + [V]_{2N} \\ &= bv^2 + 2bQ_N v^{N+1} + \sum_{k=3}^{\infty} b_k v^k - NbQ_N v^{N+1} + [v]_{N+2}. \end{aligned} \tag{4.11}$$

Hence, if we put

$$xv' = bv^2 + \sum_{k=3}^{\infty} B_k v^k, \tag{4.12}$$

then, we have

$$\begin{cases} B_k = b_k, & k \leq N, \\ B_{N+1} = b_{N+1} + (2 - N)bQ_N, & \\ \cdots. & \end{cases} \tag{4.13}$$

Therefore, if $N \neq 2$, we can take Q_N such that $B_{N+1} = 0$. On the other hand, when $N = 2$, b_{N+1} is unchanged. So we can take $Q_2 = 0$. Thus, by combine the transformations (4.9) for $N = 2, 3, \cdots$, we have a transformation

$$V = v + \sum_{k=3}^{\infty} \tilde{Q}_k v^k, \tag{4.14}$$

which reduces (4.7) to

$$xv' = bv^2 + \hat{b}v^3, \tag{4.15}$$

where $b \neq 0$, but $\hat{b}$ may vanish.

By combining (4.6) and (4.14), we have a formal transformation

$$\zeta = v + \sum_{i+k=2}^{\infty} \hat{Q}_{ik} x^i v^k, \tag{4.16}$$

which reduces the second equation of (3.11) to (4.15).

5. Simplification of the first equation. By (4.16) the first equation of (3.11) can be written as

$$x^2 \eta' = a(x, v)\eta = \left(\lambda + \sum_{i=0}^{\infty} a_i(v) x^i\right)\eta, \tag{5.1}$$

where $a_i(v)$ are expressed by formal power series in v:

$$a_i(v) = \sum_{j=0}^{\infty} a_{ij} v^j, \qquad (i = 0, 1, 2, \cdots). \tag{5.2}$$

In particular,

$$a_0(0) = 0. \tag{5.3}$$

Put

$$\lambda_0(v) = a_0(v), \qquad \lambda_1(v) = a_{10} + a_{11}v, \tag{5.4}$$

where $\lambda_1(v)$ is the first two terms of the formal power series expression of $a_1(v)$. We claim first that

PROPOSITION 5-1. *The series $\lambda_0(v) = a_0(v)$ given in (5.4) is convergent in a neighborhood of $v = 0$.*

This proposition will be proved at the end of this section.

Using Proposition 5-1, we will find a formal transformation

$$\eta = \hat{p}(x, v)u \tag{5.5}$$

such that (5.1) is reduced to

$$x^2 u' = [\lambda + \lambda_0(v) + \lambda_1(v)x]u, \tag{5.6}$$

where

$$\hat{p}(x,v) = \sum_{k=0}^{\infty} \hat{p}_k(v) x^k. \tag{5.7}$$

In particular,

$$\hat{p}_0(0) = 1. \tag{5.8}$$

From (5.5), we have

$$x^2 \eta' = x^2 \hat{p}' u + \hat{p} x^2 u', \tag{5.9}$$

which implies

$$\begin{aligned} x^2 \hat{p}' &= \hat{p}[a(x,v) - \lambda - \lambda_0(v) - \lambda_1(v)x] \\ &= \left[\sum_{k=0}^{\infty} \hat{p}_k x^k\right] \left[(a_1(v) - \lambda_1(v))x + \sum_{k=2}^{\infty} a_k(v) x^k\right]. \end{aligned} \tag{5.10}$$

On the other hand,

$$x^2 \frac{d\hat{p}}{dx} = \sum_{k=0}^{\infty} \left[k\hat{p}_k + v^2(b + \hat{b}v)\frac{d\hat{p}_k}{dv}\right] x^{k+1}. \tag{5.11}$$

Hence the coefficients $\hat{p}_k(v)$ in (5.7) must satisfy

$$\begin{aligned} v^2(b+\hat{b}v)\frac{d\hat{p}_0}{dv} &= (a_1(v) - \lambda_1(v))\hat{p}_0, \\ v^2(b+\hat{b}v)\frac{d\hat{p}_1}{dv} &= -\hat{p}_1 + [a_2(v)\hat{p}_0(v) + (a_1(v) - \lambda_1(v))\hat{p}_1(v)], \\ v^2(b+\hat{b}v)\frac{d\hat{p}_k}{dv} &= -\hat{p}_k + [a_{k+1}(v)\hat{p}_0(v) + \cdots \\ &\qquad + (a_1(v) - \lambda_1(v))\hat{p}_k(v)], \qquad (k = 2, 3, \cdots). \end{aligned} \tag{5.12}$$

Since $b \neq 0$ and by the choice of $\lambda_1(v)$ given in (5.4), we can find formal solutions of (5.12)

$$\hat{p}_k(v) = \sum_{j=0}^{\infty} \hat{p}_{kj} v^j, \tag{5.13}$$

for $k = 0, 1, 2, \cdots$.

Combine (3.10), (4.6), (4.14) and (5.5), where $\hat{p}(x,v)$ satisfies (5.7) and (5.8) with $\hat{p}_k(v)$ in formal power series satisfying (5.12), we have a formal reduction (2.1) which reduces (E.1) to (E.2). Thus Proposition 2-1 is proved.

To complete the proof of Proposition 2-1, we have to prove Proposition 5-1. By Assumption II, let the formal solutions of (E.1) be

$$\text{(5.14)}\qquad \begin{cases} y = \displaystyle\sum_{k=1}^{\infty} p_k(x,v)u^k, \\ z = q_0(x,v) + x\displaystyle\sum_{k=1}^{\infty} q_k(x,v)u^k, \end{cases}$$

where v satisfies (4.15) and u satisfies (5.6). Notice that we have formally $p_1(0,0) = 1$. Differentiating (5.14), we have

$$\text{(5.15)}\qquad \begin{aligned} x^2 y' &= \sum_{k=1}^{\infty}\left[x^2\frac{dp_k}{dx}u^k + p_k(x,v)ku^{k-1}x^2\frac{du}{dx}\right] \\ &= p_1(0,v)[\lambda + \lambda_0(v)]u + O(x) + O(u^2), \end{aligned}$$

$$\text{(5.16)}\qquad \begin{aligned} xz' &= x\frac{dq_0}{dx} + \sum_{k=1}^{\infty}\left[q_k(x,v)u^k + x^2\frac{dq_k}{dx}u^k + q_k(x,v)ku^{k-1}x^2\frac{du}{dx}\right] \\ &= v^2(b+\hat{b}v)\frac{dq_0}{dv} + q_1(0,v)u + q_1(0,v)[\lambda + \lambda_0(v)]u + O(x) + O(u^2). \end{aligned}$$

On the other hand,

$$\text{(5.17)}\qquad \begin{aligned} f(x,y,z) &= f(x,0,q_0(x,v)) + \left[\frac{\partial f}{\partial y}(x,0,q_0(x,v))p_1(x,v)\right. \\ &\qquad \left. + \frac{\partial f}{\partial z}(x,0,q_0(x,v))xq_1(x,v)\right]u + \cdots \\ &= \left[\frac{\partial f}{\partial y}(0,0,q_0(0,v))p_1(0,v)\right. \\ &\qquad \left. + \frac{\partial f}{\partial z}(0,0,q_0(x,v))xq_1(0,v) + O(x)\right]u + O(u^2) \end{aligned}$$

and

$$\text{(5.18)}\qquad \begin{aligned} g(x,y,z) &= g(0,0,q_0(0,v)) + \left[\frac{\partial g}{\partial y}(0,0,q_0(0,v))p_1(0,v)\right. \\ &\qquad \left. + \frac{\partial g}{\partial z}(0,0,q_0(0,v))xq_1(0,v) + O(x)\right]u + O(u^2). \end{aligned}$$

Note that, by Assumption II, $f(x, 0, q_0(x, v)) \equiv 0$. Put

$$q_0(0, v) = q_{00}(v), \qquad p_1(0, v) = p_{10}(v). \tag{5.19}$$

Then, by comparing (5.15) with (5.17) and (5.16) with (5.18), we have

$$p_{10}(v)[\lambda + \lambda_0(v)] = \frac{\partial f}{\partial y}(0, 0, q_{00}(v))p_{10}(v), \tag{5.20}$$

$$v^2(b + \hat{b}v)\frac{dq_{00}}{dv} = g(0, 0, q_{00}). \tag{5.21}$$

From (5.20), we have

$$\lambda_0(v) = \frac{\partial f}{\partial y}(0, 0, q_{00}(v)) - \lambda. \tag{5.22}$$

Now, by (1.6), we have

$$v^2(b + \hat{b}v)\frac{dq_{00}}{dv} = g(0, 0, q_{00}) = \sum_{k=2}^{\infty} g_{00k} q_{00}^k. \tag{5.23}$$

Note that the series in (5.23) is convergent at $q_{00} = 0$. Put

$$q_{00}(v) = vQ_{00}(v). \tag{5.24}$$

Then,

$$v(b + \hat{b}v)\frac{dQ_{00}}{dv} = -(b + \hat{b}v)Q_{00} + bQ_{00}^2 + \sum_{k=3}^{\infty} g_{00k} v^{k-2} Q_{00}^k. \tag{5.25}$$

Since $b \neq 0$, (5.25) has a regular type singularity at $v = 0$ and it has a formal solution

$$Q_{00}(v) \sim 1 + \sum_{k=1}^{\infty} c_k v^k. \tag{5.26}$$

By Briot-Bouquet theory (e.g. see H. Poincaré [P], P. F. Hsieh [Hs1] and M. Iwano[I1, I2]), the formal solution $Q_{00}(v)$ given by (5.26) is convergent in a neighborhood of $v = 0$. Thus $q_{00}(v)$ given by (5.24) is analytic at $v = 0$. Therefore $\lambda_0(v)$ given by (5.22) is analytic at $v = 0$ with its unique power series expansion (5.4). **q.e.d.**

III. The Stable Domain

6. The x- and v-domains. In order to study the analytic meaning of the formal reduction (2.1), we will study the domain in (x, v, u)–space by studying the solutions of (E.2) when x moves on a straight line from an arbitrary point x_0 in a sector

$$(6.1)\qquad |x| < \rho_1, \quad \arg\lambda - \frac{\pi}{2} + \epsilon_1 < \arg x < \arg\lambda + \frac{\pi}{2} - \epsilon_1, \qquad (\rho_1 < 1),$$

where ρ_1 and ϵ_1 are suitable small constants. The reasons of choosing these small constants will be apparent later. Rewrite the system (E.2) as

$$(6.2)\qquad x^2 u' = [\lambda + \lambda_0(v) + \lambda_1(v)x]u,$$

and

$$(6.3)\qquad xv' = v^2(b + \hat{b}v), \qquad (b \neq 0).$$

Here, $\lambda_0(v)$ is a convergent power series in v given by (5.22) and (5.4) satisfying $\lambda_0(0) = 0$ and $\lambda_1(v)$ is linear in v given by (5.4). We will study the domain in v–plane first, in two cases, then base on it to study that in u–plane in the next section.

Case 1: $\hat{b} = 0$. In this case, (6.3) becomes

$$(6.4)\qquad xv' = bv^2,$$

or, equivalently

$$(6.5)\qquad v^2\frac{dx}{dv} = \frac{1}{b}x.$$

Then the general solution of (6.5) is

$$(6.6)\qquad x(v) = C\exp\left\{-\frac{1}{bv}\right\},$$

where C is an arbitrary constant. Consider the sector

$$(6.7)\qquad |v| < \tilde{\rho}_2, \quad -\pi - \arg b + \epsilon_2 < \arg v < \pi - \arg b - \epsilon_2,$$

where $\tilde{\rho}_2$ and ϵ_2 are suitable small constants. Let x_0 and v_0 be arbitrary points in the sectors (6.1) and (6.7), respectively. The constant C is chosen such that $x(v_0) = x_0$; namely

$$(6.8)\qquad x(v) = x_0\exp\left\{\frac{1}{bv_0} - \frac{1}{bv}\right\}.$$

We will study the behavior of $x(v)$ as x moves from $x = 0$ to x_0 along the line segment $\overline{0x_0}$ in the sector (6.1). Let

$$x = x_0 s, \qquad x \in \overline{0x_0}. \tag{6.9}$$

Then, s is a real number and (6.8) is equivalent to

$$s = \exp\left\{\frac{1}{bv_0} - \frac{1}{bv}\right\}. \tag{6.10}$$

Note that, for a complex quantity A, $\Im(e^A) = 0$ if, and only if, $\Im A = 0$, (mod π). Since s is real, by (6.10), we have

$$\Im\left(\frac{1}{bv_0}\right) = \Im\left(\frac{1}{bv}\right), \tag{6.11}$$

for x on $\overline{0x_0}$ and v and v_0 in (6.7) . Put

$$v = re^{i\theta}, \qquad v_0 = r_0 e^{i\theta_0}. \tag{6.12}$$

Then, (6.11) implies

$$\frac{\sin(\theta + \arg b)}{r} = \frac{\sin(\theta_0 + \arg b)}{r_0}, \tag{6.13}$$

or equivalently

$$r = r_0 \frac{\sin(\theta + \arg b)}{\sin(\theta_0 + \arg b)}. \tag{6.14}$$

Moreover, by (6.4),

$$\frac{1}{|v|}\frac{d|v|}{ds} = \Re\left\{\frac{1}{v}\frac{dv}{ds}\right\} = \Re\left\{\frac{1}{v}\frac{dv}{dx}\frac{dx}{ds}\right\} = \Re\left\{bv\frac{1}{s}\right\} = \frac{1}{s}\Re\left\{bv\right\}. \tag{6.15}$$

Thus, $r = |v|$ is increasing for $-\frac{\pi}{2} - \arg b < \theta < \frac{\pi}{2} - \arg b$, and decreasing for $-\pi - \arg b < \theta < -\frac{\pi}{2} - \arg b$ and $\frac{\pi}{2} - \arg b < \theta < \pi - \arg b$. Also, by (6.12),

$$\frac{d\theta}{ds} = \Im\left\{\frac{1}{v}\frac{dv}{ds}\right\} = \frac{1}{s}\Im\{bv\} = \frac{|b|r}{s}\sin(\theta + \arg b). \tag{6.16}$$

Thus $\theta = \arg v$ is increasing for $-\arg b < \theta < \pi - \arg b$ and decreasing for $-\pi - \arg b < \theta <$

$-\arg b$. Let

$$r(\theta,\rho_2;\epsilon_2)=\begin{cases}\rho_2\dfrac{\sin(\theta+\arg b)}{\sin(\frac{\pi}{2}+\arg b)}, & \dfrac{\pi}{2}-\arg b<\theta<\pi-\arg b-\epsilon_2,\\ \rho_2, & -\dfrac{\pi}{2}-\arg b\le\theta\le\dfrac{\pi}{2}-\arg b,\\ \rho_2\dfrac{\sin(\theta+\arg b)}{\sin(-\frac{\pi}{2}+\arg b)}, & -\pi-\arg b+\epsilon_2<\theta<-\dfrac{\pi}{2}-\arg b,\end{cases}\tag{6.17}$$

where ρ_2 is a suitable small constant. Then we can define the stable domain in v−plane by

$$\{v=re^{i\theta}:r<r(\theta,\rho_2;\epsilon_2),\quad -\pi-\arg b+\epsilon_2<\theta<\pi-\arg b-\epsilon_2\}\tag{6.18}$$

and v stays in (6.18) if x_0 is in (6.1), v_0 is in (6.18) and x is on $\overline{0x_0}$. It is noteworthy that $\arg v$ is not monotonic when x moves on $\overline{0x_0}$ while v is in a sector containing the ray $\theta-\arg b=\pm\pi$ in its interior. Thus, we have to confine the stable domain in v−plane to the sector given by (6.18). Hence, the sector (6.18) is the stable domain in v−plane with central angle as wide as possible.

Case 2: $\hat{b}\neq 0$. In this case, consider a function

$$v=P(w),\tag{6.19}$$

where

$$xw'=bw^2.\tag{6.20}$$

Then, by (6.3), (6.19) and (6.20), $P(w)$ satisfy

$$bw^2\frac{dP}{dw}=(b+\hat{b}P)P^2.\tag{6.21}$$

Put

$$P(w)=w[1+Q(w)].\tag{6.22}$$

Substituting (6.22) into (6.21), we have

$$b[1+Q+w\frac{dQ}{dw}]=[b+\hat{b}w(1+Q)](1+Q)^2,\tag{6.23}$$

or,

$$w\frac{dQ}{dw}=Q+\frac{\hat{b}}{b}w+Q^2+\frac{3\hat{b}}{b}wQ(1+Q)+\frac{\hat{b}}{b}wQ^3.\tag{6.24}$$

Thus, $w = 0$ is a regular type singularity, by Briot-Bouquet theory (e.g. see H. Poincarè [P], P. F. Hsieh [Hs1] and M. Iwano [I1, I2]), there exists a solution

$$Q = \zeta + \sum_{j+k=2}^{\infty} q_{jk} w^j \zeta^k \tag{6.25}$$

convergent in a neighborhood of (0, 0), where ζ is a solution of

$$w\frac{d\zeta}{dw} = \zeta + \frac{\hat{b}}{b} w. \tag{6.26}$$

Namely,

$$\zeta(w) = w\left[c + \frac{\hat{b}}{b}\log w\right], \tag{6.27}$$

where c is an arbitrary constant. In particular, we can choose $c = 0$, i.e. $\zeta(1) = 0$, thus,

$$\zeta(w) = \frac{\hat{b}}{b} w \log w. \tag{6.28}$$

Substituting (6.28) into (6.25), we have

$$Q = \frac{\hat{b}}{b} w \log w + \sum_{j+k=2}^{\infty} q_{jk} w^j \left(\frac{\hat{b}}{b} w \log w\right)^k \tag{6.29}$$

and, consequently

$$P = w\left\{1 + \frac{\hat{b}}{b} w \log w + \sum_{j+k=2}^{\infty} q_{jk} w^j \left(\frac{\hat{b}}{b} w \log w\right)^k\right\}, \tag{6.30}$$

which is convergent in a neighborhood of $w = 0$.

To see the correspondence between v−domain and w−domain, note that

$$v = w\left\{1 + \frac{\hat{b}}{b} w \log w + \sum_{j+k=2}^{\infty} q_{jk} w^j \left(\frac{\hat{b}}{b} w \log w\right)^k\right\}. \tag{6.31}$$

Hence

$$\begin{aligned} \log v &= \log w + \log\left\{1 + \frac{\hat{b}}{b} w \log w + \sum_{j+k=2}^{\infty} q_{jk} w^j \left(\frac{\hat{b}}{b} w \log w\right)^k\right\} \\ &= \log w + [w, w \log w]_1, \end{aligned} \tag{6.32}$$

and

$$v \log v = w \log w + [w, w \log w]_2. \tag{6.33}$$

Thus,

$$v = w + [w, w \log w]_2. \tag{6.34}$$

By implicit function theorem,

$$w = v + [v, v \log v]_2, \qquad w \log w = v \log v + [v, v \log v]_2. \tag{6.35}$$

Hence, there is an one-to-one correspondence between v and w in a neighborhhod of $v = 0$ and that of $w = 0$. Therefore, if ρ_2 and ϵ_2 are sufficiently small, (6.18) is included in

$$\{w = \hat{r}e^{i\hat{\theta}} : \hat{r} < r(\hat{\theta}; \hat{\rho}_2, \hat{\epsilon}_2), \ -\pi - \arg b + \hat{\epsilon}_2 < \hat{\theta} < \pi - \arg b - \hat{\epsilon}_2\}. \tag{6.36}$$

On the other hand, if $\hat{\rho}_2$ and $\hat{\epsilon}_2$ are chosen to be small, then (6.36) is included in (6.18).

Thus, we can take a domain of the form (6.18) as the stable domain in v−plane. We will use (6.18) as the generic stable domain in v−plane for both cases $\hat{b} = 0$ and $\hat{b} \neq 0$.

Remark. The domains (6.18) (or (6.36)) is in the domain (6.7), and vice versa, if the constants ρ_2 and $\tilde{\rho}_2$ (or $\hat{\rho}_2$) are chosen suitably. Namely these domains are equivalent.

7. The u-domain. To find the stable domain in u−plane from (6.2), we will discuss in two cases.

Case 1. Consider first the case $\lambda_0(v) + \lambda_1(v)x \equiv 0$. Namely, (6.2) is in the form

$$x^2 u' = \lambda u, \qquad \lambda \neq 0. \tag{7.1}$$

For x_0 in the sector (6.1) and x is on the line segment $\overline{0x_0}$, by (6.9), we have

$$\frac{d(\log u)}{ds} = \frac{d(\log |u|)}{ds} + i\frac{d(\arg u)}{ds}, \tag{7.2}$$

where

$$x = s \exp\{i \arg x_0\}. \tag{7.3}$$

Thus,

$$\frac{1}{|u|}\frac{d|u|}{ds} = \Re\left[\frac{1}{u}\frac{du}{ds}\right] \tag{7.4}$$

and

$$\frac{d(\arg u)}{ds} = \Im\left[\frac{1}{u}\frac{du}{ds}\right]. \tag{7.5}$$

For x_0 in the sector (6.1)

$$\begin{aligned}\frac{1}{|u|}\frac{d|u|}{ds} &= \Re\left[\frac{x^2}{u}\frac{du}{dx}\frac{dx}{ds}\frac{1}{x^2}\right] = \Re\left[\lambda x_0\frac{1}{x_0^2 s^2}\right] = \frac{1}{s^2}\Re\left[\frac{\lambda}{x_0}\right] \\ &= \frac{|\lambda|}{s^2|x_0|}\cos\left[\arg\left(\frac{\lambda}{x_0}\right)\right] > \frac{|\lambda|}{s^2|x_0|}\sin\epsilon_1 > 0.\end{aligned} \tag{7.6}$$

Thus, when x_0 is in the sector (6.1) and x moves on the segment $\overline{0x_0}$ from 0 to x_0, $|u|$ is increasing. Hence, in order to assure the monotonicity on the behavior of $|u|$ as x moves on the segment $\overline{0x_0}$ from 0 to x_0, we have to confine the stable domain on $x-$plane to (6.1).

For u_0 in

$$|u| < \rho_3, \tag{7.7}$$

where ρ_3 is a small constant, and x_0 in (6.1), the solution of (7.1) satisfying $u(x_0) = u_0$ is given by

$$u(x) = u_0 \exp\left\{\lambda\left(\frac{1}{x_0} - \frac{1}{x}\right)\right\}. \tag{7.8}$$

Hence, when x_0 is in the sector (6.1) and u_0 is in (7.7), when x moves on the segment $\overline{0x_0}$ from 0 to x_0, the function $u(x)$ stays in (7.7). Therefore we can take (6.1) for the stable domain in $x-$plane and (7.7) for the stable domain in $u-$plane.

Remark. It is noteworthy that

$$\frac{d(\arg u)}{ds} = \frac{1}{s^2}\Im\left[\frac{\lambda}{x_0}\right] = \frac{|\lambda|}{s^2|x_0|}\sin\left[\arg\left(\frac{\lambda}{x_0}\right)\right].$$

Thus, when x moves on the segment $\overline{0x_0}$ from 0 to x_0, $\arg u$ is increasing when x_0 is in the half sector of (6.1)

$$\arg\lambda - \frac{\pi}{2} + \epsilon_1 < \arg x \leq \arg\lambda,$$

and it is decreasing when x_0 is in the other half sector of (6.1)

$$\arg\lambda \leq \arg x < \arg\lambda + \frac{\pi}{2} - \epsilon_1.$$

Case 2. When $\lambda_0(v)+\lambda_1(v)x \not\equiv 0$, we can choose ρ_1 and ρ_2 such that

$$(7.9) \qquad |\lambda_0(v)+\lambda_1(v)x| \le \frac{|\lambda|\sin\epsilon_1}{2},$$

for (x,v) in the sectors (6.1) and (6.18). Then

$$(7.10) \qquad \begin{aligned}\frac{1}{|u|}\frac{d|u|}{ds} &= \Re\left[\frac{x^2}{u}\frac{du}{dx}\frac{1}{x^2}\frac{dx}{ds}\right] = \Re\left[\{\lambda+\lambda_0(v)+\lambda_1(v)x\}\frac{1}{x^2}\frac{dx}{ds}\right]\\ &= \Re\left[\{\lambda+\lambda_0(v)+\lambda_1(v)x\}\frac{\exp[-i\arg x_0]}{|x|^2}\right]\\ &\ge \frac{|\lambda|}{|x|^2}\left\{\Re[\exp\{i(\arg\lambda-\arg x\}] - \frac{|\lambda_0(v)+\lambda_1(v)x|}{|\lambda|}\right\}\\ &= \frac{|\lambda|}{|x|^2}\left\{\cos(\arg\lambda-\arg x) - \frac{|\lambda_0(v)+\lambda_1(v)x|}{|\lambda|}\right\}\\ &\ge \frac{|\lambda|}{2|x|^2}\sin\epsilon_1 > 0,\end{aligned}$$

since $\arg x = \arg x_0$ for x on $\overline{0x_0}$. Thus, if we choose ρ_1, and ρ_2 small enough, then, $u(x,v)$ satisfying $u(x_0, v(x_0)) = u_0$ has $|u(x,v)|$ monotonic increasing and staying in (7.7) as x moves on the segment $\overline{0x_0}$ from 0 to x_0 while v is in (6.18).

IV. Asymptotic Coefficients

8. Recurrence Formulae. In order to find the solution of (E.1) in the form of (2.3), put

$$(8.1) \qquad y = P(x,v,u) = \sum_{k=1}^{\infty} P_k(x,v)u^k,$$

and

$$(8.2) \qquad z = Q(x,v,u) = Q_0(x,v) + x\sum_{k=1}^{\infty} Q_k(x,v)u^k,$$

where u and v satisfy (E.2). Differentiate (8.1) and (8.2), by (E.2), we have

$$(8.3) \qquad \begin{aligned} x^2y' &= \sum_{k=1}^{\infty}\left\{x^2\frac{dP_k}{dx} + kP_k\frac{x^2}{u}\frac{du}{dx}\right\}u^k\\ &= \sum_{k=1}^{\infty}\left\{x^2\frac{dP_k}{dx} + kP_k[\lambda+\lambda_0(v)+\lambda_1(v)x]\right\}u^k,\end{aligned}$$

and

$$
\begin{aligned}
xz' &= x\frac{dQ_0}{dx} + \sum_{k=1}^{\infty}\left\{x^2\frac{dQ_k}{dx} + kQ_k\frac{x^2}{u}\frac{du}{dx}\right\}u^k \\
&= x\frac{dQ_0}{dx} + \sum_{k=1}^{\infty}\left\{x^2\frac{dQ_k}{dx} + kQ_k[\lambda + \lambda_0(v) + \lambda_1(v)x]\right\}u^k.
\end{aligned} \tag{8.4}
$$

On the other hand, by (E.2),

$$
x^2y' = \lambda\left\{\sum_{k=1}^{\infty} P_k(x,v)u^k\right\} + f(x, P(x,v,u), Q(x,v,u)), \tag{8.5}
$$

$$
xz' = g(x, P(x,v,u), Q(x,v,u)). \tag{8.6}
$$

Put

$$
\frac{\partial f}{\partial y}(x,0,Q_0(x,v)) = H(x,v), \qquad \frac{\partial f}{\partial z}(x,0,Q_0(x,v)) = J(x,v), \tag{8.7}
$$

and

$$
\frac{\partial g}{\partial y}(x,0,Q_0(x,v)) = \hat{H}(x,v), \qquad \frac{\partial g}{\partial z}(x,0,Q_0(x,v)) = \hat{J}(x,v). \tag{8.8}
$$

Then,

$$
\begin{aligned}
f(x, P(x,v,u), &Q(x,v,u)) = f(x,0,Q_0(x,v)) \\
&+ \sum_{k=1}^{\infty}\Big[H(x,v)P_k(x,v) + xJ(x,v)Q_k(x,v) + G_k(x,v)\Big]u^k,
\end{aligned} \tag{8.9}
$$

and

$$
\begin{aligned}
g(x, P(x,v,u), &Q(x,v,u)) = g(x,0,Q_0(x,v)) \\
&+ \sum_{k=1}^{\infty}\Big[\hat{H}(x,v)P_k(x,v) + x\hat{J}(x,v)Q_k(x,v) + \hat{G}_k(x,v)\Big]u^k,
\end{aligned} \tag{8.10}
$$

where $G_k(x,v)$ and $\hat{G}_k(x,v)$ are functions depending on $P_1, Q_1, \cdots, P_{k-1}, Q_{k-1}$. In particular,

$$
G_1(x,v) \equiv 0, \qquad \hat{G}_1(x,v) \equiv 0. \tag{8.11}
$$

Note that, by Assumption II, $f(x, 0, Q_0(x, v)) \equiv 0$. Comparing the coefficients in (8.3) and (8.5) with (8.9) as well as (8.4) and (8.6) with (8.10), we have

$$x\frac{dQ_0}{dx} = g(x, 0, Q_0), \tag{8.12}$$

and

$$\begin{aligned} x^2\frac{dP_k}{dx} &= \{\lambda - k[\lambda + \lambda_0(v) + \lambda_1(v)x] + H(x,v)\}P_k + xJ(x,v)Q_k + G_k(x,v), \\ x^2\frac{dQ_k}{dx} &= \{-k[\lambda + \lambda_0(v) + \lambda_1(v)x] + x\hat{J}(x,v)\}Q_k + \hat{H}(x,v)P_k + \hat{G}_k(x,v), \\ &\qquad (k = 1, 2, 3, \cdots). \end{aligned} \tag{8.13}$$

9. The formal expansion of $Q_0(x, v)$. To find the formal expansion of the leading term $Q_0(x, v)$, put

$$Q_0(x, v) = Q_{00}(v) + \sum_{\ell=1}^{\infty} Q_{0\ell}(v)x^\ell. \tag{9.1}$$

Differentiate (9.1) formally and substitute into (8.12), by (E.2), we have

$$x\frac{dQ_0}{dx} = \frac{dQ_{00}}{dv}v^2(b + \hat{b}v) + \sum_{\ell=1}^{\infty}\left\{\frac{dQ_{0\ell}}{dx}v^2(b + \hat{b}v) + \ell Q_{0\ell}\right\}x^\ell. \tag{9.2}$$

On the other hand,

$$x\frac{dQ_0}{dx} = g(0, 0, Q_{00}) + \sum_{\ell=1}^{\infty}\left[\hat{J}_0(v)Q_{0\ell}(v) + \hat{G}_{0\ell}(v)\right]x^\ell, \tag{9.3}$$

where

$$\hat{J}_0(v) = \frac{\partial g}{\partial z}(0, 0, Q_{00}(v)), \tag{9.4}$$

and $\hat{G}_{0\ell}(v)$ ($\ell = 1, 2, \cdots$) is a polynomial in $Q_{0h}(v)$ ($h = 0, 1, \cdots, \ell - 1$). Comparing the coefficients in (9.2) and (9.3), we have

$$v^2(b + \hat{b}v)\frac{dQ_{00}}{dv} = g(0, 0, Q_{00}), \tag{9.5}$$

and

$$v^2(b + \hat{b}v)\frac{dQ_{0\ell}}{dv} = [-\ell + \hat{J}_0(v)]Q_{0\ell} + \hat{G}_{0\ell}(v), \qquad (\ell = 1, 2, \cdots). \tag{9.6}$$

To find $Q_{00}(v)$, note from (1.6) that

$$g(0,0,Q_{00}) = \sum_{j=2}^{\infty} B_j Q_{00}^j, \tag{9.7}$$

and (9.5) becomes

$$v^2(b+\hat{b}v)\frac{dQ_{00}}{dv} = \sum_{j=2}^{\infty} B_j Q_{00}^j. \tag{9.8}$$

Put

$$Q_{00}(v) = v\hat{Q}(v). \tag{9.9}$$

Then (9.8) becomes

$$v^2(b+\hat{b}v)\left[\hat{Q} + v\frac{d\hat{Q}}{dv}\right] = \sum_{j=2}^{\infty} B_j v^j \hat{Q}^j, \tag{9.10}$$

or,

$$(b+\hat{b}v)\left[\hat{Q} + v\frac{d\hat{Q}}{dv}\right] = \sum_{j=2}^{\infty} B_j v^{j-2} \hat{Q}^j. \tag{9.11}$$

Since $b \neq 0$, $v = 0$ is a regular type singularity of (9.11) and, similar to (5.25), by Briot-Bouquet theory, there exists a series solution $\hat{Q}(v)$ convergent at $v = 0$. Hence, there exists a solution $Q_{00}(v)$ of (9.5) in convergent power series of v at $v = 0$.

Suppose that we have $Q_{0h}(v)$ $(h = 0, 1, \cdots, \ell - 1)$ admitting asymptotic expansions in power series of v in the sector (6.18). To find $Q_{0\ell}(v)$ from (9.6), note that (9.6) can be written as

$$v^2\frac{dQ_{0\ell}(v)}{dv} = \left[-\frac{\ell}{b} + J_{0\ell}(v)\right]Q_{0\ell}(v) + K_{0\ell}(v), \tag{9.12}$$

where $J_{0\ell}(v)$ and $K_{0\ell}(v)$ are functions of v admitting asymptotic expansions in power series of v in the sector (6.18), and furthermore $J_{0\ell}(0) = 0$, by (1.8), (9.4) and (9.9). Note that $v = 0$ is an irregular singular point of (9.12). Since $\ell > 0$, (9.12) has a formal power series solution

$$Q_{0\ell}(v) \sim \sum_{j=0}^{\infty} Q_{0\ell j} v^j. \tag{9.13}$$

By the result of M. Hukuhara [Hu3], (see also M. Iwamo [I1]), there exists a solution $Q_{0\ell}(v)$ analytic in (6.18) and admits (9.13) as an asymptotic expansion in the sector (6.18).

In this way, we obtain a formal series solution (9.1) for (8.12). The asymptotic meaning of this formal series (9.1) will be studied in Sections 11, 13 and 14.

10. The expansions of $P_k(x,v)$ and $Q_k(x,v)$. To find the formal solutions $P_k(x,v)$ and $Q_k(x,v)$ of (8.13) for $k=1,2,\cdots$, put

$$P_k(x,v) \sim \sum_{\ell=0}^{\infty} P_{k\ell}(v)x^{\ell}, \tag{10.1}$$

and

$$Q_k(x,v) \sim \sum_{\ell=0}^{\infty} Q_{k\ell}(v)x^{\ell}. \tag{10.2}$$

Differentiate (10.1) and (10.2) formally and by (E.2), we have

$$x^2\frac{dP_k}{dx} = \sum_{\ell=0}^{\infty}\left\{\frac{dP_{k\ell}}{dv}v^2(b+\hat{b}v) + \ell P_{k\ell}\right\}x^{\ell+1}, \tag{10.3}$$

and

$$x^2\frac{dQ_k}{dx} = \sum_{\ell=0}^{\infty}\left\{\frac{dQ_{k\ell}}{dv}v^2(b+\hat{b}v) + \ell Q_{k\ell}\right\}x^{\ell+1}. \tag{10.4}$$

Let

$$\begin{aligned} H(x,v) &= \sum_{\ell=0}^{\infty} H_\ell(v)x^{\ell}, & J(x,v) &= \sum_{\ell=0}^{\infty} J_\ell(v)x^{\ell}, \\ \hat{H}(x,v) &= \sum_{\ell=0}^{\infty} \hat{H}_\ell(v)x^{\ell}, & \hat{J}(x,v) &= \sum_{\ell=0}^{\infty} \hat{J}_\ell(v)x^{\ell}, \\ G_k(x,v) &= \sum_{\ell=0}^{\infty} G_{k\ell}(v)x^{\ell}, & \hat{G}_k(x,v) &= \sum_{\ell=0}^{\infty} \hat{G}_{k\ell}(v)x^{\ell}. \end{aligned} \tag{10.5}$$

Note that

$$\begin{gathered} H_0(v) = \frac{\partial f}{\partial y}(0,0,Q_{00}(v)), \qquad J_0(v) = \frac{\partial f}{\partial z}(0,0,Q_{00}(v)), \\ \hat{H}_0(v) = \frac{\partial g}{\partial y}(0,0,Q_{00}(v)). \end{gathered} \tag{10.6}$$

By (9.9), (1.8), (9.4) and (10.6), we have

$$H_0(0) = J_0(0) = \hat{H}_0(0) = \hat{J}_0(0) = 0. \tag{10.7}$$

On the other hand, by (8.13), (10.1), (10.2) and (10.5),

$$\begin{aligned} x^2\frac{dP_k}{dx} &= \Big\{\lambda - k\lambda - k\lambda_0(v) + H_0(v) - [k\lambda_1(v) - H_1(v)]x \\ &\quad + \sum_{\ell=2}^{\infty} H_\ell(v)x^\ell\Big\}\sum_{\ell=0}^{\infty} P_{k\ell}(v)x^\ell \\ &\quad + x\Big[\sum_{\ell=0}^{\infty} J_\ell(v)x^\ell\Big]\Big[\sum_{\ell=0}^{\infty} Q_{k\ell}(v)x^\ell\Big] + \sum_{\ell=0}^{\infty} G_{k\ell}(v)x^\ell \\ &= \sum_{\ell=0}^{\infty}\Big\{\big[\lambda - k\lambda - k\lambda_0(v) + H_0(v)\big]P_{k\ell}(v) + F_{k\ell}(v)\Big\}x^\ell, \end{aligned} \tag{10.8}$$

$$\begin{aligned} x^2\frac{dQ_k}{dx} &= \Big\{ - \big[k\lambda + k\lambda_0(v) + \lambda_1(v)x\big] + x\sum_{\ell=0}^{\infty} \hat{J}_\ell(v)x^\ell\Big\}\sum_{\ell=0}^{\infty} Q_{k\ell}(v)x^\ell \\ &\quad + \Big[\sum_{\ell=0}^{\infty} \hat{H}_\ell(v)x^\ell\Big]\Big[\sum_{\ell=0}^{\infty} P_{k\ell}(v)x^\ell\Big] + \sum_{\ell=0}^{\infty} \hat{G}_{k\ell}(v)x^\ell \\ &= \sum_{\ell=0}^{\infty}\Big\{\big[-k\lambda - k\lambda_0(v)\big]Q_{k\ell}(v) + \hat{H}_0(v)P_{k\ell}(v) + K_{k\ell}(v)\Big\}x^\ell, \end{aligned} \tag{10.9}$$

where $F_{k\ell}(v)$ and $K_{k\ell}(v)$ are polynomials in $P_{kj}(v)$ and $Q_{kj}(v)$ $(j = 0, 1, \cdots, \ell-1)$. In particular, by (8.11),

$$F_{10}(v) = K_{10}(v) = 0. \tag{10.10}$$

Thus, by (10.3), (10.4), (10.8) and (10.9), we have

$$[\lambda - k\lambda - k\lambda_0(v) + H_0(v)]P_{k0}(v) + F_{k0}(v) = 0, \tag{10.11}$$

$$-k[\lambda + \lambda_0(v)]Q_{k0}(v) + \hat{H}_0(v)P_{k0}(v) + K_{k0}(v) = 0, \tag{10.12}$$

$$\begin{aligned} &[\lambda - k\lambda - k\lambda_0(v) + H_0(v)]P_{k\ell}(v) + F_{k\ell}(v) - v^2(b + \hat{b}v)\frac{dP_{k,\ell-1}}{dv} \\ &\quad - (\ell-1)P_{k,\ell-1}(v) = 0, \qquad (\ell = 1, 2, \cdots; k = 1, 2, \cdots), \end{aligned} \tag{10.13}$$

and

$$(10.14)\quad \begin{aligned}&[-k\lambda - k\lambda_0(v)]Q_{k\ell}(v) + \hat{H}_0(v)P_{k\ell}(v) + K_{k\ell}(v)\\ &- v^2(b+\hat{b}v)\frac{dQ_{k,\ell-1}}{dv} - (\ell-1)Q_{k,\ell-1}(v) = 0, \quad (\ell = 1,2,\cdots; k = 1,2,\cdots).\end{aligned}$$

For $k = 1$, comparing (5.1), (3.11), (3.5), (E.1) and (1.6), we see that

$$(10.15)\quad \sum_{i=0}^{\infty} a_i(v)x^i = \frac{\partial f}{\partial y}(x, 0, Q_0(x,v)),$$

$$(10.16)\quad \lambda_0(v) = H_0(v), \quad \lambda_1(v) = \text{first two terms of } H_1(v).$$

Thus, from (10.3) and (10.8), by (8.11), we have

$$(10.17)\quad \begin{aligned}&\left\{[H_1(v) - \lambda_1(v)] + \sum_{\ell=2}^{\infty} H_{k\ell}(v)x^{\ell-1}\right\}\sum_{\ell=0}^{\infty} P_{1\ell}(v)x^\ell\\ &+ \left[\sum_{\ell=0}^{\infty} J_\ell(v)x^\ell\right]\left[\sum_{\ell=0}^{\infty} Q_{1\ell}(v)x^\ell\right]\\ &= \sum_{\ell=0}^{\infty}\left\{\frac{dP_{1\ell}}{dv}v^2(b+\hat{b}v) + \ell P_{1\ell}(v)\right\}x^\ell.\end{aligned}$$

Thus, by (10.17) and (10.12), and due to (8.11),

$$(10.18)\quad v^2(b+\hat{b}v)\frac{dP_{10}}{dv} = [H_1(v) - \lambda_1(v)]P_{10}(v) + J_0(v)Q_{10}(v),$$

$$(10.19)\quad -[\lambda + \lambda_0(v)]Q_{10}(v) + \hat{H}_0(v)P_{10}(v) = 0,$$

and

$$(10.20)\quad v^2(b+\hat{b}v)\frac{dP_{1\ell}}{dv} = [H_1(v) - \lambda_1(v) - \ell]P_{1\ell}(v) + J_0(v)Q_{1\ell}(v) + F_{1\ell}(v), \quad (\ell = 1,2,\cdots).$$

From (10.19), we have

$$(10.21)\quad Q_{10}(v) = \frac{\hat{H}_0(v)}{\lambda + \lambda_0(v)}P_{10}(v).$$

Substitute (10.21) into (10.18), we have

$$(10.22)\quad v^2(b+\hat{b}v)\frac{dP_{10}}{dv} = \left\{\frac{\hat{H}_0(v)J_0(v)}{\lambda + \lambda_0(v)} + [H_1(v) - \lambda_1(v)]\right\}P_{10}(v).$$

By (10.7) and (10.16), the coefficient on the right hand side is of $O(v^2)$ and $v = 0$ is an ordinary point of (10.22). Thus there is a solution of $P_{10}(v)$ in a convergent power series of v with $P_{10}(0) \neq 0$, and consequently, $Q_{10}(v)$ is a convergent power series of v given by (10.21).

For $k = 1$ and $\ell \geq 1$, from (10.14), we have

$$Q_{1\ell}(v) = \frac{\hat{H}_0(v)}{\lambda + \lambda_0(v)} P_{1\ell}(v) + \tilde{K}_{1\ell}(v), \tag{10.23}$$

where $\tilde{K}_{1\ell}(v)$ is a polynomial in $P_{1j}(v), Q_{1j}(v)$ $(j = 0, 1, \cdots, \ell - 1)$ and their derivatives analytic in (6.18) and admits an asymptotic expansion in power series of v as v tends to 0 in (6.18). Substituting (10.23) into (10.20), we have

$$v^2(b + \hat{b}v)\frac{dP_{1\ell}}{dv} = \left\{ -\ell + H_1(v) - \lambda_1(v) + \frac{\hat{H}_0(v)J_0(v)}{\lambda + \lambda_0(v)} \right\} P_{1\ell}(v) + \tilde{F}_{1\ell}(v), \tag{10.24}$$

where $\tilde{F}_{1\ell}(v)$ is analytic in (6.18) and admits an asymptotic expansion in power series of v as v tends to 0 in (6.18). Since $\ell \geq 1$, there exists a formal power series solution $P_{1\ell}$ of (10.24) for each ℓ. Since $v = 0$ is an irregular singular point of (10.24), similar to that for (9.12), there exists a solution $P_{1\ell}(v)$ for (10.24) admitting the formal solution as an asymptotic expansion in the sector (6.18). Furthermore, $Q_{1\ell}(v)$ is obtained from (10.23), analytic in (6.18) and admits an asymptotic expansion in power series of v as v tends to 0 in (6.18).

For $k = 2, 3, \cdots$, $P_{k0}(v)$ is uniquely determined from (10.11) and $Q_{k0}(v)$ is uniquely determined from (10.12) each as an analytic function of v admitting an asymptotic expansion in power series of v as v tends to 0 in (6.18). For $\ell \geq 1$, $P_{k\ell}(v)$ is determined from (10.13) as an analytic function depending on $P_{kj}(v)$ and $Q_{kj}(v)$ $(j = 0, 1, \cdots, \ell - 1)$ and their derivatives admitting an asymptotic expansion in power series of v as v tends to 0 in (6.18). Consequently, $Q_{k\ell}(v)$ is determined from (10.14) analytic in (6.18) and admitting an asymptotic expansion in power series of v as v tends to 0 in the sector (6.18).

Thus we obtain the following

PROPOSITION 10-1. *The system (E.1) has a formal solution (8.1) and (8.2) where*

(i) $Q_0(x, v)$ *is given by a formal series (9.1) with* $Q_{00}(v)$ *a convergent power series of* v *and* $Q_{0\ell}(v)$ $(\ell = 1, 2, \cdots)$ *analytic in (6.18) admitting the asymptotic expansion (9.13) as* v *tends to 0 in (6.18);*

(ii) $P_k(x, v)$ *and* $Q_k(x, v)$ $(k = 1, 2, \cdots)$ *are given by the formal sereis (10.1) and (10.2), respectively, with coefficients* $P_{k\ell}(v)$ *and* $Q_{k\ell}(v)$ $(\ell = 1, 2, \cdots)$ *analytic in (6.18) admitting asymptotic expansions in power series of* v *as* v *tends to 0 in (6.18).*

The asymptotic meanings of (10.1) and (10.2) will be studied in Sections 11, 13 and 14 while that of (8.1) and (8.2) will be done in Sections 12, 15 and 16.

V. General Solutions

11. Asymptotic solutions $P_k(x,v)$ and $Q_k(x,v)$. We will investigate first the asymptotic meaning of the formal solutions $Q_0(x,v)$ of (8.12) obtained in Section 9 and $P_k(x,v)$ and $Q_k(x,v)$ $(k=1,2,\cdots)$ of (8.13) obtained in Section 10. We will establish the following two Propositions.

PROPOSITION 11-1. *Let (x_0,v_0) be an arbitrary point in*

$$(11.1)\qquad \begin{aligned} &|x|<\rho_1, \quad \arg\lambda-\frac{\pi}{2}+\epsilon_1<\arg x<\arg\lambda+\frac{\pi}{2}-\epsilon_1,\\ &|v|<r(\arg v;\rho_2,\epsilon_2), \quad -\pi-\arg b+\epsilon_2<\arg v<\pi-\arg b-\epsilon_2,\end{aligned}$$

where $\rho_1,\rho_2,\epsilon_1,\epsilon_2$ $(\rho_1<1)$ are suitable small constants. Let $V(x)$ be the solution of (6.3) such that $V(x_0)=v_0$. Then, (8.12) has the asymptotic solution $Q_0(x,V(x))$ where

$$(11.2)\qquad Q_0(x,v)\simeq Q_{00}(v)+\sum_{\ell=1}^{\infty}Q_{0\ell}(v)x^{\ell},$$

uniformly in (6.18) as x tends to 0 in the sector (6.1). Here the coefficients $Q_{0\ell}(v)$ $(\ell=0,1,\cdots)$ are given in Proposition 10-1.

PROPOSITION 11-2. *Let (x_0,v_0) be an arbitrary point in (11.1) and $V(x)$ be the solution of (6.3) such that $V(x_0)=v_0$. Then, for $k=1,2,\cdots$, (8.13) has the asymptotic solutions $\{P_k(x,V(x)),Q_k(x,V(x))\}$ where*

$$(11.3)\qquad \begin{aligned} P_k(x,v)&\simeq P_{k0}(v)+\sum_{\ell=1}^{\infty}P_{k\ell}(v)x^{\ell},\\ Q_k(x,v)&\simeq Q_{k0}(v)+\sum_{\ell=1}^{\infty}Q_{k\ell}(v)x^{\ell},\end{aligned}$$

uniformly in (6.18) as x tends to 0 in the sector (6.1). Here the coefficients $P_{k\ell}(v)$ and $Q_{k\ell}(v)$ $(\ell=0,1,\cdots;\ k=1,2,\cdots)$ are given in Proposition 10-1.

The proof of Proposition 11-1 is similar to that of Proposition 11-2. Thus, we provide only the proof of Proposition 11-2 in Sections 13 and 14.

12. Analytic solutions $P(x,v,u)$ and $Q(x,v,u)$. Utilizing the results of Propositions 11-1 and 11-2, we will establish

PROPOSITION 12-1. *Let (x_0, v_0, u_0) be an arbitrary point in*

$$
(12.1)\qquad
\begin{aligned}
&|x| < \rho_1, \quad \arg\lambda - \frac{\pi}{2} + \epsilon_1 < \arg x < \arg\lambda + \frac{\pi}{2} - \epsilon_1,\\
&|v| < r(\arg v; \rho_2, \epsilon_2), \quad -\pi - \arg b + \epsilon_2 < \arg v < \pi - \arg b - \epsilon_2,\\
&|u| < \rho_3,
\end{aligned}
$$

where $\rho_1, \rho_2, \epsilon_1, \epsilon_2$ are small constants given in Proposition 11-1 *and ρ_3 is a suitable small constant. Let $V(x)$ be the solution of (6.3) such that $V(x_0) = v_0$ and $U(x)$ be the solution of (6.2) such that $U(x_0) = u_0$. Then, (E.1) has the analytic solution $\{P(x, U(x), V(x)), Q(x, U(x), V(x))\}$ where*

$$
(12.2)\qquad
\begin{aligned}
P(x, v, u) &= \sum_{k=1}^{\infty} P_k(x, v) u^k,\\
Q(x, v, u) &= Q_0(x, v) + x \sum_{k=1}^{\infty} Q_k(x, v) u^k,
\end{aligned}
$$

converge uniformly in (7.7). Here the coefficients $Q_0(x, v)$, $P_k(x, v)$ and $Q_k(x, v)$ $(k = 1, 2, \cdots)$ are given in (11.2) and (11.3) satisfying the properties described in Propositions 11-1 *and* 11-2.

This proposition is to be proved in Sections 15 and 16.

Combining Propositions 11-1, 11-2 and 12-1, Theorem M is proved.

13. Proof of Proposition 11-2. As the situations for $k = 1$ and $k > 1$ of Proposition 11-2 are different, we have to deal with them separately. Moreover, the proof for the case $k > 1$ can be obtained by minor adjustment from that for the case $k = 1$, we will prove only the case of $k = 1$ here.

Note that when $k = 1$, by (8.1), (8.13) becomes

$$
(13.1)\qquad
\begin{aligned}
x\frac{dP_1}{dx} &= \{[H_1(v) - \lambda_1(v)] + x\tilde{H}(x, v)\}P_1 + J(x, v)Q_1,\\
x^2\frac{dQ_1}{dx} &= \{-[\lambda + \lambda_0(v) + \lambda_1(v)x] + x\hat{J}(x, v)\}Q_1 + \hat{H}(x, v)P_1,
\end{aligned}
$$

where $\tilde{H}(x, v) = x^{-2}[H(x, v) - H_0(v) - H_1(v)x]$. We obtained the formal solution (10.1) and (10.2) with $P_{10}(v)$ and $Q_{10}(v)$ analytic at $v = 0$ while $P_{1\ell}(v)$ and $Q_{1\ell}(v)$ $(\ell = 1, 2, \cdots)$ admitting asymptotic expansions in power series of v as v tends to 0 in (6.18).

Let (x_0, v_0) be an arbitrary point in (11.1). Let $V(x)$ be the solution of (6.3) such that $V(x_0) = v_0$. To show that there exists a solution $\{P_1(x, V(x)), Q_1(x, (V(x))\}$ of (13.1) satisfying Proposition 11-2, let ϵ_1, ϵ_2 and α be fixed small positive constants such that

$$
(13.2)\qquad |\lambda| \sin\epsilon_1 < 1, \qquad \alpha = \frac{|\lambda| \sin\epsilon_1}{4},
$$

and N be a positive integer satisfying

$$N \geq |\lambda| \sin \epsilon_1. \tag{13.3}$$

Pick ρ_1 and ρ_2 so small that

$$\begin{aligned} &|H_1(v) - \lambda_1(v)| + |x\tilde{H}(x,v)| \leq \alpha, \qquad |J(x,v)| \leq \alpha, \\ &|\hat{H}(x,v)| \leq \alpha, \qquad |\lambda_0(v)| + |\lambda_1(v)x| + |x\tilde{J}(x,v)| \leq \alpha, \end{aligned} \tag{13.4}$$

are satisfied for (x, v) is (11.1). Note that (7.9) follows from (13.4). Let

$$P_1^{(N)}(x,v) = \sum_{\ell=0}^{N-1} P_{1\ell}(v)x^\ell, \quad Q_1^{(N)}(x,v) = \sum_{\ell=0}^{N-1} Q_{1\ell}(v)x^\ell, \tag{13.5}$$

and make a change of variables

$$P_1 = P_1^{(N)}(x, V(x)) + \eta_N, \qquad Q_1 = Q_1^{(N)}(x, V(x)) + \zeta_N. \tag{13.6}$$

Then $\{\eta_N, \zeta_N\}$ satisfies the system of equations

$$\begin{aligned} x\frac{d\eta_N}{dx} &= \{[H_1(V(x)) - \lambda_1(V(x))] + x\tilde{H}(x, V(x))\}\eta_N \\ &\quad + J(x, V(x))\zeta_N + F_1^{(N)}(x, V(x)), \\ x^2\frac{d\zeta_N}{dx} &= \{-[\lambda + \lambda_0(V(x)) + \lambda_1(V(x))x] + x\tilde{J}(x, V(x))\}\zeta_N \\ &\quad + \hat{H}(x, V(x))\eta_N + G_1^{(N)}(x, V(x)), \end{aligned} \tag{13.7}$$

and (13.7) possesses a formal solution

$$\begin{aligned} \eta_N &\sim \sum_{\ell=N}^{\infty} P_{1\ell}(V(x))x^\ell, \\ \zeta_N &\sim \sum_{\ell=N}^{\infty} Q_{1\ell}(V(x))x^\ell. \end{aligned} \tag{13.8}$$

Moreover, $F_1^{(N)}(x, v)$ and $G_1^{(N)}(x, v)$ are analytic in (11.1), and there exists a positive constant B_N such that

$$|F_1^{(N)}(x,v)| \leq B_N|x|^N, \qquad |G_1^{(N)}(x,v)| \leq B_N|x|^N, \tag{13.9}$$

for (x, v) in (11.1). Then, Proposition 11-2 is proved if we prove the following

PROPOSITION 13-1. *Let N be an integer satisfying (13.3). Then, (13.7) has a unique solution $\{\eta_N(x, V(x)),\ \zeta_N(x, V(x))\}$ such that*

$$|\eta_N(x, V(x))| \le K_N |x|^N, \qquad |\zeta_N(x, V(x))| \le K_N |x|^N \tag{13.10}$$

for a suitable positive constant K_N whenever $(x, V(x))$ is in a domain of the form

$$\begin{aligned} |x| < \xi_N, &\quad \arg\lambda - \frac{\pi}{2} + \epsilon_1 < \arg x < \arg\lambda + \frac{\pi}{2} - \epsilon_1, \\ |v| < r(\arg v; \delta_N, \epsilon_2), &\quad -\pi - \arg b + \epsilon_2 < \arg v < \pi - \arg b - \epsilon_2. \end{aligned} \tag{13.11-N}$$

Here $0 < \xi_N < \rho_1 < 1$ and $0 < \delta_N < \rho_2$.

Proposition 11-2 follows from Proposition 13-1 in the following manner. By the transformation (13.6), (13.7) has a solution

$$\sum_{\ell=0}^{N-1} P_{1\ell}(V(x))x^\ell + \eta_N(x, V(x)), \quad \sum_{\ell=0}^{N-1} Q_{1\ell}(V(x))x^\ell + \zeta_N(x, V(x)), \tag{13.12}$$

provided that $(x, V(x))$ is in (13.11-N). Let N' be an integer greater than N. Then

$$\sum_{\ell=N}^{N'-1} P_{1\ell}(V(x))x^\ell + \eta_{N'}(x, V(x)), \quad \sum_{\ell=N}^{N'-1} Q_{1\ell}(V(x))x^\ell + \zeta_{N'}(x, V(x)), \tag{13.13}$$

is a solution of (13.7). This solution satisfies the condition (13.10) if $(x, V(x))$ is in the common part of (13.11-N) and (13.11-N'). Hence, by the uniqueness of the solution of (13.7), as assured by Proposition 13-1, (13.13) must coincide with $\{\eta_N(x, V(x)), \zeta_N(x, V(x))\}$. Thus the solution expressed by (13.12) is independent of N, provided that N satisfies (13.3). Denote this solution by $\{\tilde{\eta}(x, V(x)), \tilde{\zeta}(x, V(x))\}$. Then, by means of analytic continuation, the function $\tilde{\eta}(x, v)$ and $\tilde{\zeta}(x, v)$ are defined in the domain

$$\begin{aligned} |x| < \xi_0, &\quad \arg\lambda - \frac{\pi}{2} + \epsilon_1 < \arg x < \arg\lambda + \frac{\pi}{2} - \epsilon_1, \\ |v| < r(\arg v; \delta_0, \epsilon_2), &\quad -\pi - \arg b + \epsilon_2 < \arg v < \pi - \arg b - \epsilon_2. \end{aligned} \tag{13.14}$$

Here $\xi_0 = \sup \xi_N$ and $\delta_0 = \sup \delta_N$. Thus, Proposition 11-2 is proved.

14. Proof of Proposition 13-1. To show Proposition 13-1, let (x_1, v_1) be an arbitrary point of (13.11-N) and $V(x)$ be the solution of (6.3) such that $V(x_1) = v_1$. Note that $\{\eta_N, \zeta_N\}$

are given by the integral equations

$$(14.1)\quad \begin{aligned} \eta_N(x_1,v_1) &= \int_0^{x_1} \frac{1}{x}\Big\{[H_1(V(x)) - \lambda_1(V(x)) + x\tilde{H}(x,V(x))]\eta_N \\ &\qquad + J(x,V(x))\zeta_N + F_1^{(N)}(x,V(x))\Big\}dx, \\ \zeta_N(x_1,v_1) &= \exp\left[\frac{\lambda}{x_1}\right]\int_0^{x_1} \frac{1}{x^2}\exp\left[\frac{-\lambda}{x}\right]\Big\{[-\lambda_0(V(x)) - \lambda_1(V(x))x \\ &\qquad + x\tilde{J}(x,V(x))]\zeta_N + \hat{H}(x,V(x))\eta_N + G_1^{(N)}(x,V(x))\Big\}dx, \end{aligned}$$

where the integral is taken on the segment $\overline{0x_1}$. Consider the successive approximations:

$$(14.2)\quad \begin{aligned} &\eta_N^{(0)}(x_1,v_1) = 0, \qquad \zeta_N^{(0)}(x_1,v_1) = 0, \\ &\eta_N^{(h+1)}(x_1,v_1) = \int_0^{x_1} \frac{1}{x}\Big\{[H_1(V(x)) - \lambda_1(V(x)) + x\tilde{H}(x,V(x))]\eta_N^{(h)} \\ &\qquad + J(x,V(x))\zeta_N^{(h)} + F_1^{(N)}(x,V(x))\Big\}dx \\ &\zeta_N^{(h+1)}(x_1,v_1) = \exp\left[\frac{\lambda}{x_1}\right]\int_0^{x_1} \frac{1}{x^2}\exp\left[\frac{-\lambda}{x}\right]\Big\{[-\lambda_0(V(x)) - \lambda_1(V(x))x \\ &\qquad + x\tilde{J}(x,V(x))]\zeta_N^{(h)} + \hat{H}(x,V(x))\eta_N^{(h)} + G_1^{(N)}(x,V(x))\Big\}dx, \\ &\qquad (h = 0,1,2,\cdots). \end{aligned}$$

Note that

$$(14.3)\quad \begin{aligned} &\frac{d}{ds}|x|^N = N|x|^{N-1}, \qquad (s = |x|), \\ &\frac{d}{ds}|x|^N \exp\left[\Re\Big(-\frac{\lambda}{x}\Big)\right] = |x|^N\left\{\frac{N}{|x|} + \frac{|\lambda|}{|x|^2}\cos(\arg\lambda - \arg x)\right\}\exp\left[\Re\Big(-\frac{\lambda}{x}\Big)\right] \\ &\qquad \geq |x|^{N-2}|\lambda|\sin\epsilon_1 \exp\left[\Re\Big(-\frac{\lambda}{x}\Big)\right]. \end{aligned}$$

Let

$$(14.4)\quad K_N = \frac{2B_N}{|\lambda|\sin\epsilon_1}.$$

Then, by (13.2), we have

$$(14.5)\quad K_N \geq 2B_N.$$

Now, we can show that

$$
(14.6)\qquad
\begin{aligned}
&|\eta^{(h)}(x_1,v_1)-\eta^{(h-1)}(x_1,v_1)|\le \frac{K_N}{2^h}|x_1|^N,\\
&|\eta^{(h)}(x_1,v_1))|\le K_N\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^h}\right)|x_1|^N,\\
&|\zeta^{(h)}(x_1,v_1)-\zeta^{(h-1)}(x_1,v_1)|\le \frac{K_N}{2^h}|x_1|^N,\\
&|\zeta^{(h)}(x_1,v_1))|\le K_N\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^h}\right)|x_1|^N,\\
&\qquad\qquad\qquad\qquad\qquad\qquad (h=1,2,\cdots),
\end{aligned}
$$

for (x_1,v_1) in (13.11-N).

By means of (14.6), $\{\eta^{(h)}(x_1,v_1)\}$ and $\{\zeta^{(h)}(x_1,v_1)\}$ converge absolutely and uniformly to $\eta_N(x_1,v_1)$ and $\zeta_N(x_1,v_1)$, respectively, in (13.11-N) satisfying Proposition 13-1.

To see (14.6), by (14.2), (13.9), (14.3) and (14.5), we have

$$
(14.7)\qquad
\begin{aligned}
|\eta_N^{(1)}(x_1,v_1)| &= \left|\int_0^{x_1}\frac{1}{x}F_1^{(N)}(x,V(x))dx\right|\\
&\le B_N\left|\int_0^{x_1}|x|^{N-1}ds\right| = \frac{B_N}{N}|x_1|^N \le \frac{K_N}{2}|x_1|^N,\\
|\zeta_N^{(1)}(x_1,v_1)| &= \left|\exp\left[\frac{\lambda}{x_1}\right]\int_0^{x_1}\frac{1}{x^2}\exp\left[-\frac{\lambda}{x}\right]G_1^{(N)}(x,V(x))ds\right|\\
&\le B_N\exp\left[\Re\left(\frac{\lambda}{x_1}\right)\right]\left|\int_0^{x_1}|x|^{N-2}\exp\left[\Re\left(-\frac{\lambda}{x}\right)\right]ds\right|\\
&\le \frac{B_N}{|\lambda|\sin\epsilon_1}|x_1|^N = \frac{K_N}{2}|x_1|^N.
\end{aligned}
$$

Thus (14.6) is true for $h=1$.

Now assume that (14.6) is true for $h=j-1$. Then, by (14.2), the assumption on (14.6), (14.3), (13.2), (13.3), (13.4) and (13.9), we have $N>4\alpha$ and

$$
(14.8)\qquad
\begin{aligned}
&|\eta_N^{(j)}(x_1,v_1)-\eta_N^{(j-1)}(x_1,v_1)|\\
&\le \alpha\frac{K_N}{2^{j-1}}\left|\int_0^{x_1}|x|^{N-1}ds\right| = \frac{\alpha K_N}{2^{j-1}N}|x_1|^N \le \frac{K_N}{2^j}|x_1|^N,\\
&|\zeta_N^{(j)}(x_1,v_1)-\zeta_N^{(j-1)}(x_1,v_1)|\\
&\le \frac{K_N}{2^j}\exp\left[\Re\left(\frac{\lambda}{x_1}\right)\right]\left|\int_0^{x_1}|x|^{N-2}\exp\left[\Re\left(-\frac{\lambda}{x}\right)\right]ds\right|\\
&\le \frac{K_N}{2^{j-1}}\frac{\alpha}{|\lambda|\sin\epsilon_1}|x_1|^N \le \frac{K_N}{2^j}|x_1|^N.
\end{aligned}
$$

Thus (14.6) is true for $h = j$. Hence, by the principle of mathematical induction, (14.6) is true for all positive integers h.

15. Proof of Proposition 12-1. Let (x_0, v_0, u_0) be an arbitrary point in (12.1). Let $V(x)$ be the solution of (6.3) such that $V(x_0) = v_0$ and $U(x)$ be the solution of (6.2) such that $U(x_0) = u_0$. To show that there exists a solution $\{P(x, V(x), U(x)),\ Q(x, (V(x), U(x))\}$ of (E.1) satisfying Proposition 12-1, let ϵ_1, ϵ_2, ρ_1, ρ_2 and α be fixed small positive constants such that (13.2) and (13.4) are satisfied. Let N be a positive integer satisfying

$$N \geq \max\left\{\frac{8}{|\lambda|\sin\epsilon_1},\quad |\lambda|\sin\epsilon_1\right\}. \tag{15.1}$$

Let

$$\begin{aligned} P^{(N)}(x, V(x), U(x)) &= \sum_{k=1}^{N-1} P_k(x, V(x))U(x)^k, \\ Q^{(N)}(x, V(x), U(x)) &= Q_0(x, V(x)) + x\sum_{k=1}^{N-1} Q_k(x, V(x))U(x)^k, \end{aligned} \tag{15.2}$$

and make a change of variables

$$P = P^{(N)}(x, V(x), U(x)) + \phi_N, \qquad Q = Q^{(N)}(x, V(x), U(x)) + \psi_N. \tag{15.3}$$

Then $\{\phi_N, \psi_N\}$ satisfies the system of equations

$$\begin{aligned} x^2\frac{d\phi_N}{dx} &= \lambda\phi_N + F_N(x, V(x), U(x); \phi_N, \psi_N), \\ x\frac{d\psi_N}{dx} &= G_N(x, V(x), U(x); \phi_N, \psi_N), \end{aligned} \tag{15.4}$$

and (15.4) possesses a formal solution

$$\begin{aligned} \phi_N &\sim \sum_{k=N}^{\infty} P_k(x, V(x))U(x)^k, \\ \psi_N &\sim x\sum_{k=N}^{\infty} Q_k(x, V(x))U(x)^k. \end{aligned} \tag{15.5}$$

Moreover, $F_N(x,v,u;\phi_N,\psi_N)$ and $G_N(x,v,u;\phi_N,\psi_N)$ are analytic in

$$\begin{aligned}
&|x| < \xi_N, \quad \arg\lambda - \frac{\pi}{2} + \epsilon_1 < \arg x < \arg\lambda + \frac{\pi}{2} - \epsilon_1,\\
&|v| < r(\arg v; \delta_N, \epsilon_2), \quad -\pi - \arg b + \epsilon_2 < \arg v < \pi - \arg b - \epsilon_2,\\
&|u| < \gamma_N,\\
&|\phi_N| < \beta_N, \qquad |\psi_N| < \beta_N.
\end{aligned} \tag{15.6-N}$$

Here $0 < \xi_N < \rho_1 < 1$, $0 < \delta_N < \rho_2$ and γ_N and β_N are small constants. Furthermore, there exists a positive constant A, independent of N, and a positive constant $\hat{K}_N$ such that

$$\begin{aligned}
|F_N(x,v,u;\phi_N,\psi_N)| &\le A(|\phi| + |\psi|) + \hat{K}_N|u|^N,\\
|G_N(x,v,u;\phi_N,\psi_N)| &\le A(|\phi| + |\psi|) + \hat{K}_N|u|^N,
\end{aligned} \tag{15.7}$$

and

$$\begin{aligned}
|F_N(x,v,u;\phi_N,\psi_N) - F_N(x,v,u;\hat{\phi}_N,\hat{\psi}_N)| &\le A(|\phi_N - \hat{\phi}_N| + |\psi_N - \hat{\phi}_N|),\\
|G_N(x,v,u;\phi_N,\psi_N) - G_N(x,v,u;\hat{\phi}_N,\hat{\psi}_N)| &\le A(|\phi_N - \hat{\phi}_N| + |\psi_N - \hat{\phi}_N|),
\end{aligned} \tag{15.8}$$

for $(x,v,u;\phi_N,\psi_N)$ and $(x,v,u;\hat{\phi}_N,\hat{\psi}_N)$ in (15.6-N). Then, Proposition 12-1 is proved if we prove the following

PROPOSITION 15-1. *Let N be an integer satisfying*

$$N \ge \max\left\{\frac{16A}{|\lambda|\sin\epsilon_1}, \ \frac{8}{|\lambda|\sin\epsilon_1}, \ \frac{4}{\sin\epsilon_1}, \ |\lambda|\sin\epsilon_1\right\}. \tag{15.9}$$

Then, (15.4) has a unique solution $\{\phi_N(x,V(x),U(x)),\ \ \psi_N(x,V(x),U(x))\}$ such that

$$|\phi_N(x,V(x),U(x))| \le \hat{K}_N|U(x)|^N, \qquad |\psi_N(x,V(x),U(x))| \le \hat{K}_N|U(x)|^N \tag{15.10}$$

whenever $(x,V(x),U(x))$ is in a domain of the form (15.6-N).

Proposition 12.1 follows from Proposition 15-1 in the following manner. By the transformation (15.3), (E.1) has a solution

$$\begin{aligned}
&\sum_{k=1}^{N-1} P_k(x,V(x))U(x)^k + \phi_N(x,V(x),U(x)),\\
Q_0(x,V(x)) + x&\sum_{k=1}^{N-1} Q_k(x,V(x))U(x)^k + \psi_N(x,V(x),U(x)),
\end{aligned} \tag{15.11}$$

provided that $(x, V(x), U(x))$ is in (15.6-N). Let N' be an integer greater than N. Then

$$\begin{aligned}&\sum_{k=N}^{N'-1} P_k(x, V(x))U(x)^k + \phi_{N'}(x, V(x), U(x)),\\ x&\sum_{k=N}^{N'-1} Q_k(x, V(x))U(x)^k + \psi_{N'}(x, V(x), U(x)),\end{aligned} \tag{15.12}$$

is a solution of (15.4). This solution satisfies the condition (15.10) if $(x, V(x), U(x))$ is in the common part of (15.6-N) and (15.6-N′). Hence, by the uniqueness of the solution of (15.4), as assured by Proposotion 15-1, (15.12) must coincide with $\{\phi_N(x, V(x), U(x)), \psi_N(x, V(x), U(x))\}$. Thus the solution expressed by (15.11) is independent of N, provided that N satisfies (15.1). Denote this solution by $\{\tilde{\phi}(x, V(x), U(x)), \tilde{\psi}(x, V(x), U(x))\}$. Then, by means of analytic continuation, the function $\tilde{\phi}(x, v, u)$ and $\tilde{\psi}(x, v, u)$ are defined in the domain

$$\begin{aligned}&|x| < \xi_0, \quad \arg\lambda - \frac{\pi}{2} + \epsilon_1 < \arg x < \arg\lambda + \frac{\pi}{2} - \epsilon_1,\\ &|v| < r(\arg v; \delta_0, \epsilon_2), \quad -\pi - \arg b + \epsilon_2 < \arg v < \pi - \arg b - \epsilon_2,\\ &|u| < \gamma_0,\\ &|\phi_N| < \beta_0, \qquad |\psi_N| < \beta_0.\end{aligned} \tag{15.13}$$

Here $\xi_0 = \sup \xi_N$, $\delta_0 = \sup \delta_N$, $\gamma_0 = \sup \gamma_N$ and $\beta_0 = \sup \beta_N$. Thus, Proposition 12.1 is proved.

16. Proof of Proposition 15-1. To show Proposition 15-1, let (x_1, v_1, u_1) be an arbitrary point of (15.6-N), $V(x)$ be the solution of (6.3) such that $V(x_1) = v_1$ and $U(x)$ be the solution of (6.2) such that $U(x_1) = u_1$. Note that a positive integer N satisfying (15.10) also satisfies (15.1). Furthermore, $\{\phi_N, \psi_N\}$ are given by the integral equations

$$\begin{aligned}\phi_N(x_1, v_1, u_1) &= \exp\left[-\frac{\lambda}{x_1}\right]\int_0^{x_1} \frac{1}{x^2}\exp\left[\frac{\lambda}{x}\right] F_N(x, V(x), U(x); \phi_N, \psi_N)dx,\\ \psi_N(x_1, v_1, u_1) &= \int_0^{x_1} \frac{1}{x} G_N(x, V(x), U(x); \phi_N, \psi_N)dx,\end{aligned} \tag{16.1}$$

where the integral is taken on the segment $\overline{0x_1}$. Consider the successive approximations:

$$\begin{aligned}
&\phi_N^{(0)}(x_1,v_1,u_1)=0, \qquad \psi_N^{(0)}(x_1,v_1,u_1)=0,\\
&\phi_N^{(h+1)}(x_1,v_1,u_1)=\exp\left[-\frac{\lambda}{x_1}\right]\int_0^{x_1}\frac{1}{x^2}\exp\left[\frac{\lambda}{x}\right]F_N(x,V(x),U(x);\phi_N^{(h)},\psi_N^{(h)})dx,\\
&\psi_N^{(h+1)}(x_1,v_1,u_1)=\int_0^{x_1}\frac{1}{x}G_N(x,V(x),U(x);\phi_N^{(h)},\psi_N^{(h)})dx,\\
&\qquad\qquad (h=0,1,2,\cdots).
\end{aligned}\tag{16.2}$$

Note that, by (7.10) and (6.1), since $\rho_1<1$,

$$\frac{d}{ds}|u|^N=N|u|^{N-1}\frac{d|u|}{ds}\geq\frac{N|\lambda|\sin\epsilon_1}{2|x|^2}|u|^N>\frac{N|\lambda|\sin\epsilon_1}{2|x|}|u|^N.\tag{16.3}$$

Also, by (7.10) and (15.10),

$$\begin{aligned}
\frac{d}{ds}|u|^N\exp\left[\Re\left(\frac{\lambda}{x}\right)\right]&=\exp\left[\Re\left(\frac{\lambda}{x}\right)\right]|u|^N\left\{\frac{N}{|u|}\frac{d|u|}{ds}+\frac{d}{ds}\Re\left(\frac{\lambda}{x}\right)\right\}\\
&\geq\exp\left[\Re\left(\frac{\lambda}{x}\right)\right]|u|^N\left\{\frac{N|\lambda|}{2|x|^2}\sin\epsilon_1-\frac{|\lambda|}{|x|^2}\right\}\geq\frac{N|\lambda|\sin\epsilon_1}{4|x|^2}\exp\left[\Re\left(\frac{\lambda}{x}\right)\right]|u|^N.
\end{aligned}\tag{16.4}$$

Now, we can show that

$$\begin{aligned}
&|\phi^{(h)}(x_1,v_1,u_1)-\phi^{(h-1)}(x_1,v_1,u_1)|\leq\frac{\hat{K}_N}{2^h}|u_1|^N,\\
&|\phi^{(h)}(x_1,v_1,u_1))|\leq\hat{K}_N\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^h}\right)|u_1|^N,\\
&|\psi^{(h)}(x_1,v_1,u_1)-\psi^{(h-1)}(x_1,v_1,u_1)|\leq\frac{\hat{K}_N}{2^h}|u_1|^N,\\
&|\psi^{(h)}(x_1,v_1,u_1))|\leq\hat{K}_N\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^h}\right)|u_1|^N,\\
&\qquad\qquad (h=1,2,\cdots),
\end{aligned}\tag{16.5}$$

for (x_1,v_1,u_1) in (15.6-N).

From these inequalities, $\{\phi^{(h)}(x_1,v_1,u_1)\}$ and $\{\psi^{(h)}(x_1,v_1,u_1)\}$ converge absolutely and uniformly to $\phi_N(x_1,v_1,u_1)$ and $\psi_N(x_1,v_1,u_1)$, respectively, in (15.6-N) satisfying Proposition 15-1.

To see (16.5), by (15.8), (16.3), (16.4) and (15.10), we have

$$
\begin{aligned}
&|\phi_N^{(1)}(x_1,v_1,u_1)| \\
&= \exp\left[\Re\left(-\frac{\lambda}{x_1}\right)\right]\left|\int_0^{x_1}\frac{1}{x^2}\exp\left[\Re\left(\frac{\lambda}{x}\right)\right]F_N(x,V(x),U(x);0,0)dx\right| \\
&\le \exp\left[\Re\left(-\frac{\lambda}{x_1}\right)\right]\hat{K}_N\left|\int_0^{x_1}\frac{1}{|x|^2}\exp\left[\Re\left(\frac{\lambda}{x}\right)\right]|u|^N ds\right| \\
&\le \exp\left[\Re\left(-\frac{\lambda}{x_1}\right)\right]\frac{4\hat{K}_N}{N|\lambda|\sin\epsilon_1}\exp\left[\Re\left(\frac{\lambda}{x_1}\right)\right]|u_1|^N \le \frac{\hat{K}_N}{2}|u_1|^N,
\end{aligned} \tag{16.6}
$$

$$
\begin{aligned}
|\psi_N^{(1)}(x_1,v_1,u_1)| &= \left|\int_0^{x_1}\frac{1}{x}G_1^{(N)}(x,V(x),U(x);0,0)ds\right| \\
\le \hat{K}_N\left|\int_0^{x_1}\frac{1}{|x|}|u|^N ds\right| &\le \frac{2\hat{K}_N}{N|\lambda|\sin\epsilon_1}|u_1|^N \le \frac{\hat{K}_N}{2}|u_1|^N.
\end{aligned} \tag{16.7}
$$

Thus (16.5) is true for $h=1$.

Now assume that (16.5) is true for $h=j-1$. Then, by (16.2), the assumption on (16.5), (16.3), (16.4) and (15.10), we have

$$
\begin{aligned}
&|\phi_N^{(j)}(x_1,v_1,u_1)-\phi_N^{(j-1)}(x_1,v_1,u_1)| \\
&= \exp\left[\Re\left(-\frac{\lambda}{x_1}\right)\right]\left|\int_0^{x_1}\frac{1}{x^2}\exp\left[\Re\left(\frac{\lambda}{x}\right)\right]\right. \\
&\qquad \left.\left\{F_N(x,V(x),U(x);\phi_N^{(j-1)},\psi_N^{(j-1)})-F_N(x,V(x),U(x);\phi_N^{(j-2)},\psi_N^{(j-2)})\right\}dx\right| \\
&\le \exp\left[\Re\left(-\frac{\lambda}{x_1}\right)\right]\frac{2\hat{K}_N A}{2^{j-1}}\left|\int_0^{x_1}\frac{1}{|x|^2}\exp\left[\Re\left(\frac{\lambda}{x}\right)\right]|u|^N ds\right| \\
&\le \exp\left[\Re\left(-\frac{\lambda}{x_1}\right)\right]\frac{2\hat{K}_N A}{2^{j-1}}\frac{4}{N|\lambda|\sin\epsilon_1}\exp\left[\Re\left(\frac{\lambda}{x_1}\right)\right]|u_1|^N \le \frac{\hat{K}_N}{2^j}|u_1|^N,
\end{aligned} \tag{16.8}
$$

$$
\begin{aligned}
&|\psi_N^{(j)}(x_1,v_1,u_1)-\psi_N^{(j-1)}(x_1,v_1,u_1)| \\
&= \left|\int_0^{x_1}\frac{1}{x}\left\{G_N(x,V(x),U(x);\phi_N^{(j-1)},\psi_N^{(j-1)})\right.\right. \\
&\qquad \left.\left.-G_N(x,V(x),U(x);\phi_N^{(j-2)},\psi_N^{(j-2)})dx\right\}\right| \\
&\le \frac{2\hat{K}_N A}{2^{j-1}}\left|\int_0^{x_1}\frac{1}{|x|}|u|^N ds\right| \le \frac{2\hat{K}_N A}{2^{j-1}}\frac{4}{N|\lambda|\sin\epsilon_1}|u_1|^N \le \frac{K_N}{2^j}|u_1|^N.
\end{aligned} \tag{16.9}
$$

Thus (16.5) is true for $h = j$. Hence, by the principle of mathematical induction, (16.5) is true for all positive integers h.

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